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On the Interpretation of the Einstein–Cartan Formalism

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33.1 Introduction

Hehl and collaborators [1] have suggested the need to generalize Riemannian geometry to a metric-affine geometry, by admitting torsion and nonmetricity of the connection field. They assume that this geometry represents the microstructure of space-time, with Riemannian geometry emerging as some sort of macroscopic average over the metric-affine microstructure. They thereby generalize the earlier approach to the Einstein-Cartan formalism of Hehl et al. [2] based on a metric connection with torsion. A particularly clear statement of this point of view is found in Hehl, von der Heyde, and Kerlick [3]: "We claim that the [Einstein-Cartan] field equations . . . are, at a classical level, the correct microscopic gravitational field equations. Einstein's field equation ought to be considered a macroscopic phenomenological equation of limited validity, obtained by averaging [the Einstein-Cartan field equations]" (p. 1067).

Adamowicz takes an alternate approach [4], asserting that "the relation between the Einstein-Cartan theory and general relativity is similar to that between the Maxwell theory of continuous media and the classical microscopic electrodynamics" (p. 1203). However, he only develops the idea of treating the spin density that enters the Einstein-Cartan theory as the macroscopic average of microscopic angular momenta in the linear approximation, and does not make explicit the relation he suggests by developing a formal analogy between quantities in macroscopic electrodynamics and in the Einstein-Cartan theory.

In this paper, I shall develop such an analogy with macroscopic electrodynamics in detail for the exact, nonlinear version of the Einstein-Cartan theory. I discuss a correspondence between linearized solutions to the Einstein-Cartan equations

[1] Dedicated to Engelbert Schucking on his seventieth birthday with friendship and appreciation of his many contributions to science as a scholar and as a human being.

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and linearized solutions to the Einstein equations that strongly supports the interpretation of the Einstein-Cartan theory as a macroscopic theory. Finally, I discuss the prospects for extending these results to exact solutions to the Einstein-Cartan equations, and of generalizing the macroscopic approach of this paper to the full metric-affine theory.

33.2 The Analogy: Electromagnetism

I shall start with what, in order to use a terminology parallel to that used in the discussions of gravitation theory cited above [5], I shall call microscopic electrodynamics. But it should be born in mind that the term "microscopic" here is really a misnomer. It does not carry any implication that the 4-current density introduced is to be interpreted atomistically. It is assumed to be a continuous, differentiable function of its arguments, and insofar as the atomistic structure of charged matter is taken into account, some sort of averaging process is assumed to have already been done. What is implied by the term "microscopic" is merely that *all* sources of charge and current, both "free" and "bound," have been included in its evaluation.

In the 4-dimensional version of microscopic electrodynamics, two antisymmetric tensor fields are introduced: the covariant tensor or 2-form $F_{\alpha\beta}$, which incorporates the E and B fields in a particular inertial frame of reference into one 4-dimensional Lorentz-covariant field; and the contravariant tensor $G^{\alpha\beta}$, which similarly incorporates the D and H fields [6]. The first set of microscopic Maxwell field equations assert that curl F vanishes:

$$F_{[\alpha\beta,\tau]} = 0 \quad \text{or} \quad \mathbf{d}F = 0 \tag{1}$$

in the differential forms notation, i.e., F is a closed form. Locally, at least, this implies that the $F_{\alpha\beta}$ field can be derived from a 4-vector potential A_β, which incorporates the usual scalar and vector potentials ϕ and $\mathbf{A}$. Indeed, aside from topological complications (which do not occur in Minkowski spacetime), this first set of Maxwell equations (1) is equivalent to:

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta} \quad \text{or} \quad \mathbf{d}A, \tag{2}$$

i.e., the form F is exact [7]. These equations may be interpreted as postulating the absence of magnetic monopoles. Given the field F, the potentials are not unique, but only determined up to a gauge transformation:

$$A'_\alpha = A_\alpha + \phi_{,\alpha} \quad \text{or} \quad \mathbf{A}' = \mathbf{A} + \mathbf{d}\phi. \tag{3}$$

As we shall see, Eq. (1) and its consequences are actually common to both the microscopic and macroscopic versions of Maxwell's theory. So I shall refer to them just as the first set of Maxwell equations. The second set of microscopic Maxwell field equations assert that div G equals the total charge-current density:

$$G^{\alpha\beta}{}_{,\beta} = j^\alpha \tag{4}$$

Here j^α is the charge-current 4-vector density, which incorporates the usual charge and current densities p and j. As emphasized above, this includes both "bound" and "free" quantities.

In order to proceed with any applications of the two sets of microscopic Maxwell equations, it is necessary to introduce some relation between the F and G fields. In vacuum, the two fields are related by means of the metric tensor $g_{\alpha\beta}$:

$$G^{\alpha\beta} = g^{\alpha\delta} g^{\beta\tau} F_{\delta\tau} = F^{\alpha\beta} \tag{5}$$

(depending on the system of units used, a constant ϵ_0 representing the "polarization of the vacuum" may be introduced). So in vacuum, the two tensors are effectively equal [8].

Now we come to what I shall call macroscopic Maxwell theory, in which it is assumed that the charges and currents can be divided into "free" and "bound." This is done formally by introducing a third antisymmetric contravariant tensor field, the polarization tensor $p^{\alpha\beta}$, inside of matter. This tensor incorporates the usual **P** and $-$**M** fields, i.e., the electric and magnetic polarization vectors, respectively. Its value will depend on the properties of the matter under consideration, as well as on the electromagnetic fields to which the matter is subject. Some *Ansatz* for the form of these vectors in the rest frame at a point of the matter then must be introduced. The adequacy of the postulated relations is judged by the success of the resulting theory in explaining the observed electrodynamical properties of the matter.

It is also possible to go a step further by introducing a microscopic model of the matter, and deriving the form of the polarization tensor by some sort of averaging process over the multipole moments of the atomistic constituents of the matter. In the usual treatment of dielectrics, for example, the electric polarization vector is derived by averaging over the intrinsic or induced electric dipole moments of these constituents; while the magnetic polarization vector is similarly derived from an averaging over their magnetic dipole moments. In particular, if it is assumed that there are no intrinsic magnetic dipole moments, the averaged magnetic dipole moment will arise entirely from the circulation of microscopic charge.

More careful treatments emphasize that these are merely the first two terms in a multipole expansion, the terms of which decrease rapidly in value; and may even evaluate the next term, the contribution to the electric polarization from the electric quadrupole moments of the atomistic constituents [9].

The divergence of the polarization tensor gives the "bound" charge-current 4-vector:

$$p^{\alpha\beta}{}_{B} = j_B{}^{\alpha} \,. \tag{6}$$

By defining a new, macroscopic field G_M:

$$G_M{}^{\alpha\beta} = G^{\alpha\beta} - p^{\alpha\beta} \tag{7}$$

and using equations (6), one can rewrite the second set of microscopic Maxwell equations (4) as

$$G_M{}^{\alpha\beta}{}_{\beta} = j_F{}^{\alpha} \,, \tag{8}$$

where the free charge-current four-vector is defined by:

$$j_F{}^\alpha = j^\alpha - j_B{}^\alpha \,. \tag{9}$$

Equations (1), the first set of Maxwell equations for the F field; (5) relating F and G; (7) defining the G_M field in terms of the G and P fields, and (8), the second set of macroscopic Maxwell equations for the G_M field with the j_F field as its source, constitute the equations defining macroscopic Maxwell theory. They must be supplemented by prescriptions for the P and j_F fields, plus boundary conditions at the interface between matter and vacuum and at infinity, in order to solve particular problems.

33.3 The Analogy: Gravitation

Let us try to treat gravitation theory in an analogous way. For the microscopic Einstein theory, we introduce two fields, the Christoffel symbols of the first kind $[\alpha\beta, \tau]$ and a symmetric connection $\Gamma_E{}^\mu{}_{\alpha\beta} = \Gamma_E{}^\mu{}_{\beta\alpha}$, which we shall call the Einstein connection. The Christoffel symbols of the first kind are derived from a set of potentials $g_{\alpha\beta}$:

$$[\alpha\beta, \tau] = \frac{1}{2}(-g_{\alpha\beta,\tau} + g_{\tau\alpha,\beta} + g_{\beta\tau,\alpha}) \,. \tag{10}$$

Equation (10) is analogous to Eq. (2), the definition of the $F_{\alpha\beta}$ field in terms of the potentials, so we take the Christoffel symbols of the first kind as the gravitational analogue of the F field. Just as in the electromagnetic case, these definitions may be interpreted as postulating the absence of a gravitational analogue of magnetic monopoles. In this sense, they are more fundamental than the set of conditions on the Christoffel symbols of the first kind that are equivalent locally to the existence of the gravitational potentials $g_{\alpha\beta}$, which are easily derived from the commutativity of the second derivatives of the metric tensor:

$$([\alpha\beta, \tau] + [\alpha\tau, \beta])_{,\delta} - ([\delta\beta, \tau] + [\delta\tau, \beta])_{,\alpha} \,. \tag{11}$$

Equation (11) is then the gravitational analogue of Eq. (1), the first set of Maxwell equations.

The analogue of Eq. (4), the second set of microscopic Maxwell equations, are the Einstein equations for the Einstein connection Γ_E:

$$G_{\alpha\beta}(\Gamma_E) = T_{\alpha\beta}, \qquad G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^\tau{}_\tau, \tag{12}$$

where the Riemann tensor, Ricci tensor, Ricci scalar, and Einstein tensor are computed from the Einstein connection in the usual way (depending on the units used, there may be a coupling constant depending on G, the Newtonian gravitational constant, on the right-hand side of this equation, and in several subsequent equations). The $T_{\alpha\beta}$ field includes the gravitational analogues of both the "free" and "bound" material charges and currents. Like the $G_{\alpha\beta}$ field, the $T_{\alpha\beta}$ field is symmetric in its indices.

We see that the Einstein connection is the gravitational analogue of the G field in electrodynamics. In vacuum, the Einstein connection and the Christoffel symbols of the first kind are related by the metric tensor

$$\Gamma_E{}^{\mu}{}_{\alpha\beta} = g^{\mu\alpha}[\alpha\beta, \sigma] = \{^{\mu}{}_{\alpha\beta}\}, \tag{13}$$

where $\{^{\mu}{}_{\alpha\beta}\}$ are the Christoffel symbols of the second kind. So the Einstein connection is the symmetric, metric connection, for which $g_{\alpha\beta;E\mu} = 0$, where a semicolon followed by a subscript E denotes covariant differentiation with respect to the Einstein connection.

Note the dual role that, as usual, the tensor field $g_{\alpha\beta}$ plays in general relativity: it is both the metric tensor and the potentials for the Christoffel symbols of the first kind. Because of this dual role, the treatment of gauge transformations here differs somewhat from that in electromagnetism. Here, diffeomorphisms play the role of gauge transformations. A one-parameter family of such diffeomorphisms is generated by any vector field v^{μ}; if the value of the parameter is the infinitesimal ϵ, then the tensor field $g_{\alpha\beta}$ is dragged into

$$g'_{\alpha\beta} = g_{\alpha\beta} + \epsilon \pounds_v g_{\alpha\beta}, \tag{14}$$

where $\pounds_v$ is the Lie derivative with respect to the vector field v^{μ} [10]. Since the operations of Lie derivation and partial differentiation commute, Lie differentiation of Eq. (10) tells us that the Christoffel symbols of the first kind do not remain invariant, but are also dragged along by the diffeomorphism. But this is just what we should expect. Since the g field represents the metric as well as the gravitational potentials, when the metric is dragged by a diffeomorphism, any other field *must* also be dragged along by that diffeomorphism precisely in order to maintain the *same* values at each *physical* point [11].

Now we turn to the macroscopic Einstein-Cartan theory. In this theory, another connection Γ_c is introduced, which we shall call the Cartan connection, the gravitational analogue of the G_M, the macroscopic electromagnetic field. It is assumed that this connection is still metric, that is that the nonmetricity tensor, the covariant derivative of the metric tensor with respect to the Cartan connection, vanishes. That is, $g_{\alpha\beta;C\mu} = 0$, where a semicolon followed by a subscript C denotes the covariant derivative with respect to the Cartan connection. This implies that the difference between the Cartan and Einstein connections depends only on the torsion tensor S:

$$S^{\mu}{}_{\alpha\beta} = \frac{1}{2}(\Gamma_C{}^{\mu}{}_{\alpha\beta} - \Gamma_C{}^{\mu}{}_{\beta\alpha}), \tag{15}$$

which is obviously antisymmetric in its lower indices. If we define a tensor K, called the contorsion tensor [12], by

$$K^{\mu}{}_{\alpha\beta} = -S^{\mu}{}_{\alpha\beta} + S_{\alpha\beta}{}^{\mu} - S_{\beta}{}^{\mu}{}_{\alpha}, \tag{16}$$

where the indices of the torsion tensor are raised and lowered with the metric tensor, then it follows that

$$\Gamma_C{}^{\mu}{}_{\alpha\beta} = \{^{\mu}{}_{\alpha\beta}\} - K^{\mu}{}_{\alpha\beta}. \tag{17}$$

With this mathematical background, I return to the physical description of the macroscopic theory. It is assumed that the stress-energy tensor can be divided into "free" and "bound" portions. This is done formally by assuming that inside matter the torsion tensor S and hence the contorsion tensor K do not vanish. Indeed, from Eq. (17) we see that the contorsion tensor is the gravitational analogue of the polarization tensor (see Eq. (7)); like the latter, it is assumed to depend on the properties of the matter being considered as well as on the gravitational field to which the matter is subject. Just as in the electromagnetic case, its form may simply be postulated in a fully macroscopic theory; or an attempt may be made to derive it from a microscopic model of the medium. Here, I shall follow an intermediate course, postulating its form, but motivating the *Ansatz* by a microscopic argument.

I proceed again by analogy with the electromagnetism. In the case of most dielectrics, the elementary constituents (atoms and molecules) are electrically neutral. Therefore, their electric dipole moments (intrinsic or induced) are an invariant property (i.e., independent of the origin chosen for their evaluation), which can be averaged over a volume element to give a macroscopic electric dipole moment per unit volume. In the gravitational case, there is no evidence for the existence of negative mass, so that we cannot expect the elementary constituents of matter to have an invariant mass dipole moment. Indeed, by choosing the evaluation point for each such constituent at its center of mass, we can make the mass dipole moment vanish. Of course, the elementary constituents will have an invariant mass quadrupole moment (evaluated at the center of mass point), which can in principle contribute to the macroscopic contorsion tensor. We shall return to this point in the concluding section, but here we shall assume that this contribution may be neglected.

We shall here consider only the gravitational analogue of the magnetic dipole moment (we have seen that Eq. (10) implies the absence of gravitational analogues of magnetic monopoles), treated at the macroscopic level. There is a four-velocity field U^{μ} associated with each point inside matter, and I shall assume that, in the rest frame defined at each point by this velocity field, a spin vector exists, which has only spatial ("magnetic-type") components. This is equivalent to the so-called Weysenhoff *Ansatz* [13] for the form of the spin-tensor density field $s^{\mu}{}_{\alpha\beta}$:

$$s^{\mu}{}_{\alpha\beta} = U^{\mu} s_{\alpha\beta}, \quad \text{where} \quad s_{\alpha\beta} = -s_{\beta\alpha}, \quad s_{\alpha\beta} U^{\beta} = 0. \tag{18}$$

This means, by the usual association between an antisymmetric 3-dimensional tensor and a 3-vector, that in the 3-space orthogonal to the 4-velocity U^{μ} at each point, the spin tensor is equivalent to a spin vector. In the macroscopic Einstein-Cartan theory, it is assumed that inside matter the spin-tensor density field is equal to the modified torsion tensor $T^{\mu}{}_{\alpha\beta}$ [14]; but since the latter only differs from the torsion tensor by its trace, which vanishes with the Weyssenhoff *Ansatz* (Eq. (18)), it follows that $S^{\mu}{}_{\alpha\beta} = s^{\mu}{}_{\alpha\beta}$ [15]. Hence,

$$K^{\mu}{}_{\alpha\beta} = -s_{\alpha\beta}U^{\mu} + s_{\beta}{}^{\mu}U_{\alpha} - s^{\mu}{}_{\alpha}U_{\beta}, \tag{19}$$

from which it follows that

$$K^{\mu}{}_{\alpha\beta} = 0, \qquad g^{\alpha\beta}K^{\mu}{}_{\alpha\beta} = 0\,. \tag{20}$$

We can now rewrite the Einstein equations (9) inside matter in the form

$$G_{\alpha\beta}(\Gamma_C) = T_{F\alpha\beta}, \tag{21}$$

where $G_{\alpha\beta}(\Gamma_C)$ is the Einstein tensor formed from the Cartan connection, and $T_{F\alpha\beta}$ is the "free" portion of the stress-energy tensor of matter, to be defined in a moment. Note that, since the Cartan connection is not torsion-free, $G_{\alpha\beta}(\Gamma_C)$ is not symmetric.

It remains to discuss the division of the stress-energy tensor into "bound" and "free" portions. Inserting Eq. (17) into the definition of the Ricci tensor for the Einstein-Cartan connection, and utilizing Eq. (20), we get [16]

$$R_{\alpha\beta}(\Gamma_C) = R_{\alpha\beta}(\Gamma_E) - K^{\mu}{}_{\alpha\beta;E\mu} - s_{\sigma\tau}s^{\sigma\tau}U_{\alpha}U_{\beta}. \tag{22}$$

Taking the trace of this equation, and using Eq. (20), we get

$$R(\Gamma_C) = R(\Gamma_E) - s_{\sigma\tau}s^{\sigma\tau}, \tag{23}$$

so that

$$G_{\alpha\beta}(\Gamma_C) = G_{\alpha\beta}(\Gamma_E) - K^{\mu}{}_{\alpha\beta;E\mu} - s_{\sigma\tau}s^{\sigma\tau}\left(U_{\alpha}U_{\beta} - \frac{1}{2}g_{\alpha\beta}\right) \tag{24}$$

Thus, if we define the "bound" stress-energy tensor by

$$T_{B\alpha\beta} = K^{\mu}{}_{\alpha\beta;E\mu} + s_{\sigma\tau}s^{\sigma\tau}\left(U_{\alpha}U_{\beta} - \frac{1}{2}g_{\alpha\beta}\right), \tag{25}$$

and the "free" stress-energy tensor by

$$T_{F\alpha\beta} = T_{\alpha\beta} - T_{B\alpha\beta}, \tag{26}$$

then the macroscopic Einstein-Cartan field equations (21) follow from the microscopic Einstein field equations (12) and these definitions. Note that neither the "bound" nor "free" stress-energy tensors are symmetric.

We have now completed our description of the macroscopic Einstein-Cartan theory, which is based on the two sets of field equations (11)—or (10)—and (21), with the definition of the "free" stress-energy tensor given by equations (25) and (26), and the Weyssenhoff *Ansatz*, Eq. (18).

33.4 Discussion

In the linear approximation to each theory, Adamowicz has proved a correspondence theorem between solutions of the macroscopic Einstein-Cartan equations and the microscopic Einstein quations. He considers a finite source containing a static body with a spin tensor density of the form given by the Weyssenhoff *Ansatz*, Eq. (18), and shows that in this approximation it produces the same external gravitational field as does a rotating body without spin-density field but with a corresponding distribution of rotational angular momentum density in the linearized Einstein theory [18]. As a consequence, in the linear approximation at any rate, a static solution to the Einstein-Cartan equations for an axially symmetric body with a suitable Weyssenhoff spin-density field can produce the same external field as a stationary solution to the Einstein equations representing the same body without any spin-density field, but in rigid rotation about its axis. This is analogous to the situation in electrodynamics, where the external fields of a charged magnet treated by the macroscopic Maxwell equations, and of a rotating charged body, treated by the microscopic Maxwell equations, are the same [19]. The difference is inside the body. If we wanted to treat the magnet microscopically, we would have to associate an intrinsic magnetic moment with each element of the body, and these would add up to produce the same internal field as that of the rotating charged body.

Hence, the analogy here is between the solutions to the macroscopic Einstein-Cartan equations and macroscopic Maxwell equations on the one hand and the solutions to the microscopic Einstein equations and Maxwell equations on the other. We suggest this analogy between solutions is a strong argument for the analogy between the corresponding macroscopic equations on the one hand and the microscopic equations on the other.

This result, taken together with the exact formulation of the macroscopic gravitational-electromagnetic analogy developed in Sections 33.2 and 33.3, suggests that it should be possible to find exact static interior solutions of the macroscopic Einstein-Cartan equations that have the same stationary external fields as corresponding exact stationary solutions of the microscopic Einstein equations. It should even be possible to prove theorems relating entire classes of exact stationary solutions to the Einstein equations for rigidly rotating sources to classes of exact solutions to the Einstein-Cartan equations with the same exterior metric but a static interior solution representing a nonrotating source with corresponding spin tensor density distribution. The question also arises of generalizing the macroscopic approach from the Einstein-Cartan case to the case of metric-affine geometries discussed by Hehl and collaborators [20], in which the connection is no longer metric. Our discussion suggests that the Einstein-Cartan theory with the Weyssenhoff *Ansatz* is adequate to handle the gravitational analogue of magnetic polarization **M**, but cannot treat the gravitational analogue of the electric polarization **P**. We have argued above that no gravitational analogue of the electric dipole moment should exist (see Section 33.3). However, we certainly expect a gravitational analogue of the electric quadrupole moment. In the electromagnetic case, the divergence of the electric quadrupole moment tensor contributes to the electric polarization vector [21]. We suspect that, in the gravitational case, something like the covariant derivative of the quadrupole moment tensor should be related to the nonmetricity tensor, the nonvanishing covariant derivative of the metric [22]

Indeed, the nonmetricity tensor allows generalization of the right-hand side of Eq. (12) above to include a tensor symmetric in its lower indices. I plan to investigate the possibility further. Like Adamowicz I believe that the macroscopic approach to the Einstein-Cartan theory "may be used effectively for solving certain cosmological or astrophysical problems," [23] and the extension of the analogy to matter with intrinsic or induced quadrupole moments would considerably extend its range of applicability, in particular to problems involving interactions of gravitational radiation with matter.

On the formal side, this paper (at least its gravitational part) has utilized exclusively the tensor calculus. However, it is well known that the Einstein-Cartan theory can be rewritten more perspicaciously in Cartan's language of differential forms [24], and Hehl and collaborators have used this language for its generalization [25]. The results obtained here, as well as possible generalizations, should be rewritten in terms of differential forms.

Notes and References

1. Friedrich Hehl, J. Dermott McCrea, Eckehard W. Mielke, Yuval Ne'eman, "Metric-Affine Gauge Theory of Gravity: Field Equations, Noether Identities, World Spinors, and Breaking of Dilation Invariance," *Physics Reports* **258**, Nos. 1 & 2: 1–171 (1995).

2. See, e.g., Friedrich W. Hehl, Paul von der Heyde, G. David Kerlick, "General relativity with spin and torsion: Foundations and prospects," *Reviews of Modern Physics* **48**, 393–416 (1976).

3. Friedrich Hehl, Paul van der Heyde, and G. David Kerlick, General relativity with spin and torsion and its deviations from Einstein's theory," *Physical Review* **D10**, 1066–1069 (1974).

4. W. Adamowicz, "Equivalence between the Einstein-Cartan and General Relativity Theories in the Linear Approximation for a Classical Model of Spin," *Bulletin de l'Academie Polonaise des Sciences Serie des science, mathematiques astronomiques et physiques* **23**, 1203–1205 (1975).

5. In this section I follow an approach to electromagnetism well summarized in Attay Kovetz, *The Principles of Electromagnetic Theory* (Cambridge University Press, 1990), although I do not always follow his terminology and notation.

6. Strictly speaking, we should treat the *G* field as a tensor density field of weight one (see, e.g., John Stachel, "The Generally Covariant Form of Maxwell's Equations," in Melvin S. Berger, ed., *J.C. Maxwell: The Sesquicentennial Symposium* (Amsterdam/New York/Oxford: North-Holland, 1984), pp. 23–37. But since we are only introducing the electromagnic field for purposes of comparison, we avoid this complication here.

7. For a review of closed and exact forms, Poincare's lemma, and the possible topological complications, see John Stachel, "Globally stationary but locally static

space-times: A gravitational analogue of the Aharonov-Bohm effect," *Physical Review D* **26**, 1281–1290 (1982).

8. Microscopic electrodynamics in a homogeneous, isotropic medium can be treated by introducing an "optical metric," and even a nonlinear, phenomenological "vacuum polarization" can be introduced; both are discussed in the reference in note 6.

9. For such a careful treatment, see, e.g., Leon Rosenfeld, *Theory of Electrons* (Amsterdam: North-Holland, 1951), repr. with new Preface (New York: Dover, 1965).

10. For the definition of the Lie derivative, see e.g. Jan A. Schouten, *Ricci-Calculus* (Berlin/Gottingen/Heidelberg: Springer, 1954), pp. 102–111. For a review of the Lie derivative and its application to the Cauchy problem in general relativity, see John Stachel, "Covariant Formulation of the Cauchy Problem in Generalized Electrodynamics and General Relativity," *Acta Physica Polonica* **35**, 689–709 (1969).

11. This is essentially what general covariance means. See, e.g., John Stachel, "The Meaning of General Covariance: The Hole Story," in John Earman et al., eds., *Philosophical Problems of the Internal and External World/Essays on the Philosophy of Adolf Grünbaum* (Konstanz: Universitatsverlag/ Pittsburgh: University of Pittsburgh Press, 1993), pp. 129–160.

12. See, e.g., Hehl et al. (reference in footnote 2), p. 397. Note that I have placed the upper index first.

13. Jan Weyssenhoff and A. Raabe, "Relativistic dynamics of spinfluids and spin-particles," *Acta Physica Polonica* **9**, 7-xx (1947).

14. For its definition, see, e.g., the reference in note 2.

15. Depending on the units used, the proportionality factor G may be introduced here.

16. See Schouten, reference in note 10, p. 141, Eq. (4.23a), remembering that Schouten's T is defined with the opposite sign to K (see Eq. (4.20)), and that the trace of T vanishes in this case. Note that formula (4.23a) is true in an arbitrary affine-metric space, i.e., even if the nonmetricity tensor does not vanish, and we do not make the Weyssenhoff *Ansatz* for the torsion.

17. See reference in note 4.

18. Adamowicz speaks of a Weyssenhoff fluid, but the stress tensor of the body does not enter in the linear approximation, so it can be an arbitrary body.

19. We stipulate that the body be charged in order to have a better analogy with the gravitational case in which a body always has mass.

20. See the reference in note 1.

21. See, e.g., pp. 20, 22, Rosenfeld reference in note 9.

22. For the nonmetricity tensor, see the Hehl et al., reference in note 2, p. 397. Equations (2.8), (2.9) on that page show that the nonmetricity tensor would supply an additional term in the constitutive relation relating the metric-affine connection and the Christoffel symbols; and as noted in note 15, such a term could easily be included in the relation between the metric-affine and Christoffel Ricci tensors.

23. Reference in note 4, p. 1203.

24. See, e.g., Andrzej Trautman, "On the structure of the Einstein-Cartan equations," in *Differential Geometry, Symposia Mathematica*, v. **12** (Academic Press, London 1973), p. 139.
25. See the reference in note 1.