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A New Lagrangian for the Vacuum Einstein Equations and Its Tetrad Form[1]

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Abstract

We give a modification of the Palatini Lagrangian for the free gravitational field that yields the vanishing of the torsion as a result of the field equations and requires only the assumption of the symmetry of the metric. We transcribe this Lagrangian into the tetrad formalism and show how the tetrad form of the Einstein field equations follows from it. Some remarks on possible generalization to a theory with nonvanishing torsion in the presence of matter conclude the paper.

§(1): *Introduction*

The derivation of a set of field equations from a Lagrangian is often very helpful in discussing the symmetry properties of the equations, conservation laws, invariance properties of the field, and so on. It is useful to have as many properties as possible follow from the variational principle, without the need to impose additional constraints on the variations themselves when deriving the field equations.

As is well known (see, for example [9], p. 107), in the Palatini variational principle for the Einstein field equations, the metric and the affine connection are treated as independent variables in the variation. With the assumption that

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both are symmetric, the Einstein field equations as well as the definition of the affine connection as the metrical one, given by the Christoffel symbols, follow from these variations. As is equally well known, without the assumption of the symmetry of the affinity, the correct equations for this definition do not follow from the Palatini variational principle.

The usual way out of this difficulty is to impose the condition of symmetry on the coordinate components of the affine connection. When a tetrad formalism is used, however, the symmetry of the affine connection is *not* reflected in a similar symmetry of the anholonomic (tetrad) components of the affine connection. This difficulty has been sidestepped in the past by implicitly imposing the condition that the affinity be metric (though not symmetric), when taking variations. (See for example [10] or [5]. This point will be discussed in more detail in the concluding section.) This is certainly unsatisfactory, since it violates the spirit of the Palatini approach, which assumes no a priori relation between metric and affinity.

In addition, in the tetrad formalism used in [10] and [5] a mixed form of the affine connection is employed, with one coordinate and two tetrad indices. This is also necessary when the theory is rewritten in terms of tensor-valued exterior forms (see [11]). But the tetrad analogues of the coordinate components of the affinity are the anholonomic components of the connection (a.c.c.); and the affine geometry of a manifold can be completely described in terms of a set of tetrad vector fields and these anholonomic components. As for the metrical properties, we need only use the tetrad and the metric tensor in the tangent space, or tetrad metric as we shall call it for short, to describe these. Therefore, it should be possible to write a variational principle for the Einstein equations entirely in terms of the tetrad metric, the a.c.c., and the tetrad vectors, without ever introducing coordinate components of the affine connection.

In this paper we shall show that both these problems can be solved. First we shall prove that there is a modified Lagrangian with the following remarkable property. When we apply the Palatini variational technique to it, we obtain the correct affinity and the Einstein free gravitational field equations with only the assumption that the metric is symmetric. Then we shall show that the tetrad transcription of the Lagrangian yields, when we apply the Palatini variational technique to it, the tetrad transcription of the Einstein equations as well as the correct relation between the tetrad metric, the tetrad vectors, and the a.c.c.

As a preliminary to this second part we shall review the tetradial formalism in Section 3. In a concluding section, we shall make some remarks about the generalization of our Lagrangian to the case when matter is present.

§(2): *The Lagrangian*

We consider a four-dimensional manifold, with coordinates x^{μ} ($\mu = 0, 1, 2, 3$), on which there is defined a metric tensor field $g_{\lambda\mu}$ and an affine connection $\Gamma^\lambda_{\mu\nu}$. For the moment we shall not assume the symmetry of either field. The usual Lagrangian may now be formed:

$$\mathcal{L}_0 = \tilde{g}^{\lambda\mu} R_{\lambda\mu} \tag{2.1}$$

where $\tilde{g}^{\lambda\mu} = \sqrt{-g}\, g^{\lambda\mu}$ and

$$R_{\lambda\mu} = \Gamma^\kappa_{\lambda\mu,\kappa} - \Gamma^\kappa_{\kappa\mu,\lambda} + \Gamma^\kappa_{\kappa\rho}\Gamma^\rho_{\lambda\mu} - \Gamma^\kappa_{\lambda\rho}\Gamma^\rho_{\kappa\mu} \tag{2.2}$$

Variation of this Lagrangian in the general nonsymmetric case has been discussed in detail in connection with the Einstein-Straus unified field theory (see [1]; [9], p. 108). The results of this discussion are as follows. The variation with respect to $\Gamma^\lambda_{\mu\nu}$ yields an equation that does not completely determine the $\Gamma^\lambda_{\mu\nu}$ in terms of the $g_{\lambda\mu}$. Indeed the vector

$$\Gamma_\lambda = \tfrac{1}{2}(\Gamma^\mu_{\lambda\mu} - \Gamma^\mu_{\mu\lambda}) = \Gamma^\mu_{[\lambda\mu]} \tag{2.3}$$

(which is indeed a vector, since it is the trace of the torsion tensor $\Gamma^\kappa_{[\mu\lambda]}$) is completely undetermined by this equation, which on the other hand does impose the following condition on $\tilde{g}^{\lambda\mu}$:

$$(\tilde{g}^{\lambda\mu} - \tilde{g}^{\mu\lambda})_{,\mu} = 0 \tag{2.4}$$

If we assume the $g_{\lambda\mu}$ to be symmetric, the last equation is satisfied, but the $\Gamma^\lambda_{\mu\nu}$ still remain arbitrary to the extent of an arbitrary choice of Γ_λ. On the other hand, variation of (2.1) with respect to $\tilde{g}^{\lambda\mu}$ yields

$$R_{\lambda\mu} = 0 \tag{2.5}$$

without, of course, implying the symmetry of $R_{\lambda\mu}$ in the general case.

Now let us trace the terms in the Lagrangian that are causing the difficulty. Since the problem comes from the antisymmetric part of the affine connection, it seems reasonable to separate out the terms in (2.2) that arise from the existence of this antisymmetric part. The first and third terms will automatically be symmetrized in μ and λ when $R_{\lambda\mu}$ is contracted with $\tilde{g}^{\lambda\mu}$ (assumed symmetric from now on, except in Section 3) in the Lagrangian (2.1); so we need merely to symmetrize the remaining index pair in the third term and all the index pairs in the second and fourth terms and see what additional terms arise. In this way, we find that

$$\begin{aligned}\mathcal{L}_0 = \tilde{g}^{\lambda\mu}[\Gamma^\kappa_{(\lambda\mu),\kappa} - \Gamma^\kappa_{(\kappa\mu),\lambda} + \Gamma^\kappa_{(\kappa\rho)}\Gamma^\rho_{(\lambda\mu)} - \Gamma^\kappa_{(\lambda\rho)}\Gamma^\rho_{(\kappa\mu)}] - \tilde{g}^{\lambda\mu}\Gamma^\kappa_{[\lambda\rho]}\Gamma^\rho_{[\kappa\mu]} \\ + \tilde{g}^{\lambda\mu}[-\Gamma^\kappa_{[\kappa\mu],\lambda} + \Gamma^\rho_{(\lambda\mu)}\Gamma^\kappa_{[\kappa\rho]}]\end{aligned} \tag{2.6}$$

Note that the Lagrangian has been decomposed into three terms, each of which is itself a tensor density. The first term is just the Einstein-Palatini Lagrangian for the symmetrized affine connection, which is itself an affine connection, of course. The second term is algebraic quadratic in the torsion tensor; and as we shall see, its variation with respect to the torsion tensor implies the vanishing of the latter. The third term is the one that is causing the problem; therefore its elimination will eliminate the difficulty.

Noting that the last term is just $\tilde{g}^{\lambda\mu}$ contracted with the covariant derivative of Γ_μ, we see that if we subtract $\tilde{g}^{\lambda\mu}\Gamma_{\mu;\lambda}$ from $\mathcal{L}_0$, we get a new Lagrangian

$$\mathcal{L} = \tilde{g}^{\lambda\mu}[R_{\lambda\mu} - \Gamma_{\lambda;\mu}] = \tilde{g}^{\lambda\mu}[\Gamma^{\kappa}_{(\lambda\mu),\kappa} - \Gamma^{\kappa}_{(\kappa\mu),\lambda} + \Gamma^{\kappa}_{(\kappa\rho)}\Gamma^{\rho}_{(\lambda\mu)} - \Gamma^{\kappa}_{(\lambda\rho)}\Gamma^{\rho}_{(\kappa\mu)}] - \tilde{g}^{\lambda\mu}\Gamma^{\kappa}_{[\lambda\rho]}\Gamma^{\rho}_{[\kappa\mu]} \quad (2.7)$$

Thus, we have a Lagrangian completely separated into two terms, depending on the symmetric and antisymmetric parts of the affine connection respectively. The first term is just the usual Palatini Lagrangian for the symmetrized affine connection. With the assumption that the metric is symmetric, it will yield the usual relation between the metric and the symmetric part of the affine connection. The second term is an algebraic quadratic in the antisymmetric part of the affine connection and will yield the vanishing of that antisymmetric part upon variation.

It is clear that we may vary the symmetric and antisymmetric parts of the affine connection separately. After throwing away a total divergence, in the usual way, we get

$$\delta\int \mathcal{L}d^4x = \int \{[-\tilde{g}^{\lambda\mu}{}_{\|\kappa} + \tfrac{1}{2}(\tilde{g}^{\rho\lambda}{}_{\|\rho}\delta^{\mu}_{\kappa} + \tilde{g}^{\mu\rho}{}_{\|\rho}\delta^{\lambda}_{\kappa})]\,\delta\Gamma^{\kappa}_{(\lambda\mu)} - 2\tilde{g}^{\rho\mu}\Gamma^{\lambda}_{[\rho\kappa]}\,\delta\Gamma^{\kappa}_{[\lambda\mu]} + \delta\tilde{g}^{\lambda\mu}(R_{\lambda\mu} - \Gamma_{(\lambda;\mu)})\}\, d^4x \quad (2.8)$$

where $\|$ denotes the covariant derivation with respect to the symmetrized affine connection. Variation of the antisymmetric part thus yields

$$\tilde{g}^{\rho\mu}\Gamma^{\lambda}_{[\rho\kappa]} - \tilde{g}^{\rho\lambda}\Gamma^{\mu}_{[\rho\kappa]} = 0 \quad (2.9)$$

Multiplying this equation by $\tilde{g}_{\lambda\lambda'}g_{\mu\mu'} = g_{\lambda\lambda'}\tilde{g}_{\mu\mu'}$, where

$$\tilde{g}_{\lambda\mu}\tilde{g}^{\lambda\nu} = \delta^{\nu}_{\mu} \quad (2.10)$$

(i.e., $\tilde{g}_{\lambda\mu} = g_{\lambda\mu}/\sqrt{-g}$), we find

$$g_{\mu\mu'}\Gamma^{\mu}_{[\nu\lambda']} \equiv T_{\mu'\nu\lambda'} = T_{\lambda'\nu\mu'} \quad (2.11)$$

Thus, the tensor $T_{\mu'\nu\lambda'}$ is symmetric in μ', λ' but antisymmetric in ν, λ'; consequently it must vanish. This means that the torsion tensor vanishes and the affine connection is symmetric.

$$\Gamma^{\mu}_{[\nu\lambda]} = 0 \quad (2.12)$$

Variation of the symmetric part of the connection gives

$$-\tilde{g}^{\lambda\mu}{}_{\|\kappa} + \tfrac{1}{2}(\tilde{g}^{\rho\lambda}{}_{\|\rho}\delta^{\mu}_{\kappa} + \tilde{g}^{\mu\rho}{}_{\|\rho}\delta^{\lambda}_{\kappa}) = 0 \quad (2.13)$$

By the standard Palatini method, this reduces to the usual connection between the symmetric metric and the symmetrized affine connection.

Variation with respect to the contravariant metric tensor density $\tilde{g}^{\lambda\mu}$ yields the field equations

$$R_{(\lambda\mu)} - \Gamma_{(\lambda;\mu)} = R_{\lambda\mu}[\Gamma^{\nu}_{(\rho\sigma)}] - \Gamma^{\kappa}_{[\lambda\rho]}\Gamma^{\rho}_{[\kappa\mu]} = 0 \tag{2.14}$$

where $R_{\lambda\mu}[\Gamma^{\nu}_{(\rho\sigma)}]$ means the Ricci tensor formed from the symmetrized affine connection. Since the torsion tensor vanishes, this reduces to the usual Einstein equations.

§(3): *Tetrad Formalism*

In this section we shall review the tetrad formalism (see [7]) to show clearly what parts of the formalism require no more than a bare manifold, which parts need an affinely connected manifold, and which parts require a metric on the manifold.

We consider again a four-dimensional manifold and introduce four linearly independent contravariant vector fields z^{μ}_{m} $(m = 0, 1, 2, 3)$. This contravariant basis induces at each point of the manifold a dual covariant basis z^{m}_{μ} defined by

$$z^{\mu}_{m} z^{n}_{\mu} = \delta^{n}_{m}, \qquad z^{\mu}_{m} z^{m}_{\nu} = \delta^{\mu}_{\nu} \tag{3.1}$$

In a bare manifold we can define the curl of z^{l}_{μ} and the Lie bracket $[z_m, z_n]$ of z^{μ}_{m} and z^{ν}_{n}; and we can form scalars by projection onto the basis

$$\tfrac{1}{2}(z^{l}_{\nu,\mu} - z^{l}_{\mu,\nu})\, z^{\mu}_{m} z^{\nu}_{n} = \Omega^{l}_{mn} = -\Omega^{l}_{nm} \tag{3.2a}$$

$$[z_m, z_n]^{\nu} z^{l}_{\nu} = (z^{\mu}_{n} z^{\nu}_{m,\mu} - z^{\mu}_{m} z_{n}{}^{\nu}{}_{,\mu})\, z^{l}_{\nu} = 2\,\Omega^{l}_{mn} \tag{3.2b}$$

If Ω^{l}_{mn} vanishes, each of the covariant basis vectors has vanishing curl; consequently four scalar fields exist whose gradients form the covariant basis. Hence the name anholonomic object.

Now we assume the manifold to be affinely connected. The basic property of an affine connection $\Gamma^{\lambda}_{\mu\nu}$ is that given any vector field A^{λ} we can construct its absolute or covariant derivative

$$A^{\lambda}{}_{;\mu} = A^{\lambda}{}_{,\mu} + \Gamma^{\lambda}_{\mu\nu} A^{\nu} \tag{3.3}$$

Using any one of the tetrad vectors z^{λ}_{l} and projecting the resulting tensor on the tetrad, we get the following set of scalars:

$$\gamma^{l}_{mn} = z^{\lambda}_{n;\nu} z^{\nu}_{m} z^{l}_{\lambda} = (z^{\lambda}_{n,\nu} + \Gamma^{\lambda}_{\nu\mu} z^{\mu}_{n})\, z^{\nu}_{m} z^{l}_{\lambda} = -z^{l}_{\lambda;\nu} z^{\lambda}_{n} z^{\nu}_{m} \tag{3.4}$$

There are 64 scalars γ^{l}_{mn}, which we shall call the a.c.c. If the affine connection is metric and symmetric, they reduce to the Ricci rotation coefficients. If we use a holonomic basis, then the γ^{l}_{mn} are numerically equal to $\Gamma^{\lambda}_{\mu\nu}$, in coordinates adapted to that basis.

Taking the antisymmetric part of (3.4), and with the help of (3.2), we find

$$\gamma^l_{[mn]} = S^l_{mn} - \Omega^l_{mn} \tag{3.5}$$

where S^l_{mn} is the projection of the torsion tensor onto the tetrad

$$S^l_{mn} = \Gamma^\lambda_{[\mu\nu]} z^l_\lambda z^\mu_m z^\nu_n \tag{3.6}$$

Equation (3.5) shows that the symmetry of the affine connection by no means shows up as a symmetry of its anholonomic components. We need to find the breakup of the a.c.c. corresponding to the one used in Section 2, but we have just seen that it does not correspond to a breakup of the latter into symmetric and antisymmetric parts because of the anholonomic object. We can easily find the desired breakup, however, by inserting the breakup of the affine connection into (3.4):

$$\gamma^l_{mn} = [z^\lambda_{n},_\nu + (\Gamma^\lambda_{(\nu\mu)} + \Gamma^\lambda_{[\nu\mu]}) z^\mu_n]\, z^\nu_m z^l_\lambda = \overset{s}{\gamma}{}^l_{mn} + S^l_{mn} \tag{3.7}$$

where

$$\overset{s}{\gamma}{}^l_{mn} = (z^\lambda_{n},_\nu + \Gamma^\lambda_{(\nu\mu)} z^\mu_n) z^\nu_m z^l_\lambda = z^\lambda_{n\|\nu} z^\nu_m z^l_\lambda \tag{3.8}$$

Thus, even though not symmetric, $\overset{s}{\gamma}{}^l_{mn}$ are the a.c.c. corresponding to the symmetrized affine connection.[3] Even though it involves a slight abuse of language, we shall refer to the $\overset{s}{\gamma}{}^l_{mn}$ as the symmetrized a.c.c. to distinguish it from $\gamma^l_{(mn)}$, which we shall call the symmetric part of the a.c.c.

Using (3.5) and the fact that the γ^l_{mn} are the sum of their symmetric and antisymmetric parts, it is easy to show that

$$\overset{s}{\gamma}{}^l_{mn} = \gamma^l_{(mn)} - \Omega^l_{mn} \tag{3.9}$$

Now we shall introduce the metric into our manifold. For the moment we consider an arbitrary metric $g_{\mu\nu}$. With the given tetrad z^μ_m we can form the tetrad components of the metric:

$$\eta_{mn} = g_{\mu\nu} z^\mu_m z^\nu_n, \qquad \eta^{mn} = g^{\mu\nu} z^m_\mu z^n_\nu \tag{3.10}$$

with

$$\eta_{lm} \eta^{ln} = \delta^n_m \tag{3.11}$$

We shall call η_{lm} the tetrad metric of the manifold.

[3] They were first introduced by Schouten and Struik in 1935, as far as we can tell. See [8], p. 84.

It should be emphasized that the $\overset{s}{\gamma}{}^l_{mn}$ transform like the anholonomic components of an affine connection under anholonomic coordinate transformations, and the S^l_{mn} transform like the anholonomic components of a tensor under these transformations. On the other hand, neither $\gamma^l_{(mn)}$ nor $\gamma^l_{[mn]}$ have such transformation properties and thus cannot be considered as the anholonomic components of an affine connection or a tensor, respectively. This is the basic reason that the decomposition (3.7) is so important. Equation (3.9) shows that this breakup can be defined entirely in terms of the tetradial quantities.

In applications one often makes some special choice of the z^μ_m, leading to a simple form of η_{mn}; e.g., an orthonormal tetrad, in which case the numerical values of η_{mn} are equal to the components of the usual Minkowski metric. But in general the η_{mn} can be arbitrary functions of the coordinates.

There exists a class of affine connections characterized by the condition

$$g_{\lambda\mu;\nu} = 0, \tag{3.12}$$

which preserve the scalar product of any pair of vectors, with respect to the metric, under parallel transport. If $g_{\lambda\mu}$ is symmetric, (3.12) alone does not lead to a symmetric connection. The torsion tensor will vanish provided that the relation between metric and affine connection has been derived from a variational principle of the type discussed in Section 2.

When the condition (3.12) is valid, there is a simple relation between the tetrad metric and the a.c.c. Defining the directional derivation of a scalar field ϕ with respect to the tetrad vectors by the relation

$$\phi_{,l} = \phi_{,\lambda} z^\lambda_l \tag{3.13}$$

we find

$$\eta_{lm,n} = \eta_{lq}\gamma^q_{nm} + \eta_{qm}\gamma^q_{nl} \tag{3.14}$$

$$\eta^{lm}_{\ \ ,n} + \eta^{lq}\gamma^m_{nq} + \eta^{qm}\gamma^l_{nq} = 0 \tag{3.14a}$$

If the metric is symmetric, we can write (3.14)

$$\eta_{lm,n} - \gamma_{lnm} - \gamma_{mnl} = 0 \tag{3.14b}$$

where we have defined

$$\gamma_{lnm} = \eta_{lq}\gamma^q_{nm} \tag{3.15}$$

Equation (3.14b) shows a well-known special property of the a.c.c.: when η_{lm} = const, then γ_{lmn} is antisymmetric in l, n.

§(4): *The Lagrangian $\mathcal{L}$ Expressed in Tetrad Variables*

Since $\mathcal{L}$, as given in (2.7), is in terms of components with respect to a coordinate basis, our first task is to express it in terms of the three sets of tetrad variables z^l_λ, γ^l_{nm}, and η^{lm}. To simplify this calculation, we can take advantage of the fact that the first term in the Lagrangian is just the Ricci tensor for the symmetrized affinity $\Gamma^\lambda_{(\mu\nu)}$ contracted with the contravariant metric tensor density; while the second term is just the covariant derivative of the vector Γ_μ similarly contracted. By expressing the contravariant metric tensor density in terms of the tetrad variables, we can thus reduce the problem to finding the tetrad components of tensors.

The contravariant tensor $g^{\lambda\mu}$ is expressed in terms of η^{lm} and z^l_λ by the inverse of equation (3.7a):

$$g^{\lambda\mu} = \eta^{lm} z^\lambda_l z^\mu_m \tag{4.1}$$

We still need the determinant $g = \det g_{\lambda\mu}$, which is derived from the inverse of (3.7),

$$g_{\lambda\mu} = \eta_{lm} z^l_\lambda z^m_\mu \tag{4.1a}$$

and has the value

$$g = \eta Z^2; \qquad \eta \equiv \det \eta_{lm},\ Z \equiv \det z^l_\lambda \tag{4.2}$$

Therefore

$$\tilde{g}^{\lambda\mu} = \tilde{\eta}^{lm} z^\lambda_l z^\mu_m Z \tag{4.3}$$

where

$$\tilde{\eta}^{lm} = \eta^{lm}\sqrt{-\eta} \tag{4.4}$$

In the following we shall use as a third set of tetrad variables the $\tilde{\eta}^{lm}$ instead of η^{lm}. We notice that $\tilde{g}^{\lambda\mu}$ depends only on $\tilde{\eta}^{lm}$ and z^l_λ [the z^μ_m are functions of z^l_λ according to (3.1)]. Thus,

$$\begin{aligned} \mathcal{L} &= \tilde{g}^{\lambda\mu}\{R_{\lambda\mu}[\Gamma^\nu_{(\rho\sigma)}] - \Gamma^\kappa_{[\lambda\rho]}\Gamma^\rho_{[\kappa\mu]}\} \\ &= Z\tilde{\eta}^{lm}\{R_{lm}[\overset{s}{\gamma}{}^n_{rs}] - S^k_{lr}S^r_{km}\} \end{aligned} \tag{4.5}$$

where $R_{lm}[\overset{s}{\gamma}{}^n_{rs}]$ stands for the tetrad components of the Ricci tensor for the symmetrized a.c.c. The tetrad components of the Riemann tensor are well known (see [7], p. 172, for example), and by contraction those for the Ricci tensor follow at once.

$$R_{lm} = \gamma^k_{lm,k} - \gamma^k_{km,l} + \gamma^k_{kr}\gamma^r_{lm} - \gamma^k_{lr}\gamma^r_{km} + 2\Omega^r_{kl}\gamma^k_{rm} \tag{4.6}$$

Thus,

$$\mathcal{L} = Z\tilde{\eta}^{lm}\{\overset{s}{\gamma}{}^k_{lm,k} - \overset{s}{\gamma}{}^k_{km,l} + \overset{s}{\gamma}{}^k_{kr}\overset{s}{\gamma}{}^r_{lm} - \overset{s}{\gamma}{}^k_{lr}\overset{s}{\gamma}{}^r_{km} + 2\Omega^r_{kl}\overset{s}{\gamma}{}^k_{rm}\} - Z\tilde{\eta}^{lm}S^k_{lr}S^r_{km} \tag{4.7}$$

Again there is a clean separation in the Lagrangian between a term that is just the tetrad transcription of the Palatini Lagrangian for the symmetrized a.c.c. and a second term that is an algebraic quadratic in the torsion components. We can vary (4.7) directly in terms of the three sets of tetrad variables; but if we express the variations of the components of the metric and the affine connection in (2.8) in terms of the variations of our tetrad variables, we shall get the same answer.

Variation of (4.3) gives us the relation between variations of the tetrad variables and the variation of $\tilde{g}^{\lambda\mu}$. Similarly by noting that

$$\Gamma^{\kappa}_{\lambda\mu} = \gamma^{k}_{lm} z^{\kappa}_{k} z^{l}_{\lambda} z^{m}_{\mu} - z^{m}_{\mu} z^{\kappa}_{m,\lambda} \tag{4.8}$$

we can find the relation between variation of the tetrad variables and variation of $\Gamma^{\kappa}_{\lambda\mu}$.

§(5): *Variation of the Rotation Coefficients and Tetrad Metric*

We first vary the a.c.c., keeping the tetrad metric and tetrad vectors fixed. Equation (4.8) shows that the variation induced in $\Gamma^{\kappa}_{\lambda\mu}$ by variation of γ^{k}_{lm} is just the tensorial projection of the latter variation:

$$\delta\Gamma^{\kappa}_{\lambda\mu} = \delta\gamma^{k}_{lm} z^{\kappa}_{k} z^{l}_{\lambda} z^{m}_{\mu} \tag{5.1}$$

We now break up the variation of the γ^{k}_{lm} into variations of the $\overset{s}{\gamma}{}^{k}_{lm}$ and the S^{k}_{lm}. Since the tetrad vectors are fixed, the anholonomic object does not vary; and (3.5) and (3.9) show that the variation of the torsion tensor and the symmetrized a.c.c. now coincide respectively with the variation of the antisymmetric and symmetric parts of the a.c.c. Thus, each may be varied independently.

The variation of the torsion components then leads to the vanishing of the torsion by a proof exactly analogous to that in Section 2 for the torsion tensor.

Now we consider the variation of the symmetrized a.c.c. Looking back at (2.8) then shows that the coefficient of $\delta\overset{s}{\gamma}{}^{l}_{mn}$ is just the tetrad components of the coefficient of $\delta\Gamma^{\kappa}_{(\lambda\mu)}$. So, the first set of field equations is

$$z^{\kappa}_{k} z^{m}_{\mu} z^{l}_{\lambda} [-\tilde{g}^{\lambda\mu}{}_{\|\kappa} + \tfrac{1}{2}(\tilde{g}^{\rho\lambda}{}_{\|\rho}\delta^{\mu}_{\kappa} + \tilde{g}^{\mu\rho}{}_{\|\rho}\delta^{\lambda}_{\kappa})] = 0 \tag{5.2}$$

The standard Palatini method of solving the tensorial version of (5.2) still works: we take the trace of (5.2), which shows that

$$z^{m}_{\mu} \tilde{g}^{\rho\mu}{}_{\|\rho} = 0 \tag{5.3}$$

Thus, (5.2) reduces to

$$z^{\kappa}_{k} z^{m}_{\mu} z^{l}_{\lambda} \tilde{g}^{\lambda\mu}{}_{\|\kappa} = 0 \tag{5.4}$$

Taking two tetrad vectors into the covariant derivative and noting that

$$Z_{\|\kappa} = Z z^{\lambda}_{l} z^{l}_{\lambda\|\kappa} \tag{5.5}$$

we find that this reduces to

$$\tilde{\eta}^{mn}{}_{,l} - \tilde{\eta}^{mn}\overset{s}{\gamma}{}^{p}_{lp} + \tilde{\eta}^{mp}\overset{s}{\gamma}{}^{n}_{lp} + \tilde{\eta}^{np}\overset{s}{\gamma}{}^{m}_{lp} = 0 \tag{5.6}$$

Multiplying (5.6) by $\tilde{\eta}_{mn}$, we find

$$\overset{s}{\gamma}{}^{p}_{lp} = \tfrac{1}{2}\tilde{\eta}_{qp}\tilde{\eta}^{qp}{}_{,l}$$

Introducing this into (5.6) we get

$$\tilde{\eta}^{mp}\overset{s}{\gamma}{}^{n}_{lp} + \tilde{\eta}^{np}\overset{s}{\gamma}{}^{m}_{lp} = -\tilde{\eta}^{mn}{}_{,l} + \tfrac{1}{2}\tilde{\eta}^{mn}\tilde{\eta}_{pq}\tilde{\eta}^{pq}{}_{,l} \tag{5.7}$$

From the definition (4.4) of $\tilde{\eta}^{lm}$, we find

$$\tilde{\eta}^{mn}{}_{,l} - \tfrac{1}{2}\tilde{\eta}^{mn}\tilde{\eta}_{pq}\tilde{\eta}^{pq}{}_{,l} = \sqrt{-\eta}\,\eta^{mn}{}_{,l} \tag{5.8}$$

Therefore equation (5.7) reduces to

$$\tilde{\eta}^{mp}\overset{s}{\gamma}{}^{n}_{lp} + \tilde{\eta}^{np}\overset{s}{\gamma}{}^{m}_{lp} = -\sqrt{-\eta}\,\eta^{mn}{}_{,l} \tag{5.9}$$

which is identical with (3.14a) written for the symmetrized a.c.c. By lowering the indices m, n in (5.9), we get equation (3.14b), written for the symmetrized a.c.c. We have thus proved that (5.2) yields the correct value of $\overset{s}{\gamma}{}^{l}_{mn}$.

To find the complete expression for $\overset{s}{\gamma}{}^{l}_{mn}$ we write, besides (3.14b) for the symmetrized a.c.c., the two equations derived from it by the cyclic permutation $l \to m \to n \to l$. Combining these three equations, we find

$$2\overset{s}{\gamma}_{lnm} = \eta_{lm,n} + \eta_{ln,m} - \eta_{mn,l} + 2\Omega_{mnl} + 2\Omega_{nml} + 2\Omega_{lmn} \tag{5.10}$$

And when we raise the index l

$$2\overset{s}{\gamma}{}^{l}_{nm} = \eta^{lp}(\eta_{mp,n} + \eta_{np,m} - \eta_{mn,p} + 2\eta_{mq}\Omega^{q}_{np} + 2\eta_{nq}\Omega^{q}_{mp}) + 2\Omega^{l}_{mn} \tag{5.11}$$

The first five terms give the symmetric part of $2\overset{s}{\gamma}{}^{l}_{nm}$ and the last term its anti-symmetric part.

Variation of the tetrad metric directly yields

$$R_{lm}[\overset{s}{\gamma}{}^{k}_{rp}] - S^{k}_{lr}S^{r}_{km} = 0 \tag{5.12}$$

Since the tetrad components of the torsion tensor vanish, this reduces to the tetrad components of the Einstein equations.

§(6): *Variation of the Tetrad Vectors*

One can see without difficulty that the equation obtained by variation of the tetrad vectors z^{l}_{λ} cannot be independent of the equations that we derived in Section 5 from the variations of the a.c.c. γ^{l}_{mn} and the tetrad metric $\tilde{\eta}^{lm}$. Indeed, in the tensor description the Lagrangian depends only on $\tilde{g}^{\lambda\mu}$ and $\Gamma^{\lambda}_{\mu\nu}$; and consequently with the two equations derived from the variations $\delta\tilde{g}^{\lambda\mu}$ and $\delta\Gamma^{\lambda}_{\mu\nu}$ are sufficient for minimizing the action integral. We saw in Section 5 that in the tetrad description the equations obtained from the variations $\delta\tilde{\eta}^{lm}$ and $\delta\gamma^{l}_{mn}$ are equivalent to those obtained from $\delta\tilde{g}^{\lambda\mu}$ and $\delta\Gamma^{\lambda}_{\mu\nu}$. Consequently the two equations obtained from the variations $\delta\tilde{\eta}^{lm}$ and $\delta\gamma^{l}_{mn}$ are sufficient for extremalizing the action integral, and so the equation obtained from the variation δz^{l}_{λ} has to be satisfied automatically.

The exact relation between the left-hand sides of these three equations can be determined without difficulty. The equation obtained by variation of z_α^q is readily shown to be

$$\frac{\delta \mathcal{L}}{\delta z_\alpha^q} = 0 \tag{6.1}$$

with

$$\frac{\delta \mathcal{L}}{\delta z_\alpha^q} = \frac{\delta \mathcal{L}}{\delta \Gamma_{\mu\nu}^\lambda} \frac{\partial \Gamma_{\mu\nu}^\lambda}{\partial z_\alpha^q} - \left(\frac{\delta \mathcal{L}}{\delta \Gamma_{\mu\nu}^\lambda} \frac{\partial \Gamma_{\mu\nu}^\lambda}{\partial z_{\alpha,\beta}^q} \right)_{,\beta} + \frac{\delta \mathcal{L}}{\delta \tilde{g}^{\lambda\mu}} \frac{\partial \tilde{g}^{\lambda\mu}}{\partial z_\alpha^q} \tag{6.2}$$

The equations derived from the variations $\delta \gamma_{mn}^l$ and $\delta \tilde{\eta}^{lm}$ are

$$\frac{\delta \mathcal{L}}{\delta \gamma_{mn}^l} = 0, \qquad \frac{\delta \mathcal{L}}{\delta \tilde{\eta}^{lm}} = 0 \tag{6.3}$$

where

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \gamma_{mn}^l} &= \frac{\delta \mathcal{L}}{\delta \Gamma_{\mu\nu}^\lambda} \frac{\partial \Gamma_{\mu\nu}^\lambda}{\partial \gamma_{mn}^l} = \frac{\delta \mathcal{L}}{\delta \Gamma_{\mu\nu}^\lambda} z_l^\lambda z_\mu^m z_\nu^n \\ \frac{\delta \mathcal{L}}{\delta \tilde{\eta}^{lm}} &= \frac{\delta \mathcal{L}}{\delta \tilde{g}^{\lambda\mu}} \frac{\partial \tilde{g}^{\lambda\mu}}{\partial \tilde{\eta}^{lm}} = \frac{\delta \mathcal{L}}{\delta \tilde{g}^{\lambda\mu}} Z z_l^\lambda z_m^\mu \end{aligned} \tag{6.4}$$

The inverses of the last equations are

$$\frac{\delta \mathcal{L}}{\delta \Gamma_{\mu\nu}^\lambda} = \frac{\delta \mathcal{L}}{\delta \gamma_{mn}^l} z_\lambda^l z_m^\mu z_n^\nu, \qquad \frac{\delta \mathcal{L}}{\delta \tilde{g}^{\lambda\mu}} = \frac{\delta \mathcal{L}}{\delta \tilde{\eta}^{lm}} \cdot \frac{1}{Z} z_\lambda^l z_\mu^m \tag{6.5}$$

Introducing (6.5) in (6.2), we obtain the identity connecting the left-hand sides of the three tetrad equations (6.1) and (6.3).

The relation (6.2) shows that one can alternatively start by satisfying (6.1) and the first equation of (6.3), in which case the second of (6.3) will also be satisfied. This is of no practical interest, however, since (6.1) is more complicated, at least for the Lagrangian (2.7) used in this work.

§(7): *Conclusion*

It may be useful to compare our Lagrangian with that of Hehl et al. [2] used in their development of the Einstein-Cartan theory. In that case as well, in the absence of matter the torsion tensor vanishes. However, in their treatment the metric and the affine connection are not independent fields. In the Hehl et al. Lagrangian, the metric and affine connection are varied subject to the *condition* that the affine connection be metric, so that independent variation of both is not possible; rather, variation is subject to the *constraint* that $g_{\lambda\mu;\nu} = 0$. The basic difference between the two approaches is thus that for our Lagrangian the *sym-*

metrized affine connection is metric as a *result* of the field equations; while in the Hehl et al. approach the full affine connection is *assumed* to be metric *before* any field equations are derived.

If we add to our gravitational field Lagrangian (2.7) a matter Lagrangian $\mathcal{L}_M(g_{\alpha\rho}, \Gamma^{\kappa}_{[\mu\nu]}, D^A)$ depending on the symmetric metric tensor, the torsion tensor, and some other dynamical variables D^A, then the resulting field equations will still keep the symmetrized affine connection metric, while the torsion tensor will be determined by $\delta\mathcal{L}_M/\delta\Gamma^{\kappa}_{[\mu\nu]}$.[4] Thus, the resulting theory will differ from that of Hehl et al. We shall investigate this theory in a later paper.

In particular, it should be noted that our Lagrangian is not unique. As Trautman has indicated [11], the basic reason for the defect we have noted in Lagrangian (2.1) in the case of a nonsymmetric connection is that the Lagrangian is invariant under projective transformations of the connection; and conversely, the basic reason for the success of our new Lagrangian (2.7) is that we have broken this projective invariance. We could accomplish the same purpose, for example, by adding terms of the form $\tilde{g}^{\lambda\mu}\Gamma_\lambda\Gamma_\mu$ or $\tilde{g}^{\lambda\mu}\Gamma^{\alpha}_{[\lambda\beta]}\Gamma^{\beta}_{[\alpha\mu]}$ with arbitrary coefficients to the Lagrangian (2.7). Perhaps study of the theory with nonvanishing torsion will yield some criterion for choosing a unique Lagrangian.

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Note added in manuscript: After completion of this article, we found the recent work of Hehl, Kerlick, and von der Heyde [3, 4]. In this series they discuss the geometrical properties and possible physical significance of a space–time with a symmetric metric and an arbitrary affine connection which are a priori unconnected with each other. They also propose to break the projective invariance of the usual Palatini Lagrangian $(-g)^{1/2}R$ by adding a nonprojective invariant term to it. The only example they give, describing it as ad hoc, is the addition of the square of the trace of the covariant derivative of the metric. They are primarily interested in the case of nonvanishing sources for the fields, but their suggested Lagrangian would also lead to the vanishing of the antisymmetric part of the torsion in the absence of sources. In the presence of sources, their Lagrangian would lead to a different theory than ours, of course.

[4] In the tetradial formulation, the matter Lagrangian would depend on the tetrad metric and the tetrad components of the torsion tensor, of course. The resulting field equations will keep the symmetrized rotation coefficients metric, while the tetrad components of the torsion tensor will be determined by $\delta\mathcal{L}_M/\delta S^l_{mn}$.

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