Majority Runoff Elections: Strategic Voting and Duverger's Hypothesis*

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Abstract

We fully characterize the set of strictly perfect equilibria in large three-candidate majority runoff elections. Considering all possible distributions of preference orderings and intensities in the electorate, we prove that only two types of equilibria can exist. First, there are always incentives for all the voters to concentrate their votes on only two candidates (i.e. Duverger's Law equilibria always exist). Second, there is at most one equilibrium in which three candidates receive a positive fraction of the votes (i.e. a Duverger's Hypothesis equilibrium may exist). The characteristics of that unique Duverger's Hypothesis equilibrium challenge common beliefs about runoff elections: (i) some voters do not vote for their most preferred candidate (i.e. sincere voting is not an equilibrium), and (ii) supporters of the front-runner do not vote for a less-preferred candidate in order to influence who will face the front-runner in the second round (i.e. there is no push over equilibrium).

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1 Introduction

In a majority runoff election, a candidate wins outright in the first round if she obtains an absolute majority of the votes. If no candidate wins in the first round, then a second round is held between the two candidates with the most first-round votes. The winner of that round wins the election.

Over the past decades, most newly-minted democracies have adopted the majority runoff system to elect their presidents as well as other important government officials.[1] The majority runoff system is also widely used in long-standing democracies (see e.g. Blais et al. 1997, and Golder 2005).[2] Moreover, debates about whether it should be implemented even more widely are recurrent (see e.g. Italy's La Repubblica of June 20th 2012). These debates and the widespread inclination in favor of the majority runoff system rely both on formal and informal arguments. On the one hand, the majority runoff system is commonly believed to (i) be more conducive to preference and information revelation than plurality, and (ii) ensure a large mandate to the winner, thereby providing her with more democratic legitimacy. On the other hand, the majority runoff system suffers from a non-monotonicity problem that may induce a harmful strategic behavior in the first round called *push over*.[3]

The scant empirical literature on majority runoff elections is not widely supportive of these arguments. First, as reviewed in Bouton (2012), the evidence that the runoff system is more conducive to preference and information revelation than plurality is mixed. Second, there are many examples of majority runoff elections in which the winner is not the candidate preferred by the majority and thus lacks democratic legitimacy.[4] For instance, in Peru's presidential election in 2006, Lourdes Flores Nano (Unidad National) did not make it to the second round, despite opinion polls indicating that she was the majority candidate. Indeed, polls showed that she

[1]Besides very few exceptions, all new democracies in Eastern Europe and in Africa have adopted this system to elect their presidents (Golder 2005, Golder and Wantchekon 2004). This is also true for the new democracies in Latin America, although to a lesser extent.

[2]For instance, in the U.S., runoff primaries are a trademark in southern states, and most large cities have a runoff provision (Bullock and Johnson 1992, Engstrom and Engstrom 2008). In Italy, the majority runoff system is used for the elections of mayors in all major cities and governors in most regions.

[3]In runoff elections, an additional vote in favor of a candidate can reduce the likelihood of victory of that candidate (see e.g. Smith 1973). Therefore, there are situations in which supporters of the front-runner prefer to vote for a less-preferred candidate in order to influence who will face the front-runner in the second round. This is called *push over*. It is deemed harmful because it allows a minority group of strategic voters to influence the outcome of the elections to its advantage (see e.g. Cox 1997 and Saari 2003).

[4]In real-life majority runoff elections, anecdotal evidence suggests that when a Condorcet winner exists (i.e. a candidate that would win a one to one contest against any other candidate), she often does not reach the second round.

would have won a second round against the two other serious candidates: Ollanta Humala Tasso (Union for Peru) and Alan García Pérez (Aprista Party).[5] Finally, as far as we know, evidence of push over behavior in runoff elections has never been documented (see Dolez and Laurent 2010 for evidence against push over).

Such discrepancies beg an explanation. Arguably, part of the problem is that beliefs about the majority runoff system have either not been formally proven or have not been proven robust. Despite recent advances (Martinelli 2002, Morton and Rietz 2006, and Bouton 2012), one major caveat in the theoretical literature on majority runoff elections is that it does not provide a *complete characterization* of the set of (mixed strategy) voting equilibria in a setup allowing for all possible preference orderings and intensities. Such a complete characterization is crucial to properly establish the properties of the majority runoff system and to compare these properties with those of other electoral systems (Myerson 1996 and 2002, Cox 1997). Indeed, a model which does not consider some voter types might erroneously predict the existence or non-existence of some equilibria, thereby implying inaccurate properties. As we show, this is exactly what happened with push over and sincere voting equilibria. This paper fills this gap by studying elections where all possible preference orderings and intensities are represented. For reasons detailed below, we focus on the set of *strictly perfect equilibria* (Okada 1981) and fully characterize it.

We demonstrate that, in majority runoff elections, the set of strictly perfect equilibria features three main properties. First, a strictly perfect equilibrium always exists. Our proof is constructive: we show the existence of three *Duverger's Law* equilibria, in which only two candidates receive a positive fraction of the votes. In these equilibria, an outright victory in the first round always occurs. Second, a *Duverger's Hypothesis* equilibrium, in which three candidates receive a positive fraction of the votes, sometime exists. Third, we show that there are no other strictly perfect equilibria in majority runoff elections. The characteristics of the unique Duverger's Hypothesis equilibrium are as follows: (i) it never supports *push over*, (ii) it never supports *sincere voting* by all voters, i.e. all voters voting for their most-preferred candidate, and (iii) it can lead to the exclusion of the Condorcet Winner from the second round.

These results strongly qualify some of the aforementioned common beliefs about majority runoff elections. First, majority runoff elections are perceived to ensure a large mandate to the

[5]See Schmidt 2007 for more details. The 2007 French presidential election is another striking example (Spoon 2008).

winner, thereby providing her with more democratic legitimacy. In contrast, we show that, even when there are more than two serious candidates in the first round, the Condorcet winner is not guaranteed to participate in the second round. Therefore, the fact that the eventual winner of the election obtains more than 50% of the votes in the second round cannot be considered a strong proof of legitimacy. This only ensures that a potential Condorcet loser never wins.[6]

Second, majority runoff elections are commonly perceived to be more conducive to preference and information revelation than plurality elections. The argument is the following: since voters can use the second round to coordinate against a minority candidate, in the first round, they feel free to vote "sincerely" for their most-preferred candidate (Duverger 1954, Riker 1982, Cox 1997, Piketty 2000, Martinelli 2002). Our results reinforce Bouton (2012)'s argument that this perceived benefit of the majority runoff system is quite overrated. Indeed, we prove that (i) Duverger's Law equilibria exist even if voters have heterogeneous preference intensities, and (ii) the sincere voting equilibrium is not robust to such heterogeneity.

Third, the non-monotonicity of the runoff system is deemed problematic when it induces harmful push over tactics (see e.g. Cox 1997 and Saari 2003). We show that push over does not happen in any (strictly perfect) equilibrium. Thus, the only actual concern with non-monotonicity in the runoff system is that it might prevent sincere voting in equilibrium.

The results in this paper also make explicit the precise conditions on the distribution of preferences required for previously identified properties of the majority runoff system. For instance, both Martinelli (2002) and Bouton (2012) show, in a setup with a positive fraction of partisan/non-strategic voters (i.e. voters who always vote for their most preferred candidate) and not all voter types, that the sincere voting equilibrium may exist. Together with our result about the non-existence of the sincere voting equilibrium, this shows that the existence of the sincere voting equilibrium requires a large fraction of partisan/non-strategic voters and the absence of certain types of voters.

Our model of three-candidate majority runoff elections builds on Bouton (2012). The main generalization of the model is the introduction of heterogeneous intensities of preferences among supporters of any given candidate by assuming a continuum of voter types. Our model therefore captures all possible preference orderings and intensities over the set of candidates. As mentioned above, such a general structure of preferences ensures that we neither overlook nor falsely

[6]The Condorcet loser is a candidate that would lose a one to one contest against any other candidate.

establish any property of the majority runoff system. These risks are not innocuous. For instance, a model with a finite number of types, which does not include voters sufficiently close to be indifferent between the two runner-up candidates, would classify the Duverger's Hypothesis equilibrium as a sincere voting equilibrium. When all types of voters are included in the model, sincere voting is never an equilibrium.

Importantly, none of our results depends on the fact that our model includes all possible preference orderings and intensities over the set of candidates. Yet, our ability to prove that any given equilibrium "always exists" or "does not exist in general" depends on this features. In this sense, the inclusion of all possible types of voters in the model should be viewed as relaxing an assumption instead of making up a new one.

To keep the model tractable, we assume that, for any given pair of candidates participating in the second round, the probability of victory is exogenous, positive, and constant (i.e. independent of the size of the electorate). Though simple, this formulation is quite general. Indeed, since we allow for any probability of victory strictly between 0 and 1, it is equivalent to considering any possible strategy (sequentially rational or not) in the second round (except those for which one candidate wins for sure). This includes (but is not limited to) any "realistic" probability structure (e.g. the front runner or the candidate with the largest (expected) number of supporters being more likely to win in the second round).

Typically, there are many equilibria in multicandidate elections. In an environment as rich as the one considered in this paper, the multiplicity is even greater than usual. This is not undesirable *per se*. Indeed, equilibrium multiplicity captures the risk of coordination failure that exists in multicandidate elections (see e.g. Myerson and Weber 1993, Bouton and Castanheira 2012). Yet, it has been argued that some equilibria of voting games are neither robust nor reasonable (see e.g. Fey 1997 for a discussion of equilibrium stability in plurality elections). It is thus proper to refine the set of equilibria when studying multicandidate elections. In this paper, we focus on the set of *strictly perfect equilibria* (Okada 1981).

There are several reasons for using *strict perfection* as an equilibrium concept in Poisson voting games (see the technical appendix for proofs and more details). First, less stringent concepts such has perfection and properness have very little bite in Poisson voting games (De Sinopoli and Pimienta, 2009). For instance, they do not eliminate equilibria in plurality elections that have been deemed unstable and undesirable in terms of information and expectational stability (Fey, 1997).[7] Second, by contrast, strict perfection does rule out exactly those equilibria that have been deemed unstable and undesirable based on the concept of expectational stability.[8] Third, we show that strict perfection is equivalent to robustness to heterogeneous beliefs about the expected distribution of preferences in the electorate (see Fey 1997 for a discussion of the rationales for such a robustness requirement). Fourth, multiple strictly perfect equilibria always exist in our model, and in general in most voting games. This suggests that, though stringent, strict perfection is not too stringent a concept in voting games. Finally, we prove for a general class of Poisson games that strict perfection can be defined in a way that is simple and easy to use. Using strict perfection actually makes the complete characterization of the set of equilibria significantly simpler.

The rest of the paper is organized as follows: Section 2 lays out the model. Section 3 details how voters decide for whom to vote and defines the equilibrium concept precisely. Section 4 analyzes equilibrium behavior in majority runoff elections. Section 5 discusses the inclusion of non-strategic voters in the electorate. Section 6 concludes.

2 The Model

The electoral system works as follows. There are three candidates, $c \in \mathcal{C} \equiv \{R, S, W\}$, who all participate in a first round of the election. If, in the first round, a candidate receives more than half of the votes, then she is elected. Otherwise, the two candidates with the largest shares of votes will face each other in a second ballot. To lighten notation, we assume without loss of generality that ties for the second place are resolved by alphabetical order: R wins over both S and W, S wins over W.

We conduct the analysis under the assumption that the size of the electorate, ν, is distributed according to a Poisson distribution of mean n: $\nu \sim \mathcal{P}(n)$ (Appendix A1 summarizes some properties of Poisson games and applies them to runoff elections). Each voter has preferences over the candidates defined by her type, $t \in \mathcal{T}$, where $\mathcal{T}$ is a metric space. Types are assigned by *iid* draws from a non-atomistic distribution F with support $\mathcal{T}$. We label the set of non-atomistic distributions as $\mathcal{F}$. The utility of a voter of type t when candidate c is elected is given by $U(c \mid t)$.

[7] In our setup, any equilibrium in which all three candidates get a positive fraction of the votes is perfect and proper.

[8] See the technical appendix for a proof of the equivalence of expectational stability and strict perfection in our setup.

Voters of type t with $U(R \mid t) > U(c \mid t), \forall c = S, W$ prefer candidate R over any other candidate and we shall call them R's *supporters*. Similarly, voters with $U(S \mid t) > U(c \mid t), \forall c = R, W$ are S's supporters and voters with $U(W \mid t) > U(c \mid t), \forall c = R, S$ are W's supporters. We assume that the set $\mathcal{T}$ is rich enough in the sense that for any $x \in \mathbb{R}_+$ and any pair of candidates $c, i \in \mathcal{C}$, there exists a type $t \in \mathcal{T}$ such that $U(c \mid t)/U(i \mid t) = x$. We denote by γ_{ij} the (expected) fraction of voters with preferences $i \succ j \succ k$.

For the sake of simplicity, we do not explicitly model the second round. Yet, the probabilities of victory in that round influence the behavior of voters in the first round (Bouton, 2012). To capture this effect, we assume that, at the time of the first round, the probabilities of victory are given and constant. We denote by $\Pr(i \mid ij)$, $i, j \in \mathcal{C}$, the probability that candidate i defeats candidate j in the second round opposing these two candidates. Hence, $\Pr(j \mid ij) = 1 - \Pr(i \mid ij)$. We assume that all these second round probabilities are strictly positive, i.e. $\Pr(i \mid ij) \in (0, 1)$. Hence, at the time of the first round, the result of any eventual second round ballot is not certain. Though simple, this formulation is quite general. Indeed, we allow for any probabilities of victory strictly between 0 and 1. This is equivalent to considering any possible strategy in the second round (except those for which one candidate wins for sure), including both sequentially rational and non-sequentially rational strategies. Importantly, this includes (but is not limited to) any "realistic" restriction (e.g. the front runner or the candidate with the largest (expected) number of supporters being more likely to win in the second round).

The action set for each voter is $\{R, S, W\} = \mathcal{C}$. A voting strategy is $\sigma : \mathcal{T} \to \Delta(\mathcal{C})$, where σ_t denotes the strategy of a voter of type t. Call $\sigma \equiv \left((\sigma_t(c))_{c \in \mathcal{C}} \right)_{t \in \mathcal{T}} \in \Delta(\mathcal{C})^{\mathcal{T}}$ a profile of voting strategies. Define $\tau : \Delta(\mathcal{C})^{\mathcal{T}} \times \mathcal{F} \to \Delta(\mathcal{C})$:

$$\tau(\sigma, F) \equiv \left(\int_{\mathcal{T}} \sigma_t(c)\, dF(t) \right)_{c \in \mathcal{C}}$$

where the c-th element of $\tau(\sigma, F)$, $\tau_c(\sigma, F) \geq 0$ is the measure of voters' types voting for candidate c in the first round. This is also the expected share of votes received by candidate c in the first round. For any distribution of preferences F, a profile of voting strategies σ identifies a unique profile of expected share of votes.

The number of players who choose action c is denoted by x_c, where $c \in \mathcal{C}$. This number is random (voters do not observe it before going to the polls) and its distribution depends on the strategy, through $\tau_c(\sigma, F)$. For the sake of readability, we will often henceforth omit (σ, F) from the notation.

Without loss of generality, we assume that candidate R's expected share of votes is at least as large as the expected share of votes of any other candidate. Candidate R is the *front-runner*. Also, we assume that R has more chances to win against W than against S, i.e. $\Pr(R \mid RW) > \Pr(R \mid RS)$. Therefore, we label candidate W the *weak opponent*, while candidate S is the *strong opponent*.

3 Pivot Probabilities, Payoffs, and Equilibrium Concept

Since voters are instrumental, their behavior depends on the probability that a ballot affects the final outcome of the elections, i.e. its probability of being pivotal. This section identifies all the pivotal events. Then, we compute voters' expected payoffs of the different actions and define the best response correspondence.

As explained in detail in Appendix A1 (which summarizes the properties of Poisson games and applies them to runoff elections), the probability that a pivotal event E occurs is exponentially decreasing in n. The (absolute value of the) *magnitude* of event E, denoted $mag\,(E) \leq 0$ represents the "speed" at which the probability decreases towards zero: the more negative the magnitude, the faster the probability goes to zero. Unless two events have the same magnitude, their likelihood ratio converges either to zero or infinity when the electorate grows large. Proofs in this paper rely extensively on this property, and thus, on the comparison of magnitudes of pivotal events. Lemma 2 (in Appendix A1) computes the magnitudes of the different pivotal events. It shows that the magnitude of a pivotal event *piv* is larger when the expected outcome of that round is close to the conditions necessary for event *piv* to occur. Generally, the smaller the deviation with respect to the expected outcome required for the pivotal event to occur, the larger the magnitude.

3.1 Pivotal Events

The first round influences the final result either directly (if one candidate wins outright) or indirectly (through the identity of the candidates participating in the second round).

Due to the alphabetical order tie-breaking rule, the precise conditions for the pivotal events actually depend on the alphabetical order of the candidates. Yet, we define the different pivotal events for any candidates $i, j, k \in \{R, S, W\}$ and $i \neq j \neq k$, abstracting from the candidates'

Table 1: first-round pivotal events.

Event	Notation	Condition
Threshold pivotal i/ij	$piv_{i/ij}$	$x_i + \frac{1}{2} > \frac{1}{2}(x_i + x_j + x_k) \geq x_i \geq x_j \geq x_k$
Threshold pivotal ij/i	$piv_{ij/i}$	$\frac{1}{2}(x_i + x_j + x_k + 1) \geq x_i > \frac{1}{2}(x_i + x_j + x_k) > x_j$ $x_i \geq x_j \geq x_k$
Second-rank pivotal ki/kj	$piv_{ki/kj}$	$x_i = x_j - 1$ $\frac{1}{2}(x_i + x_j + x_k) \geq x_k > x_j$

alphabetical order. These conditions are thus necessarily loose.[9]

A ballot is *threshold pivotal* i/ij, denoted $piv_{i/ij}$, if candidate i lacks one vote (or less) to obtain a majority of the votes in the first round. Thus, without an additional vote in favor of i, a second round opposing i to j is held. The complementary event is the threshold pivotability ij/i, denoted $piv_{ij/i}$, that refers to an event in which any ballot against candidate i, i.e. in favor of either j or k, prevents an outright victory of i in the first round and ensures that a second round opposing i to j is held.

A ballot may also affect the final outcome if it changes the identity of the two candidates participating in the second round. This happens when a ballot changes the identity of the candidates who rank second and third in the first round. A ballot is *second-rank pivotal* ki/kj, denoted $piv_{ki/kj}$, when candidate k ranks first (but does not obtain an absolute majority of the votes), and candidates i and j tie for second place. An additional vote in favor of candidate i allows her, instead of j, to participate in the second round with k.

Table 1 summarizes the different first-round pivotal events that influence the first-round voting behavior.

3.2 Payoffs and Best Responses

Let $G_t(c, n\tau)$ denote the *expected gain* of playing action $c \in \mathcal{C}$ in the first round for a voters of type t, when the expected share of votes is τ. This gain depends on the voter's type and on the strategy function for all voters, σ. Strategies determine the expected number of votes received by each candidate in the first round, and thus the pivot probabilities. Given the probabilities of victory in the second round, we can determine the expected utility of a second round opposing

[9] In the third column of Table 1, depending on the candidates alphabetical order: (i) the conditions might feature weak inequality signs instead of strict ones or conversely, and (ii) the minus 1 might not be there. As proved in Myerson (2000, Theorem 2), such small approximations in the definition of the pivotal events do not matter for the computation of magnitudes.

i to j for a type t:

$$U\left(i,j\mid t\right)=\Pr\left(i\mid ij\right)U(i\mid t)+\Pr\left(j\mid ij\right)U(j\mid t).$$

For a type t, the expected gain of playing action c in the first round is:

$$\begin{aligned}G_t\left(c,n\tau\right) &= \Pr\left(piv_{ic/ij}\right)\left[U\left(i,c\mid t\right)-U(i,j\mid t)\right]+\Pr\left(piv_{jc/ji}\right)\left[U\left(j,c\mid t\right)-U(j,i\mid t)\right]+\\ &\quad \Pr\left(piv_{c/ci}\right)\left[U\left(c\mid t\right)-U(c,i\mid t)\right]+\Pr\left(piv_{c/cj}\right)\left[U\left(c\mid t\right)-U(c,j\mid t)\right]+\\ &\quad \Pr\left(piv_{ic/i}\right)\left[U\left(i,c\mid t\right)-U(i\mid t)\right]+\Pr\left(piv_{jc/j}\right)\left[U\left(j,c\mid t\right)-U(jvt)\right]+\\ &\quad \Pr\left(piv_{ij/i}\right)\left[U\left(i,j\mid t\right)-U(i\mid t)\right]+\Pr\left(piv_{ji/j}\right)\left[U\left(j,i\mid t\right)-U(j\mid t)\right],\end{aligned} \tag{1}$$

where $c,i,j\in\mathcal{C}$ and $c\neq i\neq j$. The first line in (1) reads as follows: if a ballot in favor of c is second-rank pivotal ic/ij, then the second round opposes i to c instead of i to j; if a ballot in favor of c is second-rank pivotal jc/ji, then the second round opposes j to c instead of j to i. The three last lines refer to the gains when the ballot is threshold pivotal.

By theorem 8 in Myerson (1998), when players behave according to a strategy profile σ, the number of voters voting for candidate c follows a Poisson distribution with mean $n\tau_c\left(\sigma,F\right)$. Hence, for any finite n, a strategy profile σ and a distribution F uniquely identify the probability of any event, including the probability that a single vote is pivotal between two electoral outcomes. That is, the vector of all pivot probabilities is a function of $\tau\left(\sigma,F\right)$. Hence, we can define the best response correspondence of a voter of type t to a strategy profile σ when the distribution of types is F, $\mathcal{B}:\mathcal{T}\times\Delta\left(\mathcal{C}\right)\rightrightarrows\Delta\left(\mathcal{C}\right)$:

$$\mathcal{B}_t\left(\tau\right)\equiv\arg\max_{\sigma_t\in\Delta(\mathcal{C})}\sum_{c\in\mathcal{C}}\sigma_t\left(c\right)G_t\left(c,n\tau\right).$$

3.3 Equilibrium Concept

In what follows, we fully characterize the set of strictly perfect equilibria (Okada 1981)[10] as the size of the electorate n goes to infinity. In the technical appendix, we show that for all Poisson games with infinite types set, a strictly perfect equilibrium can be defined as follows.

Definition 1 (Strictly Perfect Equilibrium)**.** A strategy profile σ^* is a strictly perfect equilibrium

[10] Okada (1981) defines strictly perfect equilibria for finite games. In the technical appendix we provide a straightforward extension to Poisson games.

if and only if $\exists \epsilon > 0$ such that $\forall \tau \in \Delta(\mathcal{C}) : |\tau - \tau(\sigma^*, F)| < \epsilon$, $\sigma_t^* \in \mathcal{B}_t(\tau)$, $\forall t \in \mathcal{T}$.

In words, the equilibrium strategy must be a best response to any (sufficiently small) perturbation of the equilibrium strategy.

Myerson (2000) shows that, in a Poisson game, the probability of an exact profile of voting shares is exponentially decreasing in the expected number of voters, n, and converges to zero at a speed proportional to its magnitude. Lemma 1 below (proved in the technical appendix, Proposition 3) shows that if a strategy profile is a best response to itself only if two pivotal events have identical magnitudes, then it is not a strictly perfect equilibrium for n large enough. Importantly, this does *not* imply that a strategy profile which does not generate a unique largest magnitude (as n goes to infinity) cannot be a strictly perfect equilibrium.

Lemma 1. *Let σ^* be a best response to $\tau(\sigma^*, F)$ as $n \to \infty$ only if two magnitudes are equal under $\tau(\sigma^*, F)$. Then, $\exists N \in \mathbb{N}$ such that $\forall n > N$, σ^* is not a strictly perfect equilibrium.*

This lemma greatly simplifies the equilibrium analysis since it reduces dramatically the number of sequences of strictly perfect equilibria to consider as $n \to \infty$.

4 Equilibrium analysis

This section analyzes the set of strictly perfect equilibria in runoff elections. We prove three main results. First, a strictly perfect equilibrium always exists. Our proof is constructive: we show that three *Duverger's Law* equilibria exist for any F.

Definition 2 (Duverger's Law Equilibrium)**.** A Duverger's law equilibrium is an equilibrium σ for F where there exists $i \in \mathcal{C}$ such that $\tau_i = 0$.

Second, we prove that a *Duverger's Hypothesis* equilibrium may exist and be strictly perfect.

Definition 3 (Duverger's Hypothesis Equilibrium)**.** A Duverger's Hypothesis equilibrium is an equilibrium σ for F where $\tau_i > 0$ $\forall i \in \mathcal{C}$.

Third, we show that there is only one type of Duverger's Hypothesis equilibrium which is strictly perfect. Interestingly, neither the *sincere voting* equilibrium nor *push over* equilibria exist.

Definition 4 (Sincere Equilibrium)**.** An equilibrium is sincere if and only if all voters vote for their most preferred candidate.

Definition 5 (Push Over Equilibrium)**.** A push over equilibrium is an equilibrium where $\tau_R \geq \tau_i$, $i \in \{S, W\}$, and $\sigma_t(W) > 0$ for some R's supporters.

4.1 Existence: Duverger's Law

In this section we prove that a strictly perfect equilibrium always exists. Our proof is constructive:

Proposition 1. *There always exist three strictly perfect Duverger's Law equilibria.*

Proof. See Appendix A2. □

The intuition behind this result is straightforward. If a voter expects only two candidates to receive a positive share of votes, as the expected number of votes grows large, his vote can only be decisive in determining which of these two candidates will be elected outright in the first round. That is because if only two candidates receive any vote, then one of them will receive a majority of the votes in all cases except when both candidates receive exactly a 50% share. There are three different Duverger's Law equilibria because there are three different combinations of two candidates receiving all votes. It is easy to show that if there are N candidates, then there are $\frac{N!}{(N-2)!2!}$ strictly perfect Duverger's Law equilibria.

Proposition 1 can be illustrated through a numerical example. Suppose that F is such that (i) there are 10% of W supporters, and (ii) if all voters who prefer R to S vote for R and all voters who prefer S to R vote for S then $\tau_R = 60\% > \tau_S = 40\% > \tau_W = 0\%$. In this case, all magnitudes are equal to -1 except for $\mu(piv_{R/RS})$ and $\mu(piv_{S/SR})$ which are equal to -0.0202. This means that, conditional on being pivotal, voters choose between an outright victory of either R or S in the first round, and a second round opposing R to S. Since both candidates have a positive chance of winning a second round, voters who prefer R to S vote for R in order to avoid the risk of S's victory in the second round. Similarly, voters who prefer S to R vote for S in order to avoid the risk of R's victory in the second round.

Importantly, these best responses would not change if $\mu(piv_{R/RS})$ and $\mu(piv_{S/SR})$ were different (but still the two largest magnitudes). Consider the case in which $\mu(piv_{R/RS}) > \mu(piv_{S/SR})$. All voters who prefer R to S will vote for R. Indeed, by ensuring an outright victory of R in the first round, they avoid the risk of a victory of S in the second round. For voters who prefer

Table 2: magnitudes

Threshold magnitudes	**Second-rank magnitudes**
$\mu(\mathbf{piv_{R/RS}}) = \mathbf{-0.005}$	$\mu(piv_{RS/RW}) = -0.0927$
$\mu(piv_{R/RW}) = -0.0927$	$\mu(piv_{SR/SW}) = -0.181$
$\mu(\mathbf{piv_{S/SR}}) = \mathbf{-0.0461}$	$\mu(piv_{WR/WS}) = -0.196$
$\mu(piv_{S/SW}) = -0.1897$	
$\mu(piv_{W/WR}) = -0.4$	
$\mu(piv_{W/WS}) = -0.40755$	

S to R, the choice is slightly more complex. If their decision is based only on this most likely scenario, then they would vote against R but be indifferent between voting for S or W. Indeed, any of these two actions would have the same result: decreasing the probability of an outright victory of R and increasing the probability of a victory of S (through a second round). Thus, their choice between S and W depends on the second most likely pivotal event, i.e. $\mu(piv_{S/SR})$ in the case under consideration. Thus, to avoid the risk of an upset victory of R in the second round, voters who prefer S to R vote for S.

The strict perfection of Duverger's Law equilibria ensue from the continuity of the magnitudes in the probability distribution over actions. Since small perturbations to the strategies generate small changes to the magnitudes, there is always a small enough deviation from σ^* such that the two largest magnitudes are $piv_{R/RS}$ and $piv_{S/SR}$.

The following example illustrates the robustness of the force underlying Duverger's Law equilibria. Consider the same F as in the previous example but now W supporters (who represent 10% of the electorate) vote for W whereas the other voters adopt the same strategy as above. Then, we for instance have: $\tau_R = 55\% > \tau_S = 35\% > \tau_W = 10\%$. As shown in Table 2, for this expected vote shares, the largest magnitude is $\mu(piv_{R/RS})$, and the second largest is $\mu(piv_{S/SR})$. This is thus not an equilibrium: W supporters prefer to vote for either R or S.

Contrasting Proposition 1 with Theorem 1 in Bouton (2012) highlights one specificity of our model. For the case of majority runoff, Bouton (2012) shows that Duverger's Law equilibria exist if the expected vote share of the candidate expected to rank second is large enough. This condition arises because, in Bouton (2012), the risk of victory of the minority candidate in the second round converges to zero when n grows large. The rate of convergence depends on the expected vote share of the minority candidate. If the expected vote share is too small, this risk converges to zero too fast and then voters disregard it. In our model, all candidates have a positive and constant probability of victory in the second round. Therefore, even with a small expected share of votes, the threat of the minority candidate in the second round is large enough to trigger a coordination in the first round.

4.2 Duverger's Hypothesis

The Duverger's Hypothesis suggests that, in runoff elections, voters have incentives to disperse their votes on more than two candidates. In this section, we show that these incentives indeed exist and that they can lead to the existence of a Duverger's Hypothesis equilibrium (in which three candidates receive a positive fraction of the votes). Moreover, as shown in Section 4.3 (Proposition 3), the only strictly perfect Duverger's Hypothesis equilibria are those identified in the following proposition:

Proposition 2. *For some distribution of preferences, there exist strictly perfect equilibria in which three candidates receive a positive share of the votes. In these equilibria, all voters who prefer the front runner to the runner-up vote for the front runner. Some, but not all, of the supporters of the weak opponent will vote for the strong opponent, regardless of which candidate is expected to receive more votes.*

Proof. See Appendix A2. □

To understand the intuition of this result, we must first understand voters' reaction when they must choose between an outright victory of R and a second round opposing R to the runner-up (i.e. either $piv_{R/RS}$ or $piv_{R/RW}$ has the largest magnitude). All voters who prefer R to the runner-up will vote for R. Indeed, by ensuring an outright victory of R in the first round, they avoid the risk of a victory of the runner-up in the second round. For voters who prefer the runner-up to R, the choice is slightly more complex. If their decision is based only on this most likely scenario, then they would vote against R but remain indifferent between S or W. Indeed, either of these two actions would have the same result: decreasing the probability of an outright victory of R and increasing the probability of a victory of the runner-up (through a second round). Thus, their choice between S and W depends on the second most likely pivotal event. There are two cases to consider: $piv_{S/SR}$ (or $piv_{W/WR}$) and $piv_{RS/RW}$.

If the threshold pivotability S/SR (or W/WR) dominates (which happens when both R and the runner-up have a large advantage with respect to the third candidate), the incentives are the same as in a Duverger's Law equilibrium: all voters who prefer the runner-up to R vote for the runner-up. Therefore, we cannot have a Duverger's Hypothesis equilibrium. Suppose, on the contrary, that the second-rank pivotability RS/RW dominates (which happens when S and W are sufficiently close to each other). In this situation, voters voting against R realize that they determine whether S or W faces R in the second round. Consider the choice of a supporter of S who prefers the runner-up to the front-runner. He casts his ballot considering what to do if R does not pass the threshold (by voting against R he actually maximizes this probability). He would surely prefer to vote for S, and for two good reasons. First, because he prefers S to W. Second, S has more chances to win in the second round than W. Consider the choice of a W supporter who prefers the runner-up to the front-runner. He prefers W to S, but he also knows that S has better chances of winning against R. Since he prefers S to R, he faces a trade off between the likelihood of a second round victory against R and how much he prefers W to S. If he is sufficiently close to indifference between S and W, then he votes for the former. Otherwise, he votes for W.

In an equilibrium such as those described in Proposition 2 the Condorcet winner might be the candidate receiving the least share of votes (she would thus be very unlikely to reach the second round if held). This happens when the Condorcet winner is the second best choice of a large fraction of the voters, but the first choice of only a minority. Hence, in the first round, a large fraction of the support she would receive in a pairwise ballot is lost in favor of a third candidate.[11]

We can illustrate this result through a numerical example. Consider the following situation: supporters of S represent 35% of the voters ($\gamma_{SW} = 15\%$ and $\gamma_{SR} = 20\%$), while the share of R's and W's supporters is equal to 25% ($\gamma_{RS} = 16\%$ and $\gamma_{RW} = 9\%$) and 40% ($\gamma_{WR} = 10\%$ and $\gamma_{WS} = 30\%$), respectively. It is easy to verify that (i) S is the Condorcet winner, (ii) R is the Condorcet loser, and (iii) S is a stronger opponent of R than W. In this case, there exists an equilibrium in which (i) the Condorcet winner, S, is expected not to reach the second round, and (ii) the weak opponent, W, is expected to defeat the front-runner in the second round. In particular, the expected vote shares in that equilibrium are $\tau_R = 45\%$, $\tau_W = 36\%$, and $\tau_S = 19\%$. For those expected vote shares, the magnitudes are given in Table 3. Since the

[11] Thus, strict perfection does not exclude coordination failures among the voters who prefer the Condorcet winner to the ultimate winner of the election. This is in stark contrast with Messner and Polborn (2007) who consider coalition-proof equilibria and find that when a Condorcet winner exists, then it is the unique coalition-proof equilibrium outcome.

Table 3: magnitudes

Threshold magnitudes	Second-rank magnitudes
$\mu(piv_{R/RS}) = -0.02968$	$\mu(\mathbf{piv_{RS/RW}}) = \mathbf{-0.02693}$
$\mu(\mathbf{piv_{R/RW}}) = \mathbf{-0.005}$	$\mu(piv_{SR/SW}) = -0.05982$
$\mu(piv_{S/SR}) = -0.2178$	$\mu(piv_{WR/WS}) = -0.05519$
$\mu(piv_{S/SW}) = -0.2178$	
$\mu(piv_{W/WR}) = -0.04$	
$\mu(piv_{W/WS}) = -0.08233$	

largest magnitude is $\mu(piv_{R/RW}) = -0.005$, we have that all voters who prefer R to W vote for R ($\gamma_{RS}+\gamma_{RW}+\gamma_{SR} = 45\%$), and that all voters who prefer W to R ($\gamma_{WR}+\gamma_{WS}+\gamma_{SW} = 55\%$) vote against R, either for W or for S. Since the second largest magnitude is $\mu(piv_{RW/RS}) = -0.0263$, the choice between S and W is determined by the utility difference between a second round opposing R to S, and a second round opposing R and W. As detailed in the proof of Proposition 2, this difference depends on (i) the intensity of the relative preference between W and S, and (ii) the probabilities of victory in the second round. Since $\Pr(R \mid RS) < \Pr(R \mid RW)$, some voters who prefer (only slightly) W to S vote for S because she is more likely than W to defeat R in the second round. There are many different combinations of distribution of preferences F, $\Pr(R \mid RS)$, and $\Pr(R \mid RW)$ such that 4% of the voters, all of whom prefer W to both R and S, vote for S.

4.3 No Other Equilibria

In the previous sections we have identified two types of strictly perfect equilibria: Duverger's Law equilibria, and the Duverger's Hypothesis equilibria as described in Proposition 2. The following proposition establishes that these are the only two types of strictly perfect equilibria.

Proposition 3. *There is no strictly perfect equilibrium other than those characterized in Propositions 1 and 2.*

Proof. See Appendix A2. □

A direct implication of Proposition 3 is that *sincere voting* and *push-over tactics*, two types of voting behavior that are commonly believed to arise in (three-candidate) runoff elections (Duverger 1957, Cox 1997, Martinelli 2002), are not supported in equilibrium. There are two main differences between our analysis and previous studies that explain why such behaviors do not arise in our model: the richness of the preference structure and the focus on strictly perfect equilibria. Our results thus show implicitly that both sincere voting and push-over tactics are not robust phenomena in runoff elections and are therefore unlikely to be observed empirically.

There are two situations to consider in order to understand why voters do not vote sincerely in the first round. First, a situation in which the most likely pivotal event is the threshold pivotability R/RS or R/RW. Conditional on being pivotal, voters thus choose between an outright victory of R and a second round opposing R to the runner-up (S or W). The incentives are thus the same as in the Duverger's Hypothesis equilibrium described in the previous section: all voters who prefer R to the runner-up prefer to vote for R. This includes voters whose most preferred candidate is the third candidate. Hence, their vote will not be sincere. Second, a situation in which the most likely pivotal event is the second-rank pivotability RS/RW. Conditional on being pivotal, voters thus choose whom of S and W will oppose R in the second round. A vote for R is irrelevant to that choice and thus useless. By contrast, ballots for S and W are valuable: voting for W increases the chances of R to win in the second round (since W is a weaker opponent of R than S) whereas voting for S decreases the chances of R in the second round but it also decreases the chances of W to win. Therefore, R-supporters prefer not to vote sincerely: they vote either for S or for W.

Push over is the incentive to vote for an unpopular candidate in the first round with the sole purpose of helping the front-runner to win in the second round.[12] It works as follows. Suppose that a voter ranks candidate R higher than both S and W. He expects R to gain enough votes to reach the second round, but not enough to win outright in the first round. In his expectations, S and W will receive a much lower share than R, but the difference between the expected shares of S and W is small. For which candidate should our voter vote? A vote for his most preferred candidate, R is of no use: it is very unlikely that such a vote will push R above the threshold of 50% (nor it is likely that a vote will be needed to ensure R's participation in the second round). On the other hand, a vote for either S or W is likely to change the composition of the second round. Since R has higher chances of winning a second round against the weak opponent, W, then our R-supporter prefers to vote for this candidate to ensure a higher chance of his most preferred candidate to win the election.

For a supporter of R to push over and vote for W in equilibrium, one cannot have that a

[12] Push over is intrinsically related to the "non-monotonicity" of runoff systems, i.e. the fact that increasing the vote share of a candidate may decrease her probability of victory (Schmidt 1973).

unique pivotal event is more likely than all others. Indeed, all R-supporter vote for R when a threshold pivotability (R/RS or R/RW) dominates, and vote for either S or W (making impossible that R is the front-runner) if a second-rank pivotability (RS/RW) dominates. We thus need two pivotal events to dominate. This requires an expected tie between the top two contenders: an impossibility in any strictly perfect equilibrium (Lemma 1). Thus, push over is not a robust phenomenon in runoff elections.

Though not supported in equilibrium, push over incentives do affect the voting behavior of voters. For instance, as explained above, there are situations in which the desire to qualify a weak opponent to the second round induces R-supporters to behave non-sincerely.

Together, Propositions 1, 2, and 3 allow us to draw a general conclusion about the nature of the support in the two rounds. The front-runner always receives the support of all the voters who prefer her to the runner-up. An implication is that the vote share of the front-runner should not increase between the first and the second round if the distribution of voters remains unchanged between the two rounds. Thus, unless the front runner wins outright in the first round, then he is expected to lose in the second round. This is not an appealing feature of our model. Indeed, such a scenario seems to happen very frequently in real life elections. For instance, Bullock and Johnson (1992) report empirical evidence on U.S. data according to which the election winner corresponds to the first-round winner approximately 70% of the times. However, it appears that this feature of our model is an artifact of the assumption that all voters are strategic. In the next section, we show that a model including non-strategic voters accommodates easily for such a phenomenon.

5 Non-Strategic Voters

The empirical literature on strategic voting (see Kawai and Watanabe 2012 and references therein) shows that the electorate is composed of both strategic and non-strategic voters. Non-strategic voters vote for their most preferred candidate no matter what other voters do, whereas strategic voters maximize their expected utility, taking the behavior of the other voters into account. Models including only one type of voters are thus at odds with empirical findings. In this section, we discuss the robustness of our results to the presence of non-strategic voters.

There is no reason to believe that voters of some types, i.e. with some given preferences, are more likely to be strategic than others. Therefore, we adopt a neutral position: we assume that each voter, no matter his type, is strategic with probability λ and non-strategic with probability $(1-\lambda)$. This implies that among the supporters of, say, R, a fraction $(1-\lambda)$ vote for R no matter what they expect others to do.

The best response of strategic voters is not affected by the presence of non-strategic voters. Yet, the presence of non-strategic voters may affect the equilibrium properties of majority runoff elections. We illustrate this influence through numerical examples. First, with non-strategic voters, the vote share of the front-runner can increase in the second round.[13] This is in stark contrast with the predictions of the model without non-strategic voters. Given the empirical evidence on U.S. data according to which the election winner corresponds to the first-round winner approximately 70% of the times, this example suggests that a model including both strategic and non-strategic voters outperforms a model including only strategic voters. Second, in the presence of non-strategic voters, push over can be supported in equilibrium. Yet, we identify a necessary condition for the existence of a push over equilibrium: it requires an unreasonably large fraction of non-strategic voters in the electorate.

5.1 Increase of the Frontrunner's Vote Share in the Second Round

In the presence of non-strategic voters in the electorate, we can prove the existence of a strictly perfect Duverger's Hypothesis equilibrium in which the vote share of the front runner increases in the second round. To compute the vote shares in the second round, we assume that voters are sequentially rational. Therefore, all voters vote for their most-preferred participating candidate.

Suppose that $\lambda = 30\%$, i.e. 70% of the voters are expected to be non-strategic. Suppose also the following expected distribution of preferences in the electorate:

Preferences	Expected Share
$R \succ S \succ W$	0.21
$R \succ W \succ S$	0.2
$S \succ R \succ W$	0.11
$S \succ W \succ R$	0.05
$W \succ S \succ R$	0.33
$W \succ R \succ S$	0.1

[13] We assume that all voters are sequentially rational. Therefore, in the second round, they all vote for their most-preferred participating candidate. See Bouton (2012) for a formal analysis of voting behavior in the second round of a runoff election.

Therefore, R is the Condorcet winner, and S is the Condorcet loser. However, S is a stronger opponent of R than W (49% of the votes in the second round for S and 48% for W when opposed to R). In this case, we can prove the existence of a Duverger's Hypothesis equilibrium as the one identified in Proposition 2. When strategic voters who prefer R to W vote for R and those who prefer W to R vote for their most preferred candidate, the expected vote shares are: $\tau_R = 0.21 + 0.2 + 0.3$*$0.11 = 0.443$, $\tau_W = 0.33 + 0.1 = 0.43$, and $\tau_S = 0.7$*$0.11 + 0.05 = 0.127$. For these expected vote shares, the largest magnitude is $\mu\left(piv_{R/RW}\right) = -0.00652$, and the second largest is $\mu\left(piv_{RW/RS}\right) = -0.08962$. Hence, the postulated strategy is indeed a best response for all strategic voters (see Section 4.2).[14] The (expected) vote share of R in the second round is 52% if opposed to W and 51% if opposed to S. This is substantially higher than the 44.3% of the votes that R is expected to receive in the first round.

5.2 Push Over

We prove two results in this subsection. First, we show that a strictly perfect push over equilibrium may exist. Second, we prove that the fraction of strategic voters must be sufficiently small for a push over equilibrium to exist.

Suppose that $\lambda = 0.11\%$, i.e. 89% of the voters are expected to be non-strategic. Suppose also the following expected distribution of preferences in the electorate:

Preferences	Expected Share
$R \succ S \succ W$	0.13
$R \succ W \succ S$	0.28
$S \succ R \succ W$	0.145
$S \succ W \succ R$	0.155
$W \succ S \succ R$	0.23
$W \succ R \succ S$	0.06

In this example, sincere voting would lead (in expectation) to a second round opposing R to S and then to an expected victory of S in the second round. Yet, there is a push over equilibrium in which S is expected to rank third. In that equilibrium, all strategic voters (even those who rank R first) vote for W if they prefer W to S and vote for S if they prefer S to W. Non-strategic voters

[14]Here we implicitly assume that all W's supporters have sufficiently strong preferences in favor of W that they prefer a second round opposing R to W instead of S even if S is a stronger opponent of R.

vote for their most preferred candidate. We therefore have: $\tau_R = 0.89 * (0.13 + 0.28) = 0.3649$, $\tau_W = 0.23 + 0.06 + 0.11*0.28 = 0.3208$, and $\tau_S = 0.145 + 0.155 + 0.11*0.13 = 0.3143$. For these expected vote shares, the largest magnitude is $\mu\left(piv_{RW/RS}\right) = -3.32633 \times 10^{-5}$. Hence, the postulated strategy is indeed a best response for all strategic voters.[15] This example highlights how, by pushing over, strategic R supporters can influence the outcome of the election to their advantage.

We now identify a necessary condition on the fraction of non-strategic voters for the push-over equilibrium to exist. To be the frontrunner, a candidate must receive strictly more than 1/3 of the votes. In a push-over equilibrium, we know that all strategic R supporters vote for W. Therefore, the fraction of non-strategic R supporters must be strictly larger than 1/3:

$$\begin{aligned}
(1-\lambda)\left(\gamma_{RS}+\gamma_{RW}\right) &> 1/3 \\
&\Leftrightarrow \\
1-\lambda &> \frac{1}{3\left(\gamma_{RS}+\gamma_{RW}\right)}
\end{aligned}$$

If the fraction of R's supporters is 50%, $\gamma_{RS} + \gamma_{RW} = 50\%$, we have that $1 - \lambda > 2/3$, i.e. the fraction of non-strategic voters in the electorate must be strictly larger than 2/3. For $\gamma_{RS}+\gamma_{RW} < 50\%$, this minimal fraction increases. It decreases for $\gamma_{RS} + \gamma_{RW} > 50\%$.

For push over to arise in equilibrium, the electorate must be composed of a large fraction of non-strategic voters. Ultimately, the existence of a push over equilibrium is thus an empirical question. Considering the fraction of strategic voters found by Kawai and Watanabe (2012), i.e. between 63.4% and 84.9% of the electorate, our model predicts that a necessary (but far from sufficient) condition for a push over equilibrium to exist is that at least 91.1% of the electorate prefer R to both S and W. Arguably, this is quite unlikely to happen. Dolez and Laurent (2010) tests directly for push over behavior. They find that (p.10): "the number of the 'ingenious' voters is zero, that is no respondent intended to desert temporarily his/her preferred party on the first round to favor it at the second". This supports our result that push over is unlikely to arise in real-life majority runoff elections.

[15] Here we implicitly assume that all W supporters have sufficiently strong preferences in favor of W that they prefer a second round opposing R to W instead of S, even if W is expected to be defeated by R in the second round. This is not necessary for the existence of a push over equilibrium. We also assume that all strategic R supporters prefer a second round of R vs. W rather than vs. S. This is satisfied if, for instance, all R supporters have sufficiently intense preferences in favor of R and against S and/or W. Note that we could relax this assumption.

6 Conclusions

In this paper, we characterized the set of strictly perfect equilibria in three candidate runoff elections. In all equilibria, the front-runner receives the votes of all the voters who prefer her to the runner-up. An equilibrium where all the remaining voters coordinate on the runner-up candidate always exists, that is, similarly to the case of plurality elections, there always exists a Duverger's Law equilibrium in which only two candidates receive a positive vote share. We also showed that there is at most one Duverger's Hypothesis equilibrium in which three candidates receive a positive fraction of the votes. The characteristics of that unique Duverger's Hypothesis equilibrium challenge common beliefs about runoff elections: (i) some voters do not vote for their most preferred candidate (i.e. the sincere voting equilibrium does not exist), (ii) supporters of the front-runner do not vote for a less-preferred candidate in order to "choose" who will face the front-runner in the second round (i.e. there is no push over equilibrium), and (iii) it can lead to the exclusion of the Condorcet Winner from the second round.

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Appendices

Appendix A1 provides a reminder of some fundamental properties of Poisson games (Myerson 2000 and 2002). Appendix A2 demonstrates the claims made in Section 4.

Appendix A1: Large Poisson Games in Runoff Elections

A Poisson game $\Gamma \equiv (n, \mathcal{T}, F, \mathcal{C}, u)$ is defined by the expected number of voters $n \in \mathbb{N}$, the set of types $\mathcal{T}$, a probability measure F defined over $\mathcal{T}$, a set of actions $\mathcal{C}$ and a vector of payoffs $u_t : \mathcal{C} \times Z(\mathcal{C}) \to \mathbb{R}$, each $t \in \mathcal{T}$, where $Z(\mathcal{C})$ is the set of all action profiles for the players.

Myerson (2000) shows that, in a Poisson game, the probability of an exact profile of vote shares is exponentially decreasing in the expected number of voters, n, and converges to zero at an exponential speed proportional to its magnitude. Furthermore, Myerson (2000, Corollary 1) shows that:

Lemma 1. *Compare two events E and E' with different magnitudes: $\mu(E) < \mu(E')$. Then, the probability ratio of the former over the later event goes to zero as n increases:*

$$\mu(E) < \mu(E') \Rightarrow \frac{\Pr(E)}{\Pr(E')} \underset{n\to\infty}{\to} 0$$

The intuition is that the probabilities of different events do not converge towards zero at the same speed. Hence, unless two events have the same magnitude, their likelihood ratio converges either to zero or to infinity when the electorate grows large. Myerson calls this result the *magnitude theorem*. Proofs in this paper rely extensively on this property of large Poisson games.

We make use of the magnitude theorem to identify the properties of the set of strictly perfect equilibria as $n \to \infty$. As explained in Section 3, there are two types of pivotal events in a majority runoff election: the threshold pivotabilities and the second-rank pivotabilities. As proven in Bouton (2012), the magnitude of a pivotal event piv is larger when the expected outcome of the first round, τ, is close to the conditions necessary for event piv to occur. For instance, the pivotal event $piv_{i/ij}$ is more likely to occur when $1/2 = \tau_i > \tau_j > \tau_k$ than when $1/2 > \tau_k > \tau_j > \tau_i$. Indeed, the occurrence of the pivotal event in the latter case requires a "larger deviation with respect to the expected outcome".

Lemma 2. *The magnitudes of the pivot probabilities are:*

(a) Threshold pivot probability* i/ij *and* ij/i*:

$$\mu\left(piv_{i/ij}\right) = \mu\left(piv_{ij/i}\right) = \begin{cases} 2\sqrt{\left(\tau_j+\tau_k\right)\tau_i} - 1 & \text{if } \frac{\tau_j}{\tau_j+\tau_k} \geq \frac{1}{2}; \\ 2\sqrt{2\tau_i\sqrt{\tau_j\tau_k}} - 1 & \text{otherwise} \end{cases} \tag{2}$$

(b) Second-rank pivot probability* ki/kj *and* kj/ki*:

$$\begin{aligned} \mu\left(piv_{ki/kj}\right) &= \mu\left(piv_{kj/ki}\right) = \\ &= \begin{cases} -\left(\sqrt{\tau_i}-\sqrt{\tau_j}\right)^2 & \text{if } 2\sqrt{\tau_i\tau_j} > \tau_k > \sqrt{\tau_i\tau_j}; \\ 2\sqrt{2\tau_k\sqrt{\tau_i\tau_j}} - 1 & \text{if } \tau_k > 2\sqrt{\tau_i\tau_j}; \\ 3\left(\tau_i\tau_j\tau_k\right)^{\frac{1}{3}} - 1 & \text{if } \sqrt{\tau_i\tau_j} > \tau_k. \end{cases} \end{aligned} \tag{3}$$

We are now in position of establishing some preliminary results on the equilibrium behavior of the magnitudes of different pivot probabilities. In particular, Lemma 3 says that the magnitude of $piv_{R/Ri}$, the event that a single vote being decisive between the front runner winning outright and a second round between the front-runner and the runner-up, is never less than the magnitude of any other first round pivot probability. Also, the magnitude of $piv_{RS/RW}$, the event that a vote is pivotal in determining which candidate will face the front runner in a second round, is never less than the magnitude of any other second round pivot probability and is strictly larger unless the front runner and the runner-up have the same expected share of votes.

Lemma 3. *There are three possible rankings of the two largest magnitudes:*

(i) $\mu\left(piv_{R/Ri}\right) \geq \mu\left(piv_{i/Ri}\right) \geq$ *any other magnitude; or*

(ii) $\mu\left(piv_{R/Ri}\right) \geq \mu\left(piv_{RS/RW}\right) \geq$ *any other magnitude; or*

(iii) $\mu\left(piv_{RS/RW}\right) > \mu\left(piv_{R/Ri}\right) \geq$ *any other magnitude.*

Proof. We first compare $\mu\left(piv_{R/Ri}\right)$ with other threshold magnitudes and show that this is the largest threshold magnitude:

$$\mu\left(piv_{R/Ri}\right) = 2\sqrt{\left(\tau_j + \tau_i\right)\tau_R} - 1 \geq 2\sqrt{\left(\tau_j + \tau_R\right)\tau_i} - 1 = \mu\left(piv_{i/Ri}\right)$$

and the expression holds with equality only if $\tau_j = 0$ or $\tau_i = \tau_R$. Also, trivially

$$\mu\left(piv_{R/Ri}\right) = 2\sqrt{\left(\tau_j + \tau_i\right)\tau_R} - 1 > 2\sqrt{\left(\tau_i + \tau_R\right)\tau_j} - 1 = \mu\left(piv_{j/Rj}\right)$$

unless $\tau_R = \tau_i$ and

$$\mu\left(piv_{i/Ri}\right) = 2\sqrt{\left(\tau_j + \tau_R\right)\tau_i} - 1 > 2\sqrt{\left(\tau_i + \tau_R\right)\tau_j} - 1 = \mu\left(piv_{j/Rj}\right)$$

unless $\tau_j = \tau_i$. We can also show that

$$\mu\left(piv_{R/Ri}\right) \quad = \quad 2\sqrt{\left(\tau_j + \tau_i\right)\tau_R} - 1 > 2\sqrt{2\tau_R\sqrt{\tau_i\tau_j}} - 1 = \mu\left(piv_{R/Rj}\right)$$

unless $\tau_i = \tau_j$. Indeed, $2\sqrt{\left(\tau_j + \tau_i\right)\tau_R} - 1 > 2\sqrt{2\tau_R\sqrt{\tau_i\tau_j}} - 1 \iff \tau_R\left(\tau_i + \tau_j\right) - 2\tau_R\sqrt{\tau_i\tau_j} > 0$. The LHS of the last inequality can be rewritten as $\tau_R\left[\tau_j + \tau_i - 2\sqrt{\tau_i\tau_j}\right]$ and

$$\tau_R\left[\tau_j + \tau_i - 2\sqrt{\tau_i\tau_j}\right] \quad = \tau_R\left(\sqrt{\tau_i} - \sqrt{\tau_j}\right)^2 > \quad 0$$

if $\tau_i > \tau_j$.

It remains to show that $\mu\left(piv_{R/Ri}\right)$ is larger than $\mu\left(piv_{i/ij}\right)$ and $\mu\left(piv_{j/ij}\right)$. The first condition is satisfied if $\tau_R > \tau_i$ or $\tau_i > \tau_j$ since

$$\begin{aligned}
\mu\left(piv_{R/Ri}\right) &> \mu\left(piv_{i/ij}\right) \\
2\sqrt{\left(\tau_j + \tau_i\right)\tau_R} - 1 &> 2\sqrt{2\tau_i\sqrt{\tau_j\tau_R}} - 1 \\
\frac{\left(\tau_j + \tau_i\right)}{2}\tau_R &> \sqrt{\tau_j\tau_i}\sqrt{\tau_i\tau_R}.
\end{aligned}$$

Notice that the first element of the LHS is greater or equal (only if $\tau_R = \tau_i = \tau_j = 1/3$) than the first element of the RHS since a geometric mean of $x, y, \ldots$ is always less than or equal to the arithmetic mean of $x, y, \ldots$, with the equality holding only if $x = y = \ldots$. Also, since $\tau_R \geq \tau_i$, the second element is also greater than or equal to the second element of the RHS. The last case, i.e. $\mu\left(piv_{R/Ri}\right) > \mu\left(piv_{j/ij}\right)$ follows a very similar argument.

Furthermore,

$$\mu\left(piv_{i/Ri}\right) = 2\sqrt{\left(\tau_j + \tau_R\right)\tau_i} - 1 > 2\sqrt{2\tau_i\sqrt{\tau_j\tau_R}} - 1 = \mu\left(piv_{i/ij}\right)$$

unless $\tau_j = \tau_R$, and similarly $\mu\left(piv_{i/Ri}\right) > \mu\left(piv_{j/ij}\right)$.

We now compare $\mu\left(piv_{RS/RW}\right)$ with other second-rank magnitudes and we show that this is the largest second-rank magnitude. First, notice that for $\tau_R \geq \tau_i \geq \tau_j$, $\tau_R \geq \sqrt{\tau_i\tau_j}$, $\tau_j \leq \sqrt{\tau_R\tau_i}$, and $\tau_i > 2\sqrt{\tau_R\tau_j} \Rightarrow \tau_R > 2\sqrt{\tau_i\tau_j}$. Hence, to prove that $\mu\left(piv_{RS/RW}\right) > \mu\left(piv_{jR/ji}\right)$ it is sufficient to show two conditions: 1) if $\tau_R < 2\sqrt{\tau_i\tau_j}$, then it is sufficient to show that $-\left(\sqrt{\tau_i} - \sqrt{\tau_j}\right)^2 > 3\left(\tau_i\tau_j\tau_R\right)^{\frac{1}{3}} - 1$. The inequality can be rewritten as (using $\tau_R + \tau_i + \tau_j = 1$)

$$\tau_i + \tau_j - 2\sqrt{\tau_i\tau_j} < \tau_i + \tau_j + \tau_R - 3\left(\tau_i\tau_j\tau_R\right)^{\frac{1}{3}}$$

and therefore as

$$\frac{\tau_R + 2\sqrt{\tau_i\tau_j}}{3} > \left(\sqrt{\tau_i\tau_j}^2\tau_R\right)^{\frac{1}{3}}.$$

The RHS and the LHS are, respectively, the weighted geometric and arithmetic means of $\sqrt{\tau_i\tau_j}$ and τ_R with weights 2 and 1. It follows that they are equal if and only if $\tau_R = \tau_i = \tau_j = \frac{1}{3}$, otherwise, the inequality holds.

2) if $\tau_R \geq 2\sqrt{\tau_i\tau_j}$, then it is sufficient to show that $2\sqrt{2\tau_R\sqrt{\tau_i\tau_j}} - 1 \geq 3\left(\tau_R\tau_i\tau_j\right)^{\frac{1}{3}} - 1$. Taking logs and simplifying, we get

$$\begin{aligned}\frac{3}{2}\ln 2 - \ln 3 &\geq \frac{\ln\tau_i\tau_j - 2\ln\tau_R}{12}\\ \ln\left[\left(\frac{\sqrt{2}^3}{3}\right)^{12}\right] &\geq \ln\left(\frac{\tau_i\tau_j}{\tau_R^2}\right)\end{aligned}$$

which simplifies to $\tau_R \geq \left(\frac{\sqrt{2}^3}{3}\right)^6\sqrt{\tau_i\tau_j}$. Notice that $\left(\frac{\sqrt{2}^3}{3}\right)^6 \approx .7023 < 2$. Hence, since $\tau_R > 2\sqrt{\tau_i\tau_j}$, we have shown that $\mu\left(piv_{RS/RW}\right) > \mu\left(piv_{jR/ji}\right)$.

To show that $\mu\left(piv_{RS/RW}\right) \geq \mu\left(piv_{iR/ij}\right)$, we divide the analysis into three cases. 1) if $\tau_i < \sqrt{\tau_R\tau_j}$, then $\mu\left(piv_{iR/ij}\right) = \mu\left(piv_{jR/ji}\right)$ and we have just shown that $\mu\left(piv_{RS/RW}\right) \geq \mu\left(piv_{jR/ji}\right)$ with equality holding only if $\tau_R = \tau_i = \tau_j = \frac{1}{3}$.

2) if $\tau_i > 2\sqrt{\tau_R \tau_j}$, then $\mu\left(piv_{RS/RW}\right) \geq \mu\left(piv_{iR/ij}\right)$ if and only if

$$\begin{aligned} 2\sqrt{2\tau_R\sqrt{\tau_i\tau_j}} - 1 &\geq 2\sqrt{2\tau_i\sqrt{\tau_R\tau_j}} - 1 \\ \tau_R\sqrt{\tau_i\tau_j} &\geq \tau_i\sqrt{\tau_R\tau_j} \\ \sqrt{\tau_R} &\geq \sqrt{\tau_i} \end{aligned}$$

which is trivially true for all $\tau_R \geq \tau_i$, with equality only if $\tau_R = \tau_i$.

3) if $\sqrt{\tau_R\tau_j} < \tau_i < 2\sqrt{\tau_R\tau_j}$ and $\tau_R < 2\sqrt{\tau_i\tau_j}$, then $\mu\left(piv_{RS/RW}\right) \geq \mu\left(piv_{iR/ij}\right)$ if and only if

$$\begin{aligned} -\left(\sqrt{\tau_i} - \sqrt{\tau_j}\right)^2 &\geq -\left(\sqrt{\tau_R} - \sqrt{\tau_j}\right)^2 \\ \sqrt{\tau_i} &\leq \sqrt{\tau_R} \end{aligned}$$

which is trivially true for all $\tau_R \geq \tau_i$, with equality only if $\tau_R = \tau_i$.

Last, we compare $\mu\left(piv_{R/Ri}\right)$ and $\mu\left(piv_{i/Ri}\right)$ with $\mu\left(piv_{RS/RW}\right)$. Notice that if $\tau_R \geq 2\sqrt{\tau_i\tau_j}$, then $\mu\left(piv_{RS/RW}\right) = \mu\left(piv_{R/Rj}\right) \leq \mu\left(piv_{R/Ri}\right)$ with the last inequality holding with strict sign unless $\tau_i = \tau_j$. Otherwise, if $\tau_R \leq 2\sqrt{\tau_i\tau_j}$, then there exist two regions of $\Delta(\mathcal{C})$ such that $\mu\left(piv_{i/Ri}\right) > \mu\left(piv_{RS/RW}\right)$ and $\mu\left(piv_{i/Ri}\right) > \mu\left(piv_{RS/RW}\right)$, respectively.

Furthermore, if $\tau_R \geq 2\sqrt{\tau_i\tau_j}$, $\mu\left(piv_{i/Ri}\right) \geq \mu\left(piv_{RS/RW}\right) \Rightarrow$

$$\begin{aligned} 2\sqrt{(\tau_j + \tau_R)\,\tau_i} - 1 &\geq 2\sqrt{2\tau_R\sqrt{\tau_i\tau_j}} - 1 \\ (\tau_j + \tau_R)\,\tau_i &\geq 2\tau_R\sqrt{\tau_i\tau_j}. \end{aligned}$$

Using $\tau_i \geq \sqrt{\tau_i\tau_j}$, we have

$$\begin{aligned} (\tau_j + \tau_R)\,\tau_i &\geq 2\tau_R\tau_i \geq 2\tau_R\sqrt{\tau_i\tau_j} \\ \tau_j &\geq \tau_R \end{aligned}$$

with equality holding only if $\tau_R = \tau_S = \tau_W = 1/3$. Otherwise, if $\tau_R \geq 2\sqrt{\tau_i\tau_j}$, there exist two regions of $\Delta(\mathcal{C})$ such that $\mu\left(piv_{i/Ri}\right) > \mu\left(piv_{RS/RW}\right)$ and $\mu\left(piv_{i/Ri}\right) > \mu\left(piv_{RS/RW}\right)$, respectively. $\square$

Appendix A2: Proofs of Section 4

Proof. [Proof of Proposition 1] In a Duverger's law equilibrium there exists $j \in \{W, S\} : \tau_j(\sigma^*, F) = 0$. That is, we need $\sigma_t^*(j) = 0$ for all but at most a measure zero of voter types. This implies $\mu\left(piv_{R/Ri}\right) = \mu\left(piv_{i/Ri}\right) >$ any other magnitude, $i \neq j$. We can construct $G_t(c, n\tau), \forall c \in \mathcal{C}$ and divide by $\Pr\left(piv_{R/Ri}\right)$ and take the limit when $n \to \infty$. To prove that there exists a Duverger's law equilibrium, it is sufficient to show that for all $t \in \mathcal{T}$, there exist $c \neq j : \lim_{n\to\infty} G_t(c, n\tau) \geq \lim_{n\to\infty} G_t(j, n\tau)$.

Let

$$\phi = \lim \frac{\Pr\left(piv_{S/Ri}\right)}{\Pr\left(piv_{R/Ri}\right)} > 0.$$

Then, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{G_t\left(j, n\tau\right)}{\Pr\left(piv_{R/Ri}\right)} &= U\left(R, i \mid t\right) - U\left(R \mid t\right) + \phi\left[U\left(R, i \mid t\right) - U\left(i \mid t\right)\right] \\
&= \left(1+\phi\right) U\left(R, i \mid t\right) - U\left(R \mid t\right) - \phi U\left(i \mid t\right);
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{G_t\left(i, n\tau\right)}{\Pr\left(piv_{R/Ri}\right)} &= U\left(R, i \mid t\right) - U\left(R \mid t\right) + \phi\left[U\left(i \mid t\right) - U\left(R, i \mid t\right)\right] \\
&= \left(1-\phi\right) U\left(R, i \mid t\right) - U\left(R \mid t\right) + \phi U\left(i \mid t\right);
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{G_t\left(R, n\tau\right)}{\Pr\left(piv_{R/Ri}\right)} &= U\left(R \mid t\right) - U\left(R, i \mid t\right) + \phi\left[U\left(R, i \mid t\right) - U\left(i \mid t\right)\right] \\
&= \left(\phi-1\right) U\left(R, i \mid t\right) + U\left(R \mid t\right) - \phi U\left(i \mid t\right).
\end{aligned}$$

Notice that $U\left(R, i \mid t\right)$ is a strict convex combination of $U\left(R \mid t\right)$ and $U\left(i \mid t\right)$. It is easy to show that $\lim_{n\to\infty} G_t\left(R, n\tau\right) \geq \lim_{n\to\infty} G_t\left(j, n\tau\right)$ if $U\left(R \mid t\right) \geq U\left(i \mid t\right)$ and $\lim_{n\to\infty} G_t\left(i, n\tau\right) \geq \lim_{n\to\infty} G_t\left(j, n\tau\right)$ otherwise. Hence, $\sigma_t^*\left(R\right) = 1$ is a best response if $U\left(R \mid t\right) > U\left(i \mid t\right)$, $\sigma_t^*\left(i\right) = 1$ is a best response if $U\left(i \mid t\right) > U\left(R \mid t\right)$, and $\sigma_t^*\left(j\right) = 0$ is a best response for all $t \in \mathcal{T}$, since for all $t \in \mathcal{T} : U\left(i \mid t\right) = U\left(R \mid t\right)$, $\mathcal{R}_t\left(\tau\left(\sigma^*, F\right)\right)(c) = [0,1], \forall c \in \mathcal{C}$. Notice also that $t \in \mathcal{T} : U\left(i \mid a, b\right) = U\left(R \mid a, b\right)$ has measure zero for all $F \in \mathcal{F}$. Hence, this is a strict equilibrium for all but a measure zero of voters. We imposed no restrictions on F, hence for any $F \in \mathcal{F}$, if $\sigma_t^*\left(j\right) = 0$, for all $t \in \mathcal{T}$ but at most $t \in \mathcal{T} : U\left(i \mid t\right) = U\left(R \mid t\right)$, then $\tau_j\left(\sigma^*, F\right) = 0$.

To show that a Duverger's law equilibrium is strictly perfect, consider any $\tau \in \Delta\left(\mathcal{C}\right) : \left|\tau - \tau\left(\sigma^*, F\right)\right| < \epsilon$ for some $\epsilon > 0$ and $\tau_j\left(\sigma^*, F\right) = 0$. Notice that all magnitude formulae are continuous. For ϵ sufficiently small, the order of the magnitudes is $\mu\left(piv_{R/Ri}\right) > \mu\left(piv_{i/Ri}\right) >$any other magnitude, $i \neq j$. In which case,

$$\lim_{n\to\infty} \frac{G_t\left(j, n\tau\right)}{\Pr\left(piv_{R/RS}\right)} = U\left(R, i \mid t\right) - U\left(R \mid t\right)$$

$$\lim_{n\to\infty} \frac{G_t\left(i, n\tau\right)}{\Pr\left(piv_{R/RS}\right)} = U\left(R, i \mid t\right) - U\left(R \mid t\right)$$

$$\lim_{n\to\infty} \frac{G_t(R, n\tau)}{\Pr\left(piv_{R/RS}\right)} \quad = \quad U(R \mid t) - U(R, i \mid t).$$

Trivially, σ^* is a best response. □

Proof. [Proof of Proposition 2] The starting point of the proof is to consider a tuple (σ^*, F) such that $\mu\left(piv_{R/Ri}\right) > \mu\left(piv_{RS/RW}\right) \geq \mu\left(piv_{R/Rj}\right)$ and $\mu\left(piv_{RS/RW}\right) >$ any other magnitude (that is, i is the runner up candidate). We then divide the rest of the proof is into three parts. First, we show that all voters with $U(R \mid t) > U(i \mid t)$ vote for candidate R. Second, we analyze the behavior of voters with $U(R \mid t) < U(i \mid t)$. We show that those with $U(R, S \mid t) \geq U(R, W \mid t)$ vote for S, whereas the others vote for W. Finally, we prove that the equilibrium is strictly perfect.

First, we can construct,

$$\lim_{n\to\infty} \frac{G_t(j, n\tau)}{\Pr\left(piv_{R/Ri}\right)} = U(R, i \mid t) - U(R \mid t)$$
$$\lim_{n\to\infty} \frac{G_t(i, n\tau)}{\Pr\left(piv_{R/Ri}\right)} = U(R, i \mid t) - U(R \mid t)$$
$$\lim_{n\to\infty} \frac{G_t(R, n\tau)}{\Pr\left(piv_{R/Ri}\right)} = U(R \mid t) - U(R, i \mid t)$$

and conclude that all the voters with $U(R|\, t) > U(i|\, t)$ vote for candidate R.

Second, divide the expected gains by $\Pr\left(piv_{RS/RW}\right)$ and assume $\mu\left(piv_{RS/RW}\right) > \mu\left(piv_{R/Rj}\right)$. Then

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{G_t(S, n\tau)}{\Pr\left(piv_{RS/RW}\right)} - \frac{G_t(W, n\tau)}{\Pr\left(piv_{RS/RW}\right)}\right] \quad &= \quad U(R, S \mid t) - U(R, W \mid t) - U(R, W \mid t) + U(S, W \mid t) \\ &= \quad 2\left[U(R, S \mid t) - U(R, W \mid t)\right].\end{aligned}$$

Hence, a voter of type t (with $U(R|\, t) < U(i|\, t)$) votes for S only if

$$U(R, S \mid t) \quad \geq \quad U(R, W \mid t)$$
$$\Longleftrightarrow$$

$$\begin{aligned}&\Pr(R \mid RS)U(R \mid t) + (1 - \Pr(R \mid RS))\, U(S \mid t) \geq \\ &\qquad \geq \Pr(R \mid RW)U(R \mid t) + (1 - \Pr(R \mid RW))\, U(W \mid t) \qquad (4)\end{aligned}$$

with any mixed strategy allowed for the measure zero of voters' types with $U(R, S \mid t) = U(R, W \mid t)$. Otherwise she votes for W. Notice that the condition in 4 is independent of which candidate, S or W, is expected to receive more votes. Furthermore, since $\Pr(R \mid RS) < \Pr(R \mid RW)$, a type t indifferent between S and W must be a W's supporter. If instead $\mu\left(piv_{RS/RW}\right) = \mu\left(piv_{R/Rj}\right)$, then

$$\begin{aligned}
\lim_{n\to\infty}\left[\frac{G_t\left(S, n\tau\right)}{\Pr\left(piv_{RS/RW}\right)} - \frac{G_t\left(W, n\tau\right)}{\Pr\left(piv_{RS/RW}\right)}\right] &= U\left(R, S \mid t\right) - U\left(R, W \mid t\right) - U\left(R, W \mid t\right) + U\left(S, W \mid t\right) + \\
&\quad +\phi'\left[U\left(R, j \mid t\right) - U\left(R \mid t\right) - U\left(R, j \mid t\right) + U\left(R \mid t\right)\right] \\
&= 2\left[U\left(R, S \mid t\right) - U\left(R, W \mid t\right)\right]
\end{aligned}$$

with

$$\phi' = \frac{\Pr\left(piv_{R/Rj}\right)}{\Pr\left(piv_{RS/RW}\right)} > 0.$$

Hence, the condition for voting for S or W is not changed.

To show that this equilibrium is strictly perfect, consider any $\tau \in \Delta\left(\mathcal{C}\right) : \left|\tau - \tau\left(\sigma^*, F\right)\right| < \epsilon$ for some $\epsilon > 0$ and $\tau_j\left(\sigma^*, F\right) : \tau : \mu\left(piv_{R/Ri}\right) > \mu\left(piv_{RS/RW}\right) \geq \mu\left(piv_{R/Rj}\right)$ and $\mu\left(piv_{RS/RW}\right) >$ any other magnitude. Notice that all magnitude formulae are continuous. For ϵ sufficiently small, the order of the magnitudes is unchanged. In which case, σ^* is a best response. □

Proof. [Proof of Proposition 3] Propositions 1 and 2 characterize the set of equilibria when the order of magnitudes is as in points 1 and 2 in lemma 3. Together, Lemmata 1 and 3 imply that no other strictly perfect equilibrium σ^* can exist unless $\tau\left(\sigma^*, F\right)$ implies point 3 in lemma 3, i.e. when $\mu\left(piv_{RS/RW}\right)$ is the (strictly) largest magnitude. Notice that this implies $\tau_R\left(\sigma^*, F\right) > 0$ (indeed, by definition, R is the front runner).

We can construct,

$$\lim_{n\to\infty}\frac{G_t\left(S, n\tau\right)}{\Pr\left(piv_{RS/RW}\right)} = U\left(R, S \mid t\right) - U\left(R, W \mid t\right)$$

$$\lim_{n\to\infty}\frac{G_t\left(W, n\tau\right)}{\Pr\left(piv_{RS/RW}\right)} = U\left(R, W \mid t\right) - U\left(R, S \mid t\right)$$

$$\lim_{n\to\infty}\frac{G_t\left(R, n\tau\right)}{\Pr\left(piv_{RS/RW}\right)} = 0.$$

Hence, voting for R is a best response for a measure zero of voter types, those with $t \in \mathcal{T} : U\left(R, S \mid t\right) = U\left(R, W \mid t\right)$. Hence, $\tau_R\left(\sigma^*, F\right) = 0$, contradicting the assumption that $\mu\left(piv_{RS/RW}\right)$ is the largest magnitude. □