Risk, Uncertainty, and Option Exercise*

Jianjun Miao[†] and Neng Wang[‡]

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Abstract

Many economic decisions can be described by an option exercise problem. The standard real options approach emphasizes the importance of uncertainty in determining option value and timing of option exercise. No distinction between risk and uncertainty is made. Motivated by the experimental evidence such as the Ellsberg Paradox, we follow Knight (1921) and distinguish risk from uncertainty. To afford this distinction, we adopt the multiple-priors utility model to analyze an option exercise problem. We also provide several examples to illustrate that our model can explain some phenomena that are hard to reconcile with the standard real options model. These examples include real investment, job search, firm exit, firm default, and youth suicide.

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[†]Department of Economics, Boston University, Boston, MA 02215. Email: miaoj@bu.edu. Tel.: 617-353-6675.

[‡]William E. Simon School of Business Administration, University of Rochester, Rochester, NY 14627. Email: wang@simon.rochester.edu; Tel.: 585-275-4896.

1 Introduction

Many economic decisions can be described by binary choices. Examples are abundant. A firm may decide whether or not to invest in a project. It may also decide whether or not to exit an industry. If a corporate firm takes on debt, it has to decide whether or not to default. A worker may decide whether or not to accept a job offer. A dismal person may decide whether or not to commit suicide.

All these decisions share three characteristics. First, the decision is irreversible. Second, there is uncertainty about future rewards. Third, agents have some leeway about the timing of decisions. These three characteristics imply that waiting has positive value. In other words, all the above problems can be viewed as a problem where agents decide when to exercise an "option" analogous to a financial call option – it has the right but not the obligation to buy an asset at some future time of its choosing. This real options approach has been widely applied in investments and corporate finance (see Dixit and Pindyck (1994)).

As argued by Dixit (1992), the standard real options approach to investment under uncertainty can be summarized as "a theory of optimal inertia". "It says that firms that refuse to invest even when the currently available rates of return are far in excess of the cost of capital may be optimally waiting to be surer that this state of affairs is not transitory. Likewise, farmers who carry large losses may be rationally keeping their operation alive on the chance that the future may be brighter" (Dixit (1992, p.109)).

However, the standard real options approach rules out the situation where agents are unsure about the likelihoods of future events. It typically adopts strong assumptions about agents' beliefs. For example, according to the rational expectations hypothesis, agents know precisely the objective probability law of the state process and their beliefs are identical to this probability law. Alternatively, according to the Bayesian approach, an agent's beliefs are represented by a subjective probability measure or Bayesian prior. There is no meaningful distinction between *risk*, where probabilities are available to guide choice, and *uncertainty*, where information is too imprecise to be summarized adequately by probabilities. By contrast, Knight (1921) emphasizes this distinction and argues that uncertainty is more common in decision-making.[1] For experimental evidence, the Ellsberg Paradox suggests that people prefer to act on known rather than unknown or ambiguous probabilities.[2] Ellsberg-type behavior contradicts the Bayesian paradigm, i.e., the existence of any prior underlying choices.

To incorporate Knightian uncertainty or ambiguity, we adopt the multiple-priors utility model (Gilboa and Schmeidler (1989), Epstein and Wang (1994)). This model is axiomatized by Gilboa and Schmeidler (1989) in a static setting and by Epstein and Schneider (2002) in a dynamic setting. In this model, the agent beliefs are represented by a set of priors. The set of priors captures both the degree of Knightian uncertainty and uncertainty aversion.[3]

We first describe an agent's option exercise decision as an optimal stopping problem. We then characterize the solution under Knightian uncertainty and compare it to that in the standard model. We find that the option value under Knightian uncertainty is smaller than that in the standard model. Moreover, an uncertainty averse agent exercises the option earlier than an expected utility maximizer.

The standard real options approach emphasizes the importance of uncertainty in determining option value and timing of option exercise. Implicitly assumed, uncertainty is identical to risk. Recognizing the difference between risk and uncertainty, we conduct comparative statics with respect to the set of priors. We find that if an

[1]Henceforth, we refer to such uncertainty as *Knightian uncertainty* or *ambiguity.*

[2]See Ellsberg (1961). One version of the story is as follows. A decision maker is a offerred a bet on drawing a red ball from two urns. The first urn contains exactly 50 red and 50 black balls. The second urn has 100 balls, either red or black, however the exact number of red or black balls is unknown. Vast majority agents choose from the first urn rather than the second.

[3]For a formal definition of uncertainty aversion, see Epstein (1999) and Epstein and Zhang (2001).

agent is more uncertainty averse or faces a higher level of Knightian uncertainty, then the option value is lower and the agent exercises the option earlier.

We provide several examples to apply our results, including real investment, job search, firm exit, firm default, and youth suicide. These examples illustrate that our model can explain some phenomena that may be hard to be reconciled with the standard real options model.

The multiple-priors utility model has been applied to asset pricing and portfolio choice problems in a number of papers.[4] To our knowledge, our paper is the first to apply the multiple-priors utility model to study real options problems.

A related approach based on robust control theory is proposed by Hansen and Sargent and their coauthors.[5] They emphasize 'model uncertainty' which is also motivated in part by the Ellsberg Paradox . We refer readers to Epstein and Schneider (2002) for further discussion on these two approaches.

The paper is organized as follows. Section 2 presents the model and states assumptions. Section 3 provides and analyzes main results. Section 4 presents several applications. Section 5 concludes.

2 The Model

2.1 Background

Before presenting the model, we first provide some background about multiple-priors utility. The static multiple-priors utility model of Gilboa and Schmeidler (1989) can be described informally as follows. Suppose uncertainty is represented by a measurable space $(S, \mathcal{F})$. The decision-maker ranks uncertain prospects or acts, maps from S into an outcome set $\mathcal{X}$. Then the multiple-priors utility $U(f)$ of any act f has the

[4]See Epstein and Wang (1994, 1995), Chen and Epstein (2001), Epstein and Miao (2003), Kogan and Wang (2002), Cao et al (2002), and Routledge and Zin (2003).

[5]See, for example, Anderson, Hansen and Sargent (2003) and Hansen and Sargent (2000).

functional form:

$$U(f) = \min_{Q \in \Delta} \int u(f)\, dQ,$$

where $u : \mathcal{X} \to \mathbb{R}$ is a von Neumann-Morgenstern utility index and Δ is a subjective set of probability measures on $(S, \mathcal{F})$. Intuitively, the multiplicity of priors models ambiguity about likelihoods of events and the minimum delivers aversion to such ambiguity. The standard expected utility model is obtained when the set of priors Δ is a singleton.

The Gilboa and Schmeidler model is generalized to a dynamic setting in discrete time by Epstein and Wang (1994). Their model can be described briefly as follows. The time t conditional utility from a consumption process $c = (c_t)_{t \geq 0}$ is defined by the recursion

$$V_t(c) = u(c_t) + \beta \min_{Q \in \mathcal{P}} E_t^Q \left[V_{t+1}(c) \right], \tag{1}$$

where $\beta \in (0,1)$ is the discount factor, E_t^Q is the conditional expectation operator with respect to measure Q, and $\mathcal{P}$ is a set of one-step-ahead conditional probability measures. An important feature of this utility is that it satisfies dynamic consistency because of the recursivity of utility (1).[6] This model will be adopted below.

2.2 Example

To understand our model and results, an illustrative example proves useful. Consider an investor who contemplates to invest irreversibly in a project in a three period setting. Assume that the investor is risk neutral and discounts cash flows by $\beta = 0.2$. The payoff in period 0 is certain, while the payoffs in periods 1 and 2 are uncertain. Investment costs $I = 145$. The initial value of the project is $x = 100$. In periods 1 and 2, the value of the project may go up or down by 50% (see Figure 1). These events are independent.

[6] See Epstein and Schneidler (2002) for further discussion on this model and dynamic consistency.

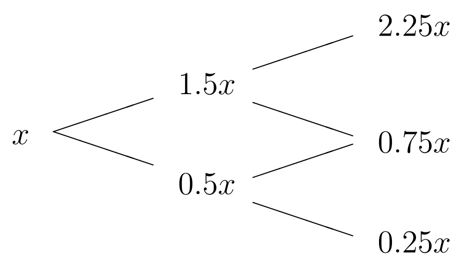

Figure 1: Investment cash flows

In standard models, the investor views the investment as purely risky. We may suppose that up and down have equal probability. Then according to the Marshallian net present value principle, the investor will not invest at period zero since the net present value of the investment opportunity is

$$NPV = x + \beta\left(1.5x + 0.5x\right)/2 + \beta^2\left(2.25x/4 + 0.75x/2 + 0.25x/4\right) - I = -21 < 0.$$

However, if the investor can postpone the investment, he can observe whether the value of the project goes up or not. He can avoid the downside loss by investing when the value goes up. We use backward induction to solve the problem. It is clear that the value of the investment at period 2, F_2, is positive only when the project value is $2.25x$,

$$F_2\left(2.25x\right) = \max\{2.25x - I, 0\} = 80.$$

Next, one can show that the value of the investment at period 1, F_1, is positive only for the project value $1.5x$,

$$F_1\left(1.5x\right) = \max\left\{1.5x - I, 0.5\beta F_2\left(2.25x\right)\right\} = 8 > 1.5x - I.$$

Finally, the value of the investment in period 0 is

$$F_0\left(x\right) = \max\left\{x - I, 0.5\beta F_1\left(1.5x\right)\right\} = 0.8 > 0 > x - I.$$

Thus, the investor will wait to invest at period 2 if and only if the value of the project goes up in both periods 1 and 2. This example illustrates the key idea of the real options approach that waiting has positive value.

Now, consider the situation under which the investor is ambiguous about the project value. Suppose he thinks that the value of the project goes up with probability 1/2 or 1/4. He is risk neutral and his preferences are represented by (1). We still use backward induction. Again, in period 2, the investment has positive value only for the project value $2.25x$,

$$V_2(2.25x) = \max\{2.25x - I, 0\} = 80.$$

In period 1, the investor compares the value of immediate investment with that if waiting until period 2. However, the investor values period 2 investment by taking the worst scenario. It can be shown that the value of the investment is positive only when the project value is $1.5x$,

$$V_1(1.5x) = \max\left\{1.5x - I, \beta\frac{1}{4}V_2(2.25x)\right\} = 5 > \beta\frac{1}{4}V_2(2.25x).$$

Finally, by a similar calculation, the period zero value of the investment is

$$V_0(x) = \max\left\{x - I, \beta\frac{1}{4}V_1(1.5x)\right\} = 0.25 > x - I.$$

The above two inequalities state that an uncertainty averse investor chooses optimally to invest in period 1, when the project value goes up. Moreover, he has a lower option value than the investor with expected utility since $V_0(x) < F_0(x)$.

In summary, this example shows that waiting has positive value for both uncertainty averse agent and the expected utility maximizer. However, the option value of the investment opportunity is lower under Knightian uncertainty than in the standard model. Therefore, an uncertainty averse agent invests earlier. We next turn to the formal model.

2.3 Setup

Consider an infinite horizon discrete time optimal stopping problem. As explained in Dixit and Pindyck (1994), the optimal stopping problem can be applied to study an agent's option exercise decision. The agent's choice is binary. In each period, he decides on whether stopping a process and taking a termination payoff, or continuing for one period, and making the same decision in the future.

Formally, uncertainty is generated by a Markov state process $(x_t)_{t\geq 0}$ taking values in $X \subset \mathbb{R}$. The probability kernel of $(x_t)_{t\geq 0}$ is given by $P : X \to \mathcal{M}(X)$, where $\mathcal{M}(X)$ is the space of probability measures on X endowed with the weak convergence topology. Continuation at date t generates a payoff $\pi(x_t)$, while stopping at date t yields a payoff $\Omega(x_t)$, where π and Ω are functions that map X into $\mathbb{R}$. Suppose the agent is risk neutral and discount future payoff flows by $\beta \in (0, 1)$.

In standard models, the agent's preferences are represented by time-additive expected utility. As in the rational expectations paradigm, P can be interpreted as the objective probability law governing the state process $(x_t)_{t\geq 0}$, and is known to the agent. The expectation in the utility function is taken with respect to this law. Alternatively, according to the Savage utility representation theorem, P is a subjective (one-step-ahead) prior and represents the agent's beliefs. By either approach, the standard stopping problem can be described by the following dynamic programming problem:

$$F(x) = \max \left\{ \Omega(x), \pi(x) + \beta \int F(x') P(dx'; x) \right\}, \tag{2}$$

where the value function F can be interpreted as an option value.

To fix ideas, we make the following assumptions. These assumptions are standard in dynamic programming theory (see Stokey and Lucas (1989)).

Assumption 1 $\pi : X \to \mathbb{R}$ *is bounded, continuous, and increasing.*

Assumption 2 $\Omega : X \to \mathbb{R}$ *is bounded, continuous and increasing.*

Assumption 3 *P is increasing and satisfies the Feller property. That is, $\int f(x') P(dx'; x)$ is increasing in x for any increasing function f and is continuous in x for any bounded and continuous function f.*

The following proposition describes the solution to problem (2).

Proposition 1 *Let Assumptions 1-3 hold. Then there exists a unique bounded, continuous and increasing function F solving the dynamic programming problem (2). Moreover, if there is a unique threshold value $x^* \in X$ such that*

$$\pi(x) + \beta \int F(x') P(dx'; x) > (<) \Omega(x), \textit{ for } x < x^*,$$

and

$$\pi(x) + \beta \int F(x') P(dx'; x) < (>) \Omega(x), \textit{ for } x > x^*,$$

then the agent continues (stops) when $x < x^*$ and stops (continues) when $x > x^*$. Finally, x^* is the solution to

$$\pi(x^*) + \beta \int F(x') P(dx'; x^*) = \Omega(x^*). \tag{3}$$

Proof. The proof is similar to that of Theorem 2. So we omit it. ■

This proposition is illustrated in Figure 2. The threshold value x^* partitions the set X into two regions – continuation and stopping regions.[7] The left diagram of Figure 1 illustrates the situation where

$$\pi(x) + \beta \int F(x') P(dx'; x) > \Omega(x), \text{ for } x < x^*,$$

and

$$\pi(x) + \beta \int F(x') P(dx'; x) < \Omega(x), \text{ for } x > x^*.$$

[7]For ease of presentation, we do not give primitive assumptions about the structure of these regions. See Dixit and Pindyck (1994, p. 129) for such an assumption. For some examples below, our assumptions can be easily verified.

In this case, we say that the continuation payoff curve crosses the termination payoff curve from above. Under this condition, the agent exercises the option when the process $(x_t)_{t\geq 0}$ first reaches the point x^* from above. The continuation region is given by $\{x \in X : x < x^*\}$ and the stopping region is given by $\{x \in X : x > x^*\}$. This case describes the upside of the agent's decision such as investment. The downside aspect such as disinvestment or exit is illustrated in the right diagram of Figure 1. The interpretation is similar.

In the above model, a role for Knightian uncertainty is excluded *a priori*, either because the agent has precise information about the probability law as in the rational expectations approach, or because the Savage axioms imply that the agent is indifferent to it. To incorporate Knightian uncertainty and uncertainty aversion, we follow the multiple-priors utility approach (Gilboa and Schmeidler (1989), Epstein and Wang (1994)) and assume that beliefs are too vague to be represented by a single probability measure and represented instead by a set of probability measures. More formally, we model beliefs by a probability kernel correspondence $\mathcal{P} : X \to \mathcal{M}(X)$. Given any $x \in X$, we think of $\mathcal{P}(x)$ as the set of conditional probability measures representing beliefs about next period's state. The multivalued nature of $\mathcal{P}$ reflects uncertainty aversion of preferences.

The stopping problem under Knightian uncertainty can be described by the following dynamic programming problem:

$$V(x) = \max\left\{\Omega(x), \pi(x) + \beta \int V(x')\,\mathcal{P}(dx'; x)\right\}, \tag{4}$$

where we adopt the notation

$$\int f(x')\,\mathcal{P}(dx'; x) \equiv \min_{Q\in\mathcal{P}(x)} \int f(x')\,Q(dx'),$$

for any Borel function $f : X \to \mathbb{R}$. Note that if $\mathcal{P} = \{P\}$, then the model reduces to the standard model (2).

To analyze problem (4), the following assumption is adopted.

Assumption 4 *The probability kernel correspondence $\mathcal{P} : X \to \mathcal{M}(X)$ is nonempty valued, continuous, compact-valued, and convex-valued, and $P(x) \in \mathcal{P}(x)$ for any $x \in X$. Moreover, given any $Q(\cdot; x) \in \mathcal{P}(x)$, $\int f(x') Q(dx'; x)$ is increasing in x for any increasing function $f : X \to \mathbb{R}$.*

This assumption is a generalization of Assumption 3 to correspondence. It ensures that $\int f(x') \mathcal{P}(dx'; x)$ is bounded, continuous, and increasing in x for any bounded, continuous, and increasing function $f : X \to \mathbb{R}$.

The next section supplies the key results of the paper and analyze the intuition behind them.

3 Main Results

The following theorem characterizes the solution to problem (4).

Theorem 2 *Let Assumptions 1-4 hold. Then there is a unique bounded, continuous, and increasing function V solving the dynamic programming problem (4). Moreover, if there exists a unique threshold value $x^{**} \in X$ such that*

$$\pi(x) + \beta \int V(x') \mathcal{P}(dx'; x) > (<) \Omega(x), \text{ for } x > x^{**},$$

and

$$\pi(x) + \beta \int V(x') \mathcal{P}(dx'; x) < (>) \Omega(x), \text{ for } x < x^{**},$$

then the agent stops (continues) when $x < x^{**}$ and continues (stops) when $x > x^{**}$. Finally, x^{**} is the solution to

$$\pi(x^{**}) + \beta \int V(x') \mathcal{P}(dx'; x^{**}) = \Omega(x^{**}). \tag{5}$$

Proof. Let $C(X)$ denote the space of all bounded and continuous functions endowed with the sup norm. $C(X)$ is a Banach space. Define an operator T as follows:

$$Tv(x) = \max\left\{\Omega(x), \pi(x) + \beta \int v(x')\,\mathcal{P}(dx';x)\right\},\ v \in C(X).$$

Then it can be verified that T maps $C(X)$ into itself. Moreover, T satisfies the Blackwell sufficient condition and hence is a contraction mapping. By the Contraction Mapping Theorem, T has a unique fixed point $V \in C(X)$ which solves the problem (4) (see Theorems 3.1 and 3.2 in Stokey and Lucas (1989)).

Next, let $C'(X) \subset C(X)$ be the set of bounded continuous and increasing functions. One can show that T maps any increasing function $C'(X)$ into an increasing function in $C'(X)$. Since $C'(X)$ is a closed subset of $C(X)$, by Corollary 1 in Stokey and Lucas (1989, p.52), the fixed point of T, V, is also increasing. The remaining part of the theorem is trivial and follows from similar intuition illustrated in Figure 2. ■

Remark: The Contraction Mapping Theorem also implies that $\lim_{n\to\infty} T^n v = V$ for any function $v \in C(X)$.

This theorem implies that the agent's option exercise decision under Knightian uncertainty has similar features to that in the standard model described in Proposition 1. It is interesting to compare the option value and option exercise time in these two models.

Theorem 3 *Let assumptions in Proposition 1 and Theorem 2 hold. Then $V \leq F$. Moreover, for both V and F, if the continuation payoff curves cross the termination payoff curves from above then $x^{**} \leq x^{*}$. On the other hand, if the continuation payoff curves cross the termination payoff curves from below, then $x^{**} \geq x^{*}$.*

Proof. Let $v \in C(X)$ satisfy $v \leq F$. Since $P(x) \in \mathcal{P}(x)$,

$$\int v(x')\,\mathcal{P}(dx';x) = \min_{Q\in\mathcal{P}(x)} \int v(x')\,Q(dx') \leq \int v(x')\,P(dx';x) \leq \int F(x')\,P(dx';x).$$

Thus,

$$\begin{aligned} Tv(x) &= \max\left\{\Omega(x), \pi(x) + \beta \int v(x') \mathcal{P}(dx'; x)\right\} \\ &\leq \max\left\{\Omega(x), \pi(x) + \beta \int F(x') P(dx'; x)\right\} \\ &= F(x). \end{aligned}$$

It follows from induction that the fixed point of T, V, must also satisfy $V \leq F$. The remaining part of the theorem follows from this fact and Figure 3. ■

In the standard model, an expected utility maximizer views the world as purely risky. For the upside such as investment decision, waiting has value because the agent can avoid the downside risk, while realizing the upside potential. Similarly, for the downside such as disinvestment or exit, waiting has value because the agent hopes there is some chance that the future may be brighter. Now, if the agent has imprecise knowledge about the likelihoods of the state of the world and hence perceives the future as ambiguous, then waiting will have less value for an uncertainty averse agent because he acts on the worst scenario.

The threshold value under Knightian uncertainty can be either bigger or smaller than that in the standard model, depending on the shapes of continuation payoff and termination payoff curves. More specifically, if the continuation payoff curves cross the termination payoff curves from above, then the threshold value under Knightian uncertainty is smaller than that in the standard model. The opposite conclusion can be obtained if the continuation payoff curves cross the termination payoff curves from below. For both cases, an uncertainty averse agent exercises the option earlier than an agent with expected utility because the former has less option value.

The final theorem concerns comparative statics.

Theorem 4 *Let the assumptions in Theorem 2 hold. Consider two probability kernel correspondences $\mathcal{P}$ and $\mathcal{Q}$. Let the corresponding value functions be $V^{\mathcal{P}}$ and $V^{\mathcal{Q}}$ and* *the corresponding threshold values be $x^{\mathcal{P}}$ and $x^{\mathcal{Q}}$. If $\mathcal{P}(x) \subset \mathcal{Q}(x)$, then $V^{\mathcal{P}} \geq V^{\mathcal{Q}}$. Moreover, if the continuation payoff curves cross the termination payoff curves from above (below), then $x^{\mathcal{P}} \geq (\leq) x^{\mathcal{Q}}$.*

Proof. Define the operator $T^{\mathcal{P}} : C(X) \to C(X)$ by

$$T^{\mathcal{P}} v(x) = \max \left\{ \Omega(x), \pi(x) + \beta \int v(x') \mathcal{P}(dx'; x) \right\}, \ v \in C(X).$$

Similarly, define an operator $T^{\mathcal{Q}} : C(X) \to C(X)$ corresponding to $\mathcal{Q}$. Take any functions $v_1, v_2 \in C(X)$ such that $v_1 \geq v_2$, it can be shown that $T^{\mathcal{P}} v_1(x) \geq T^{\mathcal{Q}} v_2(x)$. By induction, the fixed points $V^{\mathcal{P}}$ and $V^{\mathcal{Q}}$ must satisfy $V^{\mathcal{P}} \geq V^{\mathcal{Q}}$. The remaining part of the theorem follows from Figure 4. ■

It is intuitive that the set of priors $\mathcal{P}(x)$ captures uncertainty aversion and the degree of uncertainty aversion.[8] A larger set of priors means that the agent has more imprecise knowledge and is less confident to assign probabilities to the world. Hence, he is more uncertainty averse. This theorem says that the option value is lower if the agent is more uncertainty averse, and hence the agent exercises the option earlier.

4 Applications

This section applies our results to several widely studied examples, which include real investment, job search, firm exit, firm default, and youth suicide.

4.1 Real Investment

A classic application of real-options approach is irreversible investment decisions.[9] The standard real-options approach makes the analogy of corporate investment to

[8]See Gilboa and Schmeidler (1989) and Epstein (1999).

[9]See Bernanke (1983), Brennan and Schwartz (1985), McDonald and Siegel (1985, 1986), Titman (1986), Williams (1991), and Grenadier (1996) for important contributions. See Dixit and Pindyck (1994) for a textbook treatment.

the exercising of an American call option on the underlying project. Once that analogy is made, then we may apply the methodology developed in the financial options literature to corporate investment problems. Formally, we follow McDonald and Siegel (1986) and cast the investment problem into our framework by setting

$$\Omega(x) = x - I, \ \pi(x) = 0,$$

where x represents the value of the project (for example, the real estate property constructed on the land), and I represents the known and fixed investment cost to develop the project (for example, construction cost).

The standard real-options model predicts that there is an option value of waiting, because investment is irreversible and flexibility in timing has value. Empirical evidence supports the option value of waiting in corporate investment. For example, Summers (1987) finds that hurdle rates for investment range from 8 to 30 percent, with a median of 15 percent and a mean of 17 percent, much higher than the commonly used risk-adjustment cost of capital.

Another main prediction of the standard real investment model is that an increase in risk raises the option value and the investment threshold (see Pindyck and Dixit (1994)). This derives from the fundamental insight behind the option pricing theory, in that firms may capture the upside gains and minimizes the downside loss by waiting for the risk of project value to be partially resolved.

While the standard real-options model predicts a monotonic relationship between the investment threshold and the volatility of project value, our model makes an important distinction between risk and uncertainty, and argues that risk (which can be described by a single probability measure) and uncertainty (multiplicity of priors) have different effects on investment timing. Specifically, our model predicts that Knightian uncertainty lowers option value of waiting (see Figure 5). This implies that an uncertainty averse investor invests earlier than the standard expected utility maximizer. The intuition is that the decision maker's aversion to ambiguity leads him to be more "pessimistic" about the expected future value of project by waiting. Therefore, ambiguity aversion (Knightian uncertainty) has the opposite effect on investment timing decision than risk does. The effect of ambiguity aversion thus partially offsets the effect of risk on investment threshold.

4.2 Job Search

In job search models, a worker's decision can be described by a stopping problem (See Ljungqvist and Sargent (2000)). As an illustration, we present McCall's (1970) model. In each period, the worker has a job offer with wage $x \in [0, B] \subset \mathbb{R}$ drawn from a fixed distribution. The worker has the option of rejecting the offer, in which case he receives unemployment compensation c this period and waits until next period to draw another offer. Alternatively, the worker can accept the offer to work at x, in which case he receives a wage of x per period forever. Neither quitting nor firing is permitted. This model can be cast into our framework by setting

$$\Omega(x) = \frac{x}{1-\beta},\ \pi(x) = c.$$

A major prediction of McCall's model is that an increase in risk in the sense of mean-preserving spread raises the reservation wage.[10] This means that a worker tends to wait to get a higher wage offer in riskier situations. Again the intuition comes from the option pricing theory that the value of an option is increasing in risk. However, this model cannot explain the phenomenon that workers tend to accept low wages in economic downturns where risk seems to be higher. Our theory offers an explanation. In economic downturns, a worker is more ambiguous about job offers. He is less sure about the likelihood of getting a job and obtaining favorable wages. Given that the worker is uncertainty averse, our theory predicts that the value of the option is lower, and hence the reservation wage is smaller in recessions than in expansions (see Figure 6). This also implies that the worker tends to accept job offers earlier in recessions.

[10] See Ljungqvist and Sargent (2000, p.92) for a proof.

4.3 Firm Exit

For the firm exit problem, the process $(x_t)_{t\geq 0}$ could be interpreted as a demand shock or a productivity shock. Let the outside opportunity value be a constant $\gamma \geq 0$. Stay in business generates profits $\Pi(x) > 0$ and incurs a fixed cost $c_f > 0$. Then the problem fits into our framework by setting[11]

$$\Omega(x) = \gamma,\ \pi(x) = \Pi(x) - c_f.$$

According to the standard real options approach, the exit trigger point is lower than that predicted by the textbook Marshallian net present value principle. This implies that firms stay in business for lengthy periods while absorbing operating losses. Only when the upside potential is bad enough, will the firm not absorb losses and abandon operation. The standard real options approach also predicts that an increase in risk in the sense of mean preserving spread raises the option value, and hence lowers the exit trigger. This implies that firms should stay in business longer in riskier situations, even though they suffer substantial losses. However, this prediction seems to be inconsistent with the large amount of quick exit in IT industry in recent years.

Our theory provides an explanation. In recent years, due to economic recessions, firms are more ambiguous about the industry demand and their productivity. They are less sure about the likelihoods of when the economy will recover. Intuitively, the set of probability measures that workers may conceive is larger in recessions. Thus, by Theorem 4, the value of the firm is lower and the exit trigger is higher. This induces firms to exit earlier (see Figure 7).

[11]See Dixit (1989) for a continuous time model of entry and exit and Hopenhayn (1992) for a dicrete time industry equilibrium model of entry and exit.

4.4 Firm Default

Let $(x_t)_{t\geq 0}$ be the cash flow process of the firm. Suppose the firm issues corporate bond. The bond contract specifies a perpetual flow of coupon payments b to bondholders. The remaining cash flows from operation accrue to shareholders. If the firm defaults on its debt obligations, it is immediately liquidated. Upon default, bondholders get the liquidation value and shareholders get nothing. Shareholders' decision is to choose a default time to maximize equity value. Default is determined by bond covenants. If debt has positive net-worth covenants, then default occurs when the firm's net worth becomes negative. Alternatively, if debt has no protective covenants, then default occurs only when the firm cannot meet the required coupon payment by issuing additional equity: that is, when equity value falls to zero. Formally, the default problem under unprotected debt covenants is similar to the exit problem (see Figure 7) and can be cast into our framework by setting

$$\Omega(x) = 0, \ \pi(x) = x - b.$$

According to the standard real options approach, the firm continues to meet coupon payments even though it suffers losses.[12] When the cash flow is low enough, the firm defaults on its debt. Viewing equity value as an option, the standard model also predicts that equity value increases with risk (volatility) of cash flows, and hence the default boundary decreases with it. This implies that we should see less default in recessions. However, as documented in Moody (2000), default rates are countercyclical.

According to our theory, investors and firms are more ambiguous about the economic prospects in recessions than in expansions. Thus, equity value is lower and default boundary is higher in recessions than in expansions. Our theory immediately delivers countercyclical default rates.

[12]See Leland (1994) for a single firm model and Miao (2003) for an industry equilibrium model. See Huang and Huang (2003) for evidence.

The standard real options approach to corporate finance emphasizes the importance of credit risk in determining default rate, credit spread, and capital structure. Our theory opens a new door for thinking about the determinant of these variables. In particular, we argue that Knightian uncertainty is an important determinant.

4.5 Youth Suicide

There are two theories about economics of suicide. One is the Beckerian view, proposed by Hamermesh and Soss (1974). They argue that an individual will commit suicide when the expected life-time utility falls short of the benchmark utility level that he receives from ending his life. However, this Beckerian view of the world obviously ignores the option value of being alive. For example, the very poor might get lucky and wins a big lottery. Dixit and Pindyck (1994) argue that Hamermesh-Soss theory provide the individual agent with not enough rationality. Suicide (if successful) is a completely irreversible decision and forgoes all the possible future up-side scenarios. Naturally, option theory based argument suggests that the agent shall not terminate his life simply based on the present value calculations as in Hamermesh and Soss (1974). Conventional wisdom seems capture the option value of being alive very well. For example, a famous Chinese saying (from ancient days) states that "Dying with dignity is worse than surviving with misery."

Undoubtedly, option value of being alive is crucial in the economics of suicide. However, a standard real option theory also confronts empirical difficulties. Culter et al. (2001) find that suicide rates are quite flat across individuals in all age groups.[13] The almost constant suicide rate across different age groups poses a strong challenge to the standard real-option explanation of suicide, because the young has a higher expected life-time earnings than the old. Moreover, it is even harder to explain why

[13] For example, the suicide rates (in 1990) are 15 out of 100,000 for the age groups 20-24, 25-34, 35-44, and 45-54. The suicide rate is 21 out of 100,000, for age group 65+, slightly higher, most likely due to health reasons and loneliness.

college students attempt and commit suicides. After all, college students are young and have high expected future earnings opportunities.

Our theory provides an explanation to the observed high suicide rate among young people, including college students. We argue that those who commit suicide are often overly concerned about the prospects of their lives. They are not confident about their future earnings prospect and the happiness of their future lives. In our language, there is much greater perceived (Knightian) uncertainty in their minds. Thus, these young individuals perceive very low option value of continuing to be alive, which possibly motivates them to commit suicide.

We now formally map our theory to the suicide example. In order to capture the differences in life expectancy between the young and the old in a simplest possible setting, we adopt the perpetual-youth model of Yaari (1965) and Blanchard (1985). Let p_y and p_o denote the survival probabilities per period for the young and the old, respectively. Obviously, by definition, we have $p_y > p_o$. Then, the agent decides whether to terminate his life by solving the following exit problem:

$$V_i(x) = \max \left\{ \gamma,\, \pi(x) + \beta p_i \int V_i(x') \mathcal{P}_i(dx'; x) \right\}, \tag{6}$$

where $\pi(x)$ represents current earnings of being alive and γ is the level of utility attained by ending one's life, and where $i = y$ and $i = o$ stand for the "young" and the "old", respectively. We argue that the young agent who commits suicide has a more pessimistic view of life. Formally, our model states that the set of priors $\mathcal{P}_y$ for the young who commit suicide is larger than the set of priors $\mathcal{P}_o$ for the old, and contains the pessimistic view of the world. Although the young has a higher discount factor than the old ($p_y > p_o$), the value function V_y for the young may still be lower than the value function V_o for the old, due to $\mathcal{P}_o \subset \mathcal{P}_y$ (See Theorem 4).

Our model has interesting policy implications. Based on our theory, offering counselling services to college students helps them lower their perceived level of Knightian uncertainty, and thus reduces the set of priors. As a result, they will value their lives less pessimistically.

5 Conclusion

Uncertainty plays an important role in real options problems. In standard models, there is no meaningful distinction between risk and uncertainty. To afford this distinction, we have applied the multiple-priors utility model to analyze an option exercise problem. We have shown that the option value under Knightian uncertainty is smaller than that in the standard model. Moreover, an uncertainty averse agent exercises the option earlier than the standard expected utility maximizer. We have also shown that if the agent is more uncertainty averse or faces a higher level of Knightian uncertainty, then he exercises the option earlier. Several examples have revealed that our model can explain some phenomena that are inconsistent with the predictions of the standard model. Our model has a number of empirical implications. Further empirical test should be urgent in the research agenda.

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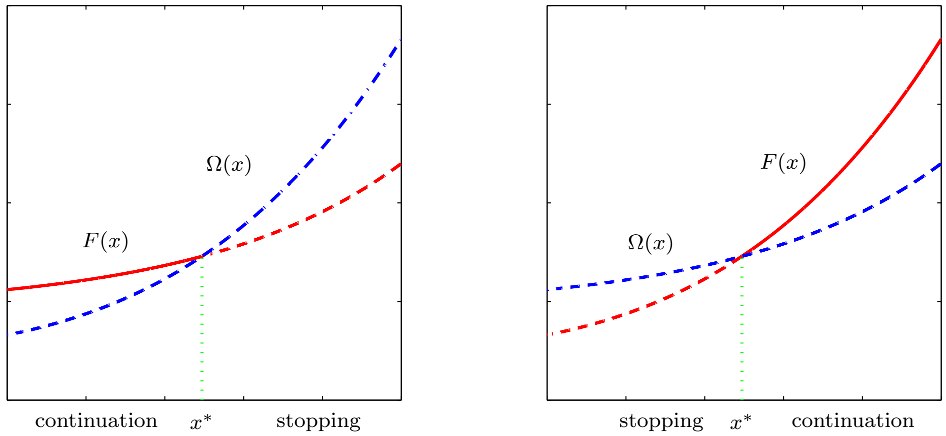

Figure 2: The value function and exercising threshold in the standard model.

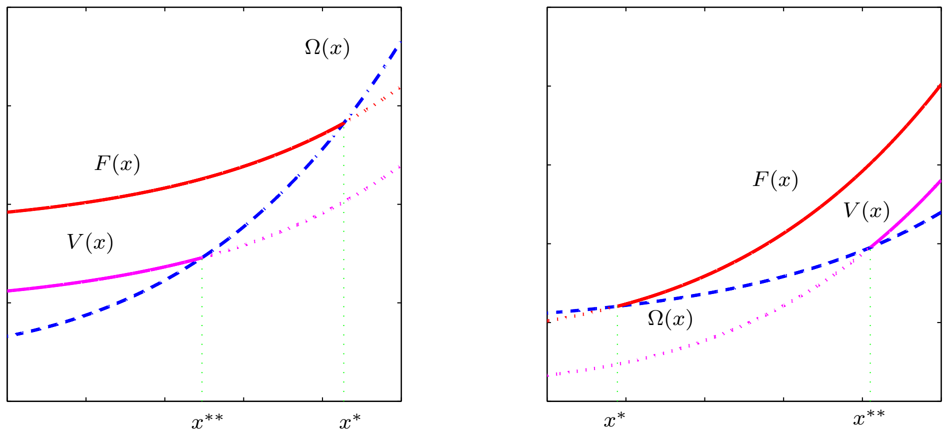

Figure 3: Comparison of the standard model and the model under Knightian uncertainty.

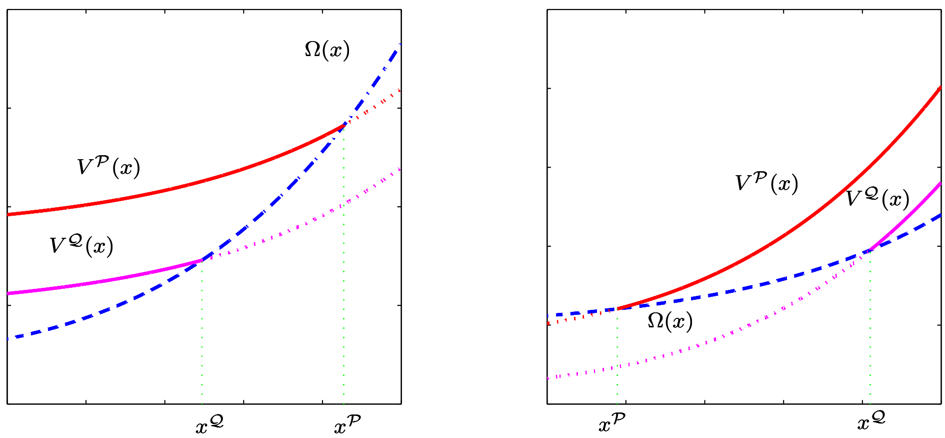

Figure 4: Option value and exercising thresholds under Knightian uncertainty for two different sets of priors $\mathcal{P} \subset \mathcal{Q}$.

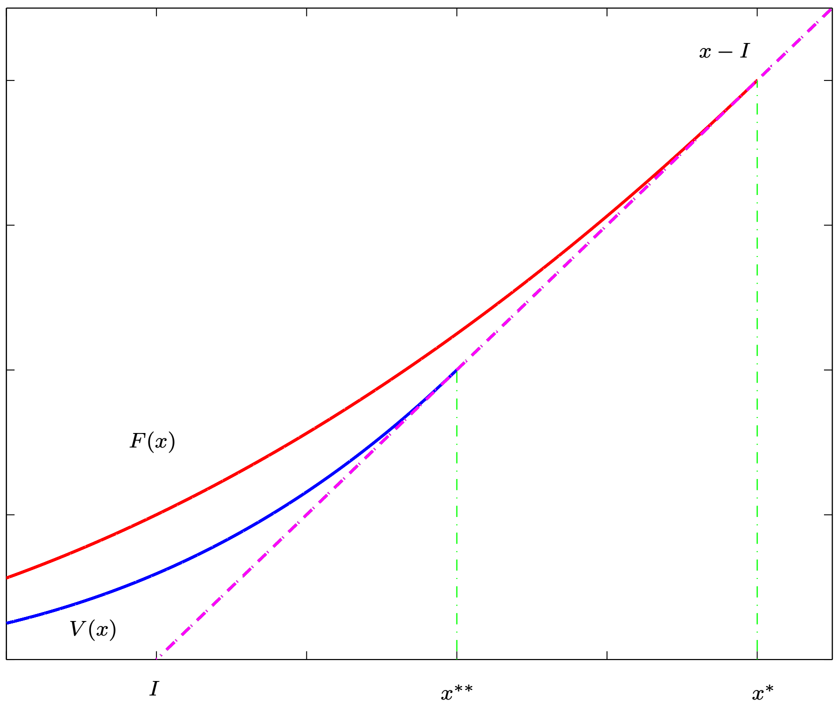

Figure 5: Investment timing under Knightian uncertainty and in the standard model. The upper (dashed) curve corresponds to the value function $F(x)$ in the standard model. The lower (solid) curve corresponds to the value function $V(x)$ under Knightian uncertainty.

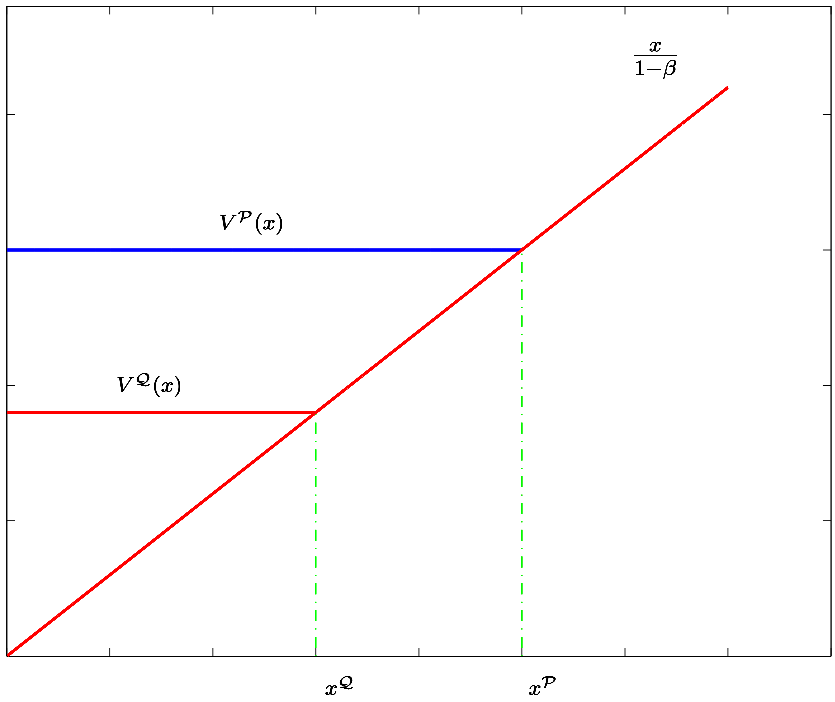

Figure 6: Job search under different degrees of Knightian uncertainty. The upper (solid) curve corresponds to the value function $V^{\mathcal{P}}(x)$ and the lower (solid) curve corresponds to the value function $V^{\mathcal{Q}}(x)$ where $\mathcal{P} \subset \mathcal{Q}$.

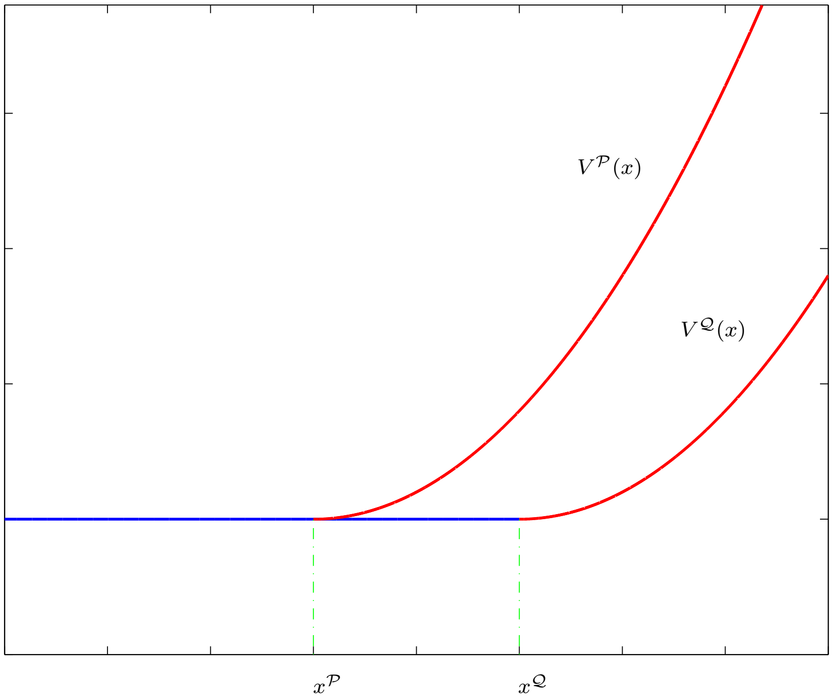

Figure 7: Firm Exit/Default under different degrees of Knightian uncertainty. The upper (dashed) curve corresponds to the value function $V^{\mathcal{P}}(x)$ and the lower (solid) curve corresponds to the value function $V^{\mathcal{Q}}(x)$ where $\mathcal{P} \subset \mathcal{Q}$.