BEAUTY IS A BEAST, FROG IS A PRINCE: ASSORTATIVE MATCHING WITH NONTRANSFERABILITIES*

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Abstract

We present sufficient conditions for monotone matching in environments where utility is not fully transferable between partners. These conditions involve complementarity in types not only of the total payoff to a match, as in the transferable utility case, but also in the degree of transferability between partners. We apply our conditions to study some models of risk sharing and incentive problems, deriving new results for predicted matching patterns in those contexts.

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1. Introduction

For the economist analyzing household behavior, firm formation, or the labor market, the characteristics of matched partners are paramount. The educational background of men and women who are married, the financial positions of firms that are merging, or the productivities of agents who are working together, all matter for understanding their respective markets. Matching patterns serve as direct evidence for theory, figure in the econometrics of selection effects, facilitate theoretical analysis, and are even treated as policy variables.

Much is known about characterizing matching in the special case of transferable utility (TU). For instance, if the function representing the total payoff to the match satisfies increasing (decreasing) differences in the partners' attributes, then there will always be positive (negative) assortative matching, whatever the distribution of types. Because they are distribution-free, results of this sort are very powerful and easy to apply.

But in many areas of economic analysis, the utility among individuals is not fully transferable ("non-transferable," or NTU, in the parlance). Partners may be risk averse with limited insurance possibilities; incentive or enforcement problems may restrict the way in which the joint output can be divided; or policy makers may impose rules about how output may be shared within relationships. As Becker (1973) pointed out long ago, rigidities that prevent partners from costlessly dividing the gains from a match may change the matching outcome, even if the level of output continues to satisfy monotone differences in type.

While interest in the issues represented by the non-transferable case is both long-standing and lively (see for instance Farrell-Scotchmer, 1988 on production in partnerships; Rosenzweig-Stark, 1989 on risk sharing in households; and more recently, Lazear, 2000 on incentive schemes for workers; Ackerberg-Botticini, 2002 on sharecropping; and Chiappori-Salanié, 2003 on the empirics of contracts), for the analyst seeking to characterize the equilibrium matching pattern in such settings, there is little theoretical guidance.

The purpose of this paper is to offer some. We present sufficient and necessary conditions for assortative matching that are simple to express, intuitive to understand, and, we hope, tractable to apply. We illustrate their use with some examples that are of independent interest.

The class of models we consider are two-person matching games without search frictions in which the utility possibility frontier for any pair of agents is a strictly decreasing function (the framework also extends to the limiting case in which the weak Pareto frontier for a matched pair is a point). After introducing the model and providing formal definitions of the monotone matching patterns, we review the logic of the classical transferable utility results, a close of examination of which leads us to propose the "generalized difference conditions" (GDC) that suffice to guarantee monotone matching for any type distribution (Proposition 1). We then apply them to a simple model of risk sharing within households.

As it is often easier to verify properties of functions locally than globally, we also present necessary and sufficient differential conditions for monotone matching (Proposition 4). The differential conditions offer additional insight into the forces governing matching. In particular, they highlight the role not only of the complementarity in partners' types that figures in the TU case, but also of a *complementarity between an agent's type and his partner's payoff* that is the new feature in the NTU case. This second complementarity entails that the degree of transferability (i.e., slope of the frontier) be monotone in type. Even if the output satisfies increasing differences in types, failure of the type-payoff complementarity may overturn the predictions of the TU model and lead instead to negative assortative matching or some more complex and/or distribution-dependent pattern.

We use the differential conditions to study a model in which principals with different monitoring technologies are matched to agents with different wealths, one interpretation of which may address some puzzling results concerning the assignment of peasants to crop types in the empirical sharecropping literature. In the example, the type-transferability relationship is responsible for the predicted matching pattern, which goes in a (possibly) unexpected direction.

We then go on to discuss other techniques that facilitate application of the generalized difference conditions. For instance, the truth of the GDC depends only on the ordinal properties of preferences (Proposition 1); this fact broadens the scope of applicability of the local conditions (Corollary 1). We also consider the relationship between the GDC and more familiar difference conditions, including lattice theoretic notions (Propositions 5 and 6), and devote some attention to necessary conditions for monotone matching (Proposition 2).

The next section delves further into the ideas underlying the general theoretical analysis by examining a very simple example. It then introduces the two models that will be used to illustrate the application of our results.

2. Issues and Examples

How do nontransferabilities affect the matching pattern? Consider the following example, which is inspired by the one in Becker (1973).

Example 1. *Suppose there are two types of men, $l < h$, and two types of women, $L < H$. The total "output" they produce when matched, as a function of the partners' types, is described by the matrix*

$$\begin{array}{ccc} & L & H \\ l & 4 & 7 \\ h & 7 & 9 \end{array} .$$

Since $9 - 7 < 7 - 4$, *the joint payoff function satisfies* decreasing differences (DD). *If payoffs are fully transferable, then it is well known that decreasing differences implies that a stable outcome will always involve negative assortative matching (NAM): high types will match with low types. If to the contrary we had a positive match of the form* $\langle l, L\rangle$, $\langle h, H\rangle$ *with equilibrium payoffs* (u_l, u_L) *and* (u_h, u_H), *then there would always be a pair of types that could do strictly better for themselves:* $u_l + u_h = (4 - u_L) + (9 - u_H) < (7 - u_H) + (7 - u_L)$; *thus* $u_l < 7 - u_H$ *or* $u_h < 7 - u_L$; *l could offer H (or h could offer*

L) slightly more than her current payoff and still get more for himself, destabilizing the positive match. The negative matching outcome maximizes total output.

Suppose instead that utility is not perfectly transferable, and consider the extreme case in which any departure from equal sharing within the marriage is impossible. For instance, the payoff to the marriage could be generated by the joint consumption of a local public good. Thus each partner in $\langle h, H\rangle$ *gets 4.5, each in* $\langle h, L\rangle$ *gets 3.5, etc. Now the only stable outcome is positive assortative matching (PAM): each* h *is better off matching with* H *(4.5) than with* L *(3.5), and thus the "power couple" blocks a negative assortative match. As Becker noted, with nontransferability, the match changes, and aggregate performance suffers as well.*

Of course this extreme form of nontransferability is not representative of most situations of economic interest, and we wonder what happens in the intermediate cases.

Example 2. *Modify the previous example by introducing a dose of transferability: some compensation, say through the return of favors, makes it possible to depart from equal sharing. Consider two simple cases. In the first, the high types are "difficult," while the low types are "easy,": beauty is a beast, frog is a prince. That is, utility is perfectly transferable between* l *and* L, l *can transfer to* H, *but not vice versa, and* L *can transfer to* h *but not vice versa. In the second case, the high types are easy and the low types difficult. See Figure 2.1, which depicts the utility possibility frontiers between pairs of types, assuming feasible transfers are made starting from the equal sharing point.*

In the first case, the degree of transferability is decreasing *in type, and in particular is changing in the same direction as (marginal) productivity. The unique outcome is NAM in this case: if things were otherwise, a high type could promise a low type almost 2.5, garnering a bit over 4.5 for itself, and the low type will be happy to accept the offer (the only way this could not happen is if both* l *and* L *were getting at least 2.5, which is an impossibility).*

In the second case, the degree of transferability is increasing *in type, opposite the direction that productivity increases, and this opposition between productivity and*

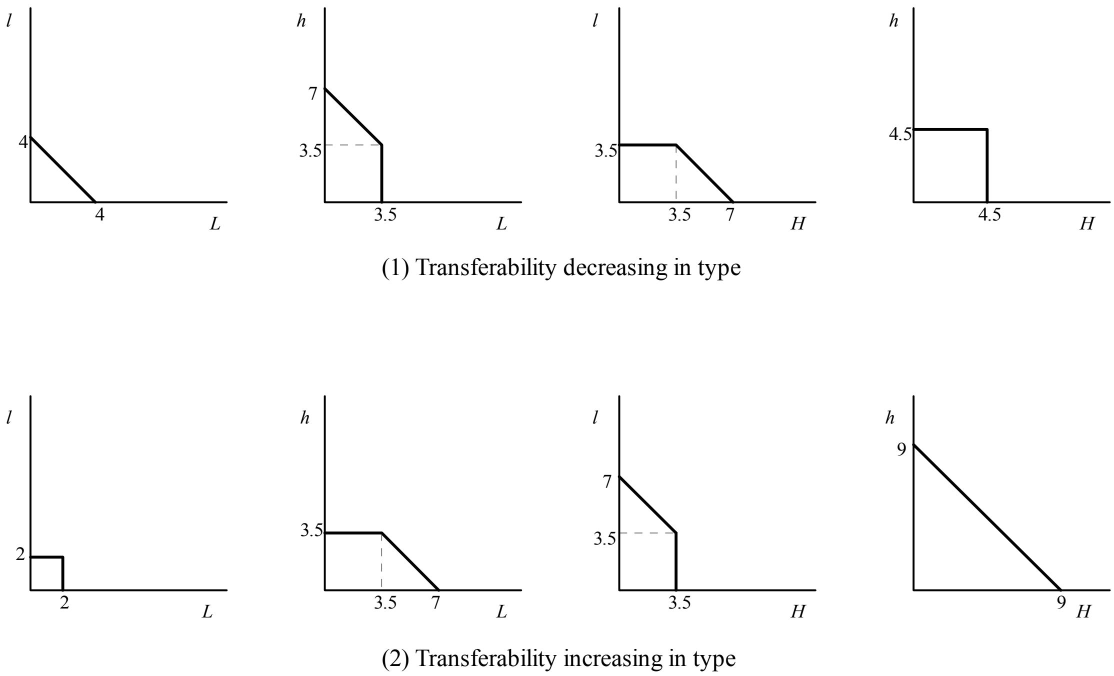

Figure 2.1: Utility possibility frontiers for Example 2.

transferability is enough to overturn the TU outcome. The easygoing high types now can get no more than 3.5 out of a mixed relationship, so they prefer a match with each other, wherein 4.5 would be available to each.[1]

The basic intuitions contained in this second example carry over to the general case, and are in a nutshell the content of our main results, Propositions 1 and 4.[2] Transferability, and its dependence on type, can be as important as productivity in determining the nature of sorting.

In the remainder of this section we present two less-contrived examples that are representative of those considered in the literature. The first is a marriage market model

[1] This begs the question of how much transferability is needed for NAM. We leave it to the reader to verify that if low types can transfer at a rate β, then there is NAM when $\beta > \beta^*$ with $\beta^* \in (0,1)$ and there is PAM when $\beta < \beta^*$.

[2] The only difference is that we shall require the frontiers to be strictly decreasing; the above examples could easily be modified to conform to this requirement without changing any conclusion.

in which partners vary in initial wealth (and risk attitude) and must share risks within their households. Although this topic has attracted considerable attention in the development literature and economics of the family, we are not aware of any attempts to establish formally what the pattern of matching among agents with differing risk attitudes would be, something which is obviously important for empirical identification, say of risk-sharing versus income-generation motives for marriage and migration (Rosenzweig-Stark, 1989).

The second is a principal-agent model in which agents vary in their initial wealth (and therefore risk aversion), and principals vary in the riskiness of their projects. Sorting effects in this sort of model are of direct interest in some applications (e.g., Newman, 1999; Prendergast, 2002) and are important considerations in the econometrics of contracting (Ackerberg-Botticini, 2002).

Example 3. *(Risk sharing in households). Consider a stylized marriage market model in which the primary desideratum in choosing a mate is suitability for risk sharing. There are two sides to the market for households, and we denote by p the characteristics of the men and by a the characteristics of the women. Suppose that household production is random, with two possible outcomes $w_i > 0$, $i = 1, 2$, and associated probabilities π_i. Everyone is an expected utility maximizer; income y yields utility $U(p, y)$ to a man of type p and $V(a, y)$ to a woman of type a. Unmatched agents get utility zero. We assume that U and V are twice differentiable, strictly increasing and strictly concave in income for all p and a. The characteristics p and a are interpreted as the indices of absolute of risk tolerance: if $p > p'$, then $-U_{22}(p, y)/U_2(p, y) < -U_{22}(p', y)/U_2(p', y)$ for all y, and $a > a'$ implies $-V_{22}(a, y)/V_2(a, y) < -V_{22}(a', y)/V_2(a', y)$.*
For informational or enforcement reasons, the only risk sharing possibilities in this economy lie within a household consisting of two agents. When partners match, their (explicit or implicit) contract specifies how each realization of the output will be shared between them.
Consider a household (p, a); the maximum expected utility the man can achieve if the woman requires a level of expected utility v to participate is given by the value function

ϕ for the optimal risk sharing problem:

$$\phi(p,a,v) \equiv \max_{\{s_i\}_{i=1,2}} \Sigma_i \pi_i U(p, w_i - s_i) \text{ s.t. } \Sigma_i \pi_i V(a, s_i) \geq v. \tag{2.1}$$

Since ϕ is generally not linear in v, utility is only imperfectly transferable: the cost to p of transferring a small amount to a depends on how much each partner already has. The same is true of the following example.

Example 4. *(Matching principals and agents). Principals' projects have a common expected return but differ in their risk characteristics; they must match with agents, who differ in initial wealth. Agents have declining absolute risk aversion, and the question is whether the safest projects are tended by the most or the least risk averse, i.e. the poorest or wealthiest agents.*
Risk-neutral principals have type indexed by $p \in (0,1]$, and agents have type index $a > 1$. Agents' unobservable effort e can either be 1 or 0. The principal's type indexes the success yield and probability of his project: it yields R/p with probability p and 0 with probability $1-p$ provided his agent exerts $e = 1$; it yields 0 with probability 1 if $e = 0$. Thus, conditional on the agent exerting effort, all tasks yield the same expected return R, but higher p implies lower risk. An agent of type a has utility $V(a+y)$ from income y; their type represents initial wealth, and we assume that $V(\cdot)$ displays increasing absolute risk tolerance.

The frontier for a principal of type p who is matched to an agent of type a is given by

$$\begin{gathered} \phi(p,a,v) = \max \ R - ps_1 - (1-p)s_0 \\ \text{s.t. } pV(a+s_1) + (1-p)V(a+s_0) - 1 \geq V(a+s_0), \\ pV(a+s_1) + (1-p)V(a+s_0) - 1 \geq v \end{gathered}$$

where s_1 and s_0 are the wages paid in case of success and failure respectively. The first inequality is the incentive compatibility condition that ensures the agent takes high effort.

Intuition might suggest that since wealthier agents are less risk averse, they should be matched to riskier tasks while the more risk averse agents should accept the safer tasks (i.e. there should be NAM in (p, a)). As we shall see in Section 6.2, this intuition is incomplete, and indeed misleading, and the complete analysis can offer an explanation for some seemingly puzzling results in the empirical literature.

Examples 3 and 4 are nontransferable utility models in which the frontiers, though downward sloping, do not have constant unit slope. As we have suggested, the traditional techniques for determining matching patterns do not apply in these cases. We shall revisit these examples as we illustrate the application of our general results.

3. Theoretical Preliminaries

The economy is populated by a continuum of agents who differ in type, which is taken to be a real-valued attribute such as skill, wealth, or risk attitude. For expositional ease we focus on *two-sided* models, that is when agents are also distinguished by a binary "gender" (man-woman, firm-worker, etc.), but as we show in Section 7.3 the results apply equally well to "one-sided models". Payoffs exceeding that obtained in autarchy, which for the general analysis we normalize to zero for all types,[3] are generated only if agents of opposite gender match.

Characteristics of agents are drawn from compact real intervals P and A with Lebesgue measure. We assume that for any economy under consideration, the support of the distribution of types is finite.

The object of analytical interest to us is the utility possibility frontier (since in equilibrium agents will always select an allocation on this frontier) for each possible pairing of agents. This frontier will be represented by a bounded continuous function $\phi : P \times A \times \mathbb{R}_+ \to \mathbb{R}_+; \phi(p, a, v)$ denotes the maximum utility generated by a type $p \in P$ on side 1 in a match with a type $a \in A$ on side 2 who receives utility v. The maximum

[3] In many applications, the autarchy payoff varies with type. For instance, in the principal-agent example it is natural to assume that an unmatched agent a gets $\ln a$. The analysis extends to this case almost without modification: see Section 7.1.

equilibrium payoff that p could ever get in a match with a is $\phi(p, a, 0)$, since a would never accept a negative payoff.

Let $\psi(a, p, \cdot)$ be the "inverse" of $\phi(p, a, \cdot)$: for any $p \in P, a \in A, v \in \mathbb{R}_+, \phi(p, a, \psi(a, p, u)) = u$ for $u \leq \phi(p, a, 0)$ and 0 for $u > \phi(p, a, 0)$; $\psi(a, p, u)$ is the maximum expected utility of a side 2 agent with characteristic a when his partner on side 1 has type p and requires a level of utility u to participate. Notice that if $u > \phi(p, a, 0)$, we define $\psi(a, p, u) = 0$.

We take ϕ to be a primitive of the model for the general analysis, although as in the examples we have presented, it will typically be derived from more fundamental assumptions about technology, preferences and choices made by the partners after they match. We shall sometimes refer to the first argument of ϕ and ψ as "*own type,*" the second argument as "*partner's type*," and the third argument as "*payoff.*"

We assume throughout that $\phi(p, a, v)$ is continuous and strictly decreasing in v when $v \in [0, \phi(p, a, 0)]$. In general, of course, $\phi(p, a, w) \neq \psi(a, p, w)$.

The notation reflects two further assumptions of matching models, namely (1) that the payoff possibilities depend only on the types of the agents and not on their individual identities; and (2) the utility possibilities of the pair of agents do not depend on what other agents in the economy are doing, i.e., there are no externalities across coalitions.

The model encompasses the case of transferable utility (TU), in which there exists a production function $h(a, p)$ such that the frontier can be written

$$\phi(p, a, v) = \begin{cases} h(p, a) - v & \text{if } v \in [0, h(p, a)] \\ 0 & \text{if } v > h(p, a). \end{cases}$$

In all other cases we have nontransferable utility (NTU).

An extreme instance of NTU, which strictly speaking lies outside our framework, is when each agent has a fixed utility from a given match (see Roth and Sotomayor, 1990 for a survey of the literature on this case): if coalition (p, a) forms, the type p agent obtains a utility of $f(p, a)$ and the a agent obtains a utility of $g(a, p)$. In terms of our notation, the frontier in this "strict NTU" case is[4]

$$\phi(p,a,v) = \begin{cases} f(p,a) & \text{if } v \leq g(a,p) \\ 0 & \text{if } v > g(a,p). \end{cases}$$

We discuss this case in Section 7.2.

3.1. Equilibrium

An equilibrium specifies both the way types are matched to each other and the utility payoff to each type. The matching correspondence is $\mathfrak{M} : P \rightrightarrows A$; the payoff allocation is a pair of functions (u,v), with $u : P \to \mathbb{R}$ and $v : A \to \mathbb{R}$ specifying the equilibrium utility achieved by each type. The important property of equilibrium for our analysis is *stability*: $\mathfrak{M}$ is stable given (u,v) if there do not exist (p,a) and $v > v(a)$ such that $\phi(p,a,v) > u(p)$. See the Appendix for precise definitions and discussion of existence of equilibrium.

3.2. Descriptions of Equilibrium Matching Patterns

When $\mathfrak{M}$ is a monotone correspondence, matching is *monotone*. We consider only a few types of monotone matching patterns in this paper. Matching satisfies *positive assortative matching* (PAM) if for all p, p' on side 1 with $p > p', a \in \mathfrak{M}(p), a' \in \mathfrak{M}(p') \Longrightarrow a \geq a'$. Matching satisfies *negative assortative matching* (NAM) if for all p, p' on side 1, $p > p'$, $a \in \mathfrak{M}(p), a' \in \mathfrak{M}(p') \Longrightarrow a' \geq a$.

Say that an equilibrium is *payoff equivalent* to another if almost all agents have the same payoff. We say an economy satisfies PAM if each equilibrium is payoff equivalent to one in which the match satisfies PAM.

[4]Note that $\phi(p,a,v)$ is not strictly decreasing on $v \in [0, g(a,p)]$. This "strict NTU" case can be viewed as the limit of environments in which each agent can transfer at a rate $\beta > 0$, that is when the frontier is defined by

$$\phi(p,a,v) = \begin{cases} f(p,a) + \beta g(a,p) - \beta v & \text{if } v \leq g(a,p) \\ \frac{1}{\beta}\left(g(a,p) + \beta f(p,a) - v\right) & \text{if } v \geq g(a,p), \end{cases}$$

for these environments the frontier is strictly decreasing on $[0, g(a,p)]$.

4. Sufficient Conditions for Monotone Matching

4.1. Logic of the TU Case

Before proceeding, let's recall the nature of the conventional transferable utility result and why it is true, as that will provide us with guidance to the general case. In the TU case, only the total payoff $h(p, a)$ is relevant. The assumption that is often made about h is that it satisfies *increasing differences* (ID): whenever $p > p'$ and $a > a'$, $h(a, p) - h(a', p) \geq h(a, p') - h(a', p')$. Why does this imply positive assortative matching (segregation in the one-sided case), irrespective of the distribution of types? Usually, the argument is made by noticing that the total output among the four types is maximized (a necessary condition of equilibrium in the TU case, but not, we should emphasize, in the case of NTU) when p matches with a and p' with a': this is evident from rearranging the ID condition.

However, it is more instructive to analyze this from the equilibrium point of view. Suppose that a and a' compete for the right to match with p rather than p'. The increasing difference condition says that a can outbid a' in this competition, since the incremental output produced if a switches to p exceeds that when a' switches. In particular, this is true *whatever* the level of utility u that p' might be receiving: rewrite ID as $h(a, p) - [h(a, p') - u] \geq h(a', p) - [h(a', p') - u]$; this is literally the statement that a's willingness to pay for p, given that p' is getting u, exceeds a''s. Thus a situation in which p is matched with a' and p' with a is never stable: a will be happy to offer more to p than the latter is getting with a'. The ID result is *distribution free*: the type distribution will affect the equilibrium payoffs, but the argument just given shows that p's partner must be larger than p''s regardless of what those payoffs might be.

The convenient feature of TU is that if a outbids a' at one level of v, he does so for all v. Such is not the case with NTU. Our sufficient condition will require explicitly that a can outbid a' for *all* levels of u. If this requirement seems strong, recall that the nature of the result sought, namely monotone matching regardless of the distribution, is also strong. By the same token, it is weaker than ID, and includes TU as a special case.

In an NTU model, the division of the surplus between the partners cannot be separated from the level that they generate. Switching to a higher type partner may not be attractive if it is also more costly to transfer utility to a high type, that is, if the frontier is steeper. A sufficient condition for PAM is that not only is there the usual type-type complementarity in the production of surplus, but also there is a type-payoff complementarity: frontiers are flatter, as well as higher, for higher types. This will perhaps be more apparent from the local form of our conditions.

4.2. Generalized Difference Conditions

Let $p > p'$ and $a > a'$ and suppose that p' were to get u. Then the above reasoning would suggest that p would be able to outbid p' for a if

$$\phi(p, a, \psi(a, p', u)) \geq \phi(p', a', \psi(a', p', u)). \tag{4.1}$$

The left-hand side is a's willingness to "pay" (in utility terms) for p rather than p', given that p' receives u (a then receives $v = \psi(a, p', u)$, so p would get $\phi(a, p, v)$ if matched with a). The right-hand side is the counterpart expression for p'. Thus the condition says in effect that a can outbid a' in an attempt to match with p instead of p'.[5]

If this is true for any value of u then we expect that an equilibrium will never have p matched with a while p' is matched with a'. But this is all that is meant by PAM: p's partner can never have a type smaller than p''s.

When satisfied by any u, $p > p'$, and $a > a'$, condition (4.1) is called *Generalized Increasing Differences* (GID).[6] The concept is illustrated in Figure 1. The frontiers for the matched pairs $\langle p', a' \rangle$,$\langle p', a \rangle$, $\langle p, a \rangle$, and $\langle p, a' \rangle$ are plotted in a four-axis diagram.

[5] Obviously the reasoning can be made in terms of p and p' competing for a; in this case the (GID) condition is

$$\psi\left(a, p, \phi\left(p, a', v\right)\right) \geq \psi\left(a, p', \phi\left(p', a', v\right)\right);$$

it is straightforward to verify that the two conditions are equivalent.

[6] The designation Generalized Increasing Differences is motivated as follows. Let A be the type

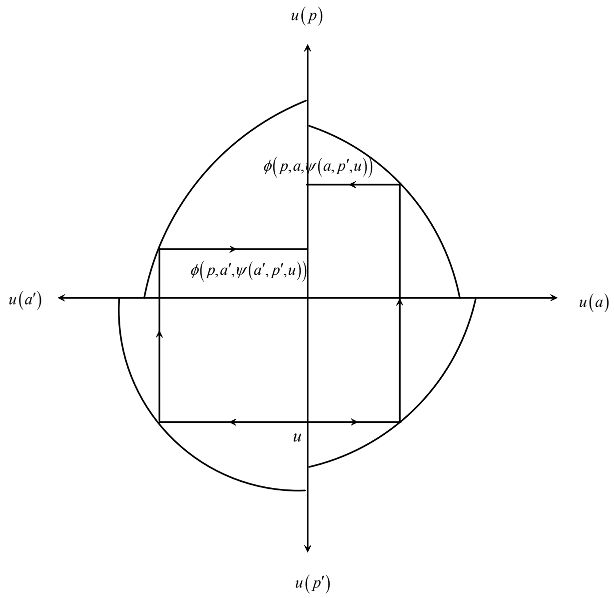

Figure 4.1: Generalized Increasing Differences

The compositions in (4.1) are indicated by following the arrows around from a level of utility u for p'. Note that the utility p obtains on the "a side" exceeds that on the a' side of the diagram.

Our main result states that GID is sufficient for PAM in the sense that all equilibria are payoff equivalent to a PAM equilibrium. There is an analogous condition, *Generalized Decreasing Differences* (GDD), for NAM.

Proposition 1. *(1) A sufficient condition for an economy to satisfy PAM is that ϕ*

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space and G be a (partially) ordered group with operation $*$ and order $\succsim$. We are interested in maps $\psi : A^2 \to G$.

Consider the condition

$$a > b \text{ and } c > d \text{ implies } \psi(c,a) * \psi(d,a)^{-1} \succsim \psi(c,b) * \psi(d,b)^{-1},$$

where $\psi(\cdot,\cdot)^{-1}$ denotes the inverse element under the group operation. When $G = \mathbb{R}$, $\succsim$ = the usual real order, and $*$ = real addition, this is just ID. GID corresponds to the case in which G = monotone functions from $\mathbb{R}$ to itself, $\succsim$ = the pointwise order, and $*$ = functional composition.

satisfies generalized increasing differences (GID) on $[0, \phi(p', a', 0)]$*: whenever* $p > p'$, $a > a'$, *and* $u \in [0, \phi(p', a', 0)]$, $\phi\left(p, a, \psi(a, p', u)\right) \geq \phi\left(p, a', \psi(a', p', u)\right)$.

(2) A sufficient condition for an economy to satisfy NAM is that ϕ *satisfies generalized decreasing differences (GDD) on* $[0, \phi(p', a', 0)]$*: whenever* $p > p'$, $a > a'$, *and* $u \in [0, \phi(p', a', 0)]$, $\phi\left(p, a, \psi(a, p', u)\right) \leq \phi\left(p, a', \psi(a', p', u)\right)$.

We now apply this result to our model of risk sharing within households.

Example 5. *We claim that in the two-state risk sharing model, there is always NAM in risk attitude, provided individuals are ordered by the Arrow-Pratt measure of absolute risk tolerance. To show this, one need only verify GDD. We have*

$$\phi(p, a, \psi(a, p', u)) \equiv \max_{\{s_i\}} \Sigma_i \pi_i U(p, w_i - s_i) \text{ s.t. } \Sigma_i \pi_i V(a, s_i) \geq \psi(a, p', u) \quad (4.2)$$

$$\psi\left(a, p', u\right) \equiv \max_{\{t_i\}} \Sigma_i \pi_i V(a, t_i) \text{ s.t. } \Sigma_i \pi_i U(p', w_i - t_i) \geq u; \quad (4.3)$$

$$\phi(p, a', \psi(a', p', u)) \equiv \max_{\{s_i\}} \Sigma_i \pi_i U(p, w_i - s_i) \text{ s.t. } \Sigma_i \pi_i V(a', s_i) \geq \psi(a', p', u) \quad (4.4)$$

$$\psi\left(a', p', u\right) \equiv \max_{\{t_i\}} \Sigma_i \pi_i V(a', t_i) \text{ s.t. } \Sigma_i \pi_i U(p', w_i - t_i) \geq u \quad (4.5)$$

Call the solution to the program in (4.2) s^{pa}*, the solution to the program in (4.3)* $s^{p'a}$*, etc. These are illustrated in the figure; without loss of generality, we assume* $w_1 < w_2$.

Also, observe that any optimal sharing rule s^* *is monotone for both partners: by the first-order conditions and strict concavity of the utilities, we must have* $s_1^* < s_2^*$ *and* $w_1 - s_1^* < w_2 - s_2^*$.

Showing GDD amounts to showing that

$$\Sigma_i \pi_i U(p, w_i - s_i^{pa'}) > \Sigma_i \pi_i U(p, w_i - s_i^{pa}).$$

First, note that if $\Sigma_i \pi_i V(a', s^{pa}) \geq \Sigma_i \pi_i V(a', s^{p'a'})$*, we are done, since* $w - s^{pa}$ *is in the feasible set for the problem that determines* $w - s^{pa'}$*. Thus assume, as in the figure, that*

$$\Sigma_i \pi_i V(a', s^{pa}) < \Sigma_i \pi_i V(a', s^{p'a'}). \quad (4.6)$$

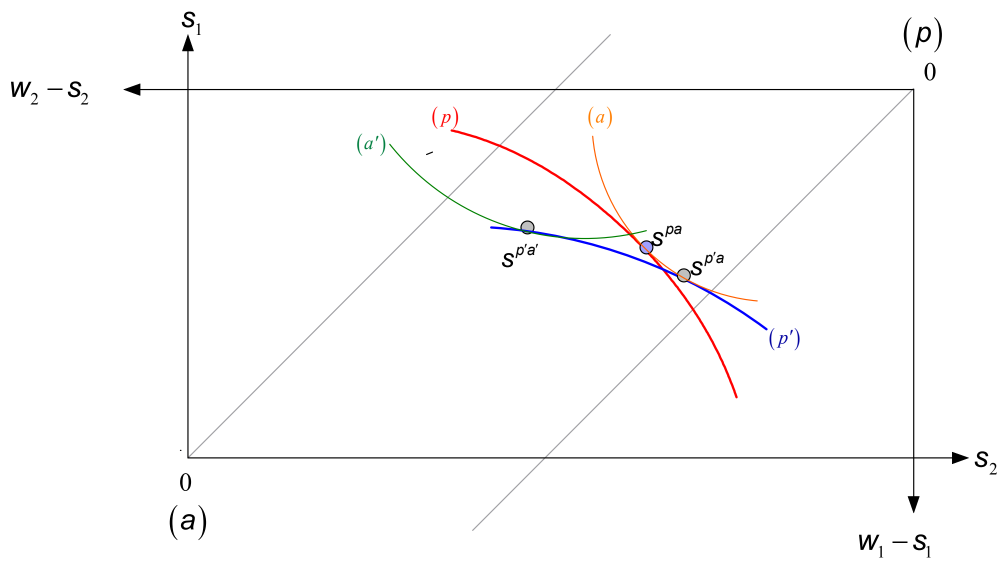

Figure 4.2: Risk Sharing

Note, by revealed preference, that

$$\Sigma_i \pi_i V(a, s_i^{pa}) \geq \Sigma_i \pi_i V(a, s_i^{p'a'}). \tag{4.7}$$

And $\Sigma_i \pi_i U(p', w_i - s_i^{p'a'}) = \Sigma_i \pi_i U(p', w_i - s_i^{p'a}) = \max_{\{s_i\}} \Sigma_i \pi_i U(p', w_i - s_i)$ *s.t.* $\Sigma_i \pi_i V(a, s_i) = \Sigma_i \pi_i V(a, s_i^{pa})$*, so that*

$$\Sigma_i \pi_i U(p', w_i - s_i^{p'a'}) \geq \max_{\{s_i\}} \Sigma_i \pi_i U(p', w_i - s_i^{pa}). \tag{4.8}$$

Now by Pratt's Theorem, $U(p, y)$ *and* $V(a, y)$ *are (strict) convexifications of* $U(p', y)$ *and* $V(a', y)$ *respectively. Now use the following*

Fact *If* $y_1 < y_1' < y_2' < y_2$*, and* $\Sigma_i \pi_i V(a', y_i) \geq \Sigma_i \pi_i V(a', y_i')$ *then* $\Sigma_i \pi_i V(a, y_i) > \Sigma_i \pi_i V(a, y_i')$.

This follows from Pratt's theorem and the fact that $(V(a', y_1), V(a', y_2))$ *is just a mean-nondecreasing spread of* $(V(a', y_1'), V(a', y_2'))$.

We now show that $s_1^{pa} < s_1^{p'a'} < s_2^{p'a'} < s_2^{pa}$. *The middle inequality follows from the* *above observation about monotonicity of optimal sharing rules. If only one of the other inequalities is violated, then either both a and a' prefer s^{pa}, contradicting (4.6), or both prefer $s^{p'a'}$, contradicting (4.7). Thus suppose that $s_1^{p'a'} < s_1^{pa} < s_2^{pa} < s_2^{p'a'}$(ignore the trivial case of weak inequalities). Then by the Fact, a prefers $s^{p'a'}$, contradicting (4.7). Thus $w_1 - s_1^{p'a'} < w_1 - s_1^{pa} < w_2 - s_2^{pa} < w_2 - s_2^{p'a'}$, and by (4.8) and Fact* **??**, *we have shown that*

$$\Sigma_i \pi_i U(p, w_i - s_i^{p'a'}) > \Sigma_i \pi_i U(p, w_i - s_i^{pa}),$$

from which it follows that $\Sigma_i \pi_i U(p, w_i - s_i^{pa'}) > \Sigma_i \pi_i U(p, w_i - s_i^{pa})$, again by revealed preference.

Thus GDD is indeed satisfied, and we conclude that in the two-state risk-sharing economy men and women will always match negatively in risk attitude. *This is intuitive: a risk-neutral agent is willing to offer a better deal for insurance than is a risk averse one, so those demanding the most insurance (the most risk averse) will share risk with the least risk averse, while the moderately risk averse share with each other.*

5. Necessity

A natural issue to consider at this point is the strength of the sufficient conditions we have given for monotone matching: is GID necessary for an economy to satisfy PAM? An affirmative answer is provided by

Proposition 2. *In a two-sided model, if the equilibrium outcome is payoff equivalent to PAM (NAM) for every distribution of types, then the frontier function ϕ satisfies GID (GDD).*

Proof. Consider PAM, as the case for the necessity of GDD for NAM is similar. Suppose there are $p, p' \in P$, $p > p'$ and $a, a' \in A$, $a > a'$, and a payoff level u such that $\phi(p, a, \psi(a, p', u)) < \phi(p, a', \psi(a', p', u))$. Then we can find a distribution of types such that there is an equilibrium that is not payoff equivalent to PAM. To see this, put an equal measure at each of the four types p, p', a, a'. Then there is $\epsilon > 0$ such that $\langle p', a\rangle$ with payoffs $(u, \psi(a, p', u))$ and $\langle p, a'\rangle$ with payoffs $(\phi(p, a', \psi(a', p', u) + \epsilon), \psi(a', p', u) +$

ϵ)) is an equilibrium. To verify stability, note that by continuity of ϕ in u, for ϵ small enough, $\phi(p, a, \psi(a, p', u)) < \phi(p, a', \psi(a', p', u) + \epsilon)$. Thus p would be strictly worse off switching to a as long as a receives at least his equilibrium payoff. Similarly a' would lose ϵ by switching to p'. Finally, the match is not payoff equivalent to PAM because p cannot generate $\phi(p, a', \psi(a', p', u) + \epsilon)$ in a match with a without offering a less than $\psi(a, p', u)$ And no match with PAM could support these payoffs, since the same inequality implies that p cannot generate his equilibrium payoff in a match with a. ■

In other words, if GID is not satisfied, there are economies in which the matching pattern will not be PAM. If GDC isn't satisfied either, then matching need not be monotone. In some cases, matching will be positive assortative for some type distributions, negative assortative for others, and nonmonotone for others still. See Legros and Newman (2002) for examples.

6. Computational Aids

A number of useful computational techniques follow from the sufficiency of the GID and GDD. We first illustrate the use of the GDC's for a case in which ϕ has a closed form representation. Next, we present a set of differential conditions. In addition to being easy to apply, they help sharpen the intuition about the trade-offs at work in NTU matching problems.

We then note that GID and GDD are preserved under ordinal transformations of types' preferences. This implies that the analyst is free to choose whichever representation of preferences is most convenient, and leads to a weakening of the differential conditions. In a number of cases, the NTU model even admits a TU representation, in which case GID and GDD reduce to ID and DD of the joint payoff function induced by the representation.

Finally, we develop the lattice-theoretic versions of our conditions and conclude the section with a remark on models with type-dependent autarchy payoffs.

6.1. Risk Sharing in Closed Form

A plausible conjecture is that the result in Example 5, in which there is negative assortative matching in risk aversion if income assumes two possible levels, extends to the case of an arbitrary (finite) number of income levels. A (somewhat involved) extension of the argument given in Section 4.2 can be used to show this is indeed the case, at least for models in which all agents have utilities in a class that includes CARA (ordered by absolute risk tolerance) or CRRA (ordered by relative risk tolerance, with common wealth).

Alternatively, we can verify this conjecture directly in certain cases. For instance, if U and V are from the CARA class, the frontier admits a closed-form representation for which verification of GDD is straightforward:

Example 6. *Modify Example 5 by letting the number of possible income realizations be any finite number n. The utilities are assumed to be of the form $U(p,y) = 1 - e^{-py}$, $V(a,y) = 1 - e^{-ay}$. Then program (2.1) becomes*

$$\phi_{\exp}(p,a,v) \equiv \max_{\{s_i\}_{i=1}^n} \Sigma_i \pi_i (1 - e^{-p(w_i - s_i)}) \ s.t. \ \Sigma_i \pi_i (1 - e^{-as_i}) \geq v,$$

and routine computation yields

$$\begin{aligned} \phi_{\exp}(p,a,v) &= 1 - (1-v)^{-\frac{p}{a}} (\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}})^{\frac{a+p}{a}} \\ \psi_{\exp}(a,p,u) &= 1 - (1-u)^{-\frac{a}{p}} (\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}})^{\frac{a+p}{p}}. \end{aligned}$$

Let $p > p'$ and $a > a'$; then the GDD condition $\phi_{\exp}(p,a,\psi_{\exp}(a,p',u)) \leq \phi_{\exp}(p,a',\psi_{\exp}(a',p',u))$ becomes

$$\begin{aligned} & 1 - ((1-u)^{-\frac{a}{p'}} (\Sigma_i \pi_i e^{-\frac{ap'w_i}{a+p'}})^{\frac{a+p'}{p'}})^{-\frac{p}{a}} (\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}})^{\frac{a+p}{a}} \\ \leq \ & 1 - ((1-u)^{-\frac{a'}{p'}} (\Sigma_i \pi_i e^{-\frac{a'p'w_i}{a'+p'}})^{\frac{a'+p'}{p'}})^{-\frac{p}{a'}} (\Sigma_i \pi_i e^{-\frac{a'pw_i}{a'+p}})^{\frac{a'+p}{a'}}, \end{aligned}$$

or

$$(\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}})^{\frac{a+p}{ap}} (\Sigma_i \pi_i e^{-\frac{a'p'w_i}{a'+p'}})^{\frac{a'+p'}{a'p'}} \geq (\Sigma_i \pi_i e^{-\frac{ap'w_i}{a+p'}})^{\frac{a+p'}{ap'}} (\Sigma_i \pi_i e^{-\frac{a'pw_i}{a'+p}})^{\frac{a'+p}{a'p}},$$

i.e. the function $(\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}})^{\frac{a+p}{ap}}$ *has log decreasing differences in* (a, p). *This condition is easily verified, as*

$$\frac{\partial^2}{\partial a \partial p} \log(\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}})^{\frac{a+p}{ap}} = \frac{ap}{(a+p)^3} \left[\frac{\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}} w_i^2}{\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}}} - \left(\frac{\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}} w_i}{\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}}} \right)^2 \right] > 0.$$

6.2. Differential Conditions

The GID condition states that whenever $p > p'$ and $u \in [0, \phi(p', a, 0)]$, the function $\phi(p, a, \psi(a, p', u))$ is increasing in a. When ϕ is smooth, this monotonicity condition is equivalent to the following condition.

Proposition 3. *If* ϕ *is smooth, a necessary and sufficient condition for GID is that for all types* $p, p' \in P$ *with* $p \geq p'$ *and all* $a \in A$ *and utilities* $u \in [0, \phi(p', a, 0)]$,

$$\phi_2(p, a, \psi(a, p', u)) + \phi_3(p, a, \psi(a, p', u)) \cdot \psi_1(a, p', u) \geq 0. \tag{6.1}$$

A necessary and sufficient condition for GDD is that for all types $p, p' \in P$ *with* $p \geq p'$, *and all* $a \in A$ *and all utilities* $u \in [0, \phi(p', a, 0)]$,

$$\phi_2(p, a, \psi(a, p', u)) + \phi_3(p, a, \psi(a, p', u)) \cdot \psi_1(a, p', u) \leq 0. \tag{6.2}$$

Proof. GID requires that $\phi(p, a, \psi(a, p', u))$ is nondecreasing in a for $p > p'$. Therefore $0 \leq \frac{d}{da}\phi(p, a, \psi(a, p', u)) = \phi_2(p, a, \psi(a, p', u)) + \phi_3(p, a, \psi(a, p', u)) \cdot \psi_1(a, p', u)$. This proves necessity. For sufficiency, integrate the left hand side of (6.1) with respect to a on a closed interval to obtain GID. ■

Condition (6.1) may be difficult to verify in practice because the condition must be verified globally (that is for different values of p') and the arguments of the different partial derivatives are not the same. For this reason, we present sufficient differential conditions that are computationally convenient and also illuminate the type-payoff complementarity property alluded to above. In this subsection we suppose that $\phi(p, a, v)$ is twice differentiable (except of course at $v = 0$ and $v = \psi(a, p, 0)$).

Proposition 4. *(i) A sufficient condition for the economy to satisfy PAM is that for all* $(p, a) \in P \times A$ *and* $v \in [0, \psi(a, p, 0))$,

$$\phi_{12}(p, a, v) \geq 0, \phi_{13}(p, a, v) \geq 0 \text{ and } \phi_2(p, a, v) \geq 0. \tag{6.3}$$

(ii) A sufficient condition for the economy to satisfy NAM is that for all $(p, a) \in P \times A$ *and* $v \in [0, \psi(a, p, 0))$,

$$\phi_{12}(p, a, v) \leq 0, \phi_{13}(p, a, v) \leq 0 \text{ and } \phi_2(p, a, v) \geq 0. \tag{6.4}$$

Proof. We show that the local conditions imply the generalized difference conditions. We first show that $\phi_2(p, a, \cdot) \geq 0$ implies that $\psi_1(a, p, \cdot) \geq 0$. By definition, $\psi(a, p, \phi(p, a, v)) = v$;differentiating both sides with respect to a yields $\psi_1 + \phi_2\psi_3 = 0$. Since $\psi_3 \leq 0$, the conclusion follows. Fix u, $p > p', a > a'$, and consider the case for PAM. Then $\phi_{12} > 0$ implies that for any $\hat{p} \in [p', p]$. $\phi_1(\hat{p}, a, \psi(a', p', u)) \geq \phi_1(\hat{p}, a', \psi(a', p', u))$. Since $\phi_2 \geq 0$, we have $\psi_1 \geq 0$ and therefore $\psi(a, p', u) \geq \psi(a', p', u)$. The assumption $\phi_{13} \geq 0$ in turn implies $\phi_1(\hat{p}, a, \psi(a, p', u)) \geq \phi_1(\hat{p}, a, \psi(a', p', u))$, so that $\phi_1(\hat{p}, a, \psi(a, p', u)) \geq \phi_1(\hat{p}, a', \psi(a', p', v))$. Integrating this expression with respect to $\hat{p}$ over $[p', p]$ and using $\phi(p', a, \psi(a, p', u)) = \phi(p', a', \psi(a', p', u)) = u$ yields GID. ■

Obviously, with TU, $\phi_{13} = 0$, so (6.3) reduces to the standard condition in that case. The extra term reflects the fact that changing the type results in a change in the slope of the frontier. For PAM, the idea is that higher types can transfer utility to their partners more easily (ϕ_3 is less negative, hence flatter).

The conditions in Proposition 4 illustrate the separate roles of both the usual type-type complementarity and the type-payoff complementarity we have mentioned. In terms of the bidding story we mentioned in Section 4.1, if two different types are competing for a higher type partner, both will be willing to offer her more than they would a partner with a lower type ($\phi_2 > 0$); if the higher type's frontier is flatter than the lower's frontier ($\phi_{13} \geq 0$), it will cost the higher type less to do this than it will the lower one; meanwhile if the high type is also more productive on the margin ($\phi_{12} > 0$) then he is sure to win, in effect being both more productive and having lower costs.

Remark 1. *As long as we assume type complementarity ($\phi_{12} \geq 0$) and all p preferring, ceteris paribus, to match with higher types a ($\phi_2 \geq 0$), condition $\phi_{13} \geq 0$ is in fact minimal to obtain PAM for all type distributions. Indeed, consider the following example: the frontier is $\phi(p, a, v) = ap - pv$. Note that $\phi_3 = -p < 0, \phi_2 = p > 0, \phi_{23} = 0, \phi_{12} = 1$, and $\phi_{13} = -1 < 0$. Since $\psi(a, p', u) = a - \frac{u}{p'}$, $\psi_1(a, p', u) = \frac{u}{p'^2}$ and for all $u \leq ap'$,we have*

$$\phi_2(p, a, \psi(a, p', u)) + \phi_3(p, a, \psi(a, p', u)) \cdot \psi_1(a, p', u) = p\frac{p'^2 - u}{p'^2},$$

which is negative when $u < p'^2$. Therefore (6.1) is violated whenever the type distribution admits pairs (p', a) such that $p' < a$ since then there exists $u \in \left(p'^2, ap'\right)$ and thus by Propositions 3 and 2 there are distributions for which the equilibrium outcome is not positive assortative.

We now apply the sufficient conditions to the principal-agent model.

Example 7. *Earlier we conjectured that the most risk averse agents ought to match with the safest tasks, since the latter are optimally contracted as fixed wages. But the following application of Proposition 4 shows otherwise.*
Recall that the problem of interest is[7]

$$\begin{aligned} \phi(p, a, v) = \max \ & R - ps_1 - (1-p)s_0 \\ s.t. \ & pV(a + s_1) + (1-p)V(a + s_0) - 1 \geq V(a + s_0), \\ & pV(a + s_1) + (1-p)V(a + s_0) - 1 \geq v \end{aligned}$$

By standard arguments, both constraints bind. Let $h(\cdot) \equiv V^{-1}(\cdot)$. Thus $V(a+s_1) = \frac{1}{p}+ V(a + s_0)$, and $V(a + s_0) = v$, from which

$$\phi(p, a, v) = R + a - ph(\frac{1}{p} + v) - (1-p)h(v).$$

[7] In this example, the agents' autarky payoffs are not zero, at least if it assumed that they can consume their initial wealth. As we note in Section 7.1, this generalization presents no particular difficulty.

Thus,

$$\phi_2(p,a,v) = 1 > 0,$$
$$\phi_{12} = 0,$$
$$\phi_{13}(p,a,v) = \frac{1}{p}h''(\frac{1}{p}+v) - h'(\frac{1}{p}+v) + h'(v).$$

Notice that if h' is convex (equivalently V''/V'^3 is decreasing; see Jewitt, 1988 for a discussion – all utilities of the CRRA class that are weakly more risk averse than logarithmic utility satisfy this condition), then $\phi_{13} > 0$: there is PAM in (p,a) : as long as risk aversion does not decline "too quickly," agents with lower risk aversion (higher wealth) are matched to principals with projects that are safer, i.e., more likely to succeed. *This result may appear surprising, since empirically we tend to associate (financially) riskier tasks to wealthier workers.*[8]

The explanation is that in the standard version of the principal-agent model with utility additively separable in income and effort, incentive compatibility for a given effort level entails that the amount of risk borne by the agent increases with wealth ($\frac{ds_1}{ds_0} > 0$ along the incentive compatibility constraint). This effect arises from the diminishing marginal utility of income. Though wealthier agents tolerate risk better than the poor, they must accept more risk on a given task; with h' convex, the latter effect dominates, and the wealthy therefore prefer the safer tasks. Put another way, a less risky task allows for a reduction in risk borne by the agent; given the increasing risk effect of incentive compatibility, the benefit of the risk reduction is greater for the rich than for the poor, and this generates a complementarity between safety and wealth.

The result offers a possible explanation for the finding in Ackerberg-Botticini (2002) that in medieval Tuscany, wealthy peasants were more likely than poor peasants to tend safe crops (cereals) rather than risky ones (vines).

This example is instructive because the entire effect comes from the nontransferability of the problem. There is no direct "productive" interaction between principal

[8] Of course, if risk aversion declines fast enough – h' is concave – then the "intuitive" negative matching pattern obtains, though this entails a possibly implausibly high level of risk tolerance.

type and agent type ($\phi_{12} = 0$); only the type-payoff complementarity plays a role in determining the match.

As is apparent from their derivation, the local sufficient conditions (6.3), (6.4) are stronger than generalized difference conditions, even restricting to smooth frontier functions. This is of practical as well as logical interest: another version of the n-state risk sharing example that admits a closed form is one in which the utilities are logarithmic, with type representing each individual's initial wealth (received as a parental bequest upon marriage, so that autarky payoff is zero): $U(p, y) = \log(1 + p + y), V(a, y) = \log(1 + a + y)$. One obtains $\phi_{\log}(p, a, v) = \ln(1 - e^{v-\Sigma_{pa}}) + \Sigma_{pa}$,where Σ_{pa} denotes $\Sigma_i \pi_i \ln(w_i + p + a + 2)$.Straightforward computations show that $\phi_{\log_2} > 0$, $\phi_{\log_{12}} < 0$, and $\phi_{\log_{13}} > 0$, while it can be shown (see Example 8 below) that it does satisfy GDD.

Thus, for some models, the generalized difference conditions may apply while the local sufficient conditions do not. However, as we shall see in the next subsection, in such cases, it may be possible to find an alternate representation of agents' preferences in which the frontiers do satisfy the local conditions.

6.3. Order Preserving Transformations

The core of an economy is independent of the cardinal representation of preferences; in particular the matching pattern must not depend on how one represents the preferences of the agents. However, some representations may be easier to work with after monotone transformations of types' utilities; in some instances it may be possible to recover a transferable utility representation.

Let $F(u; p)$ be a strictly increasing transformation applied to type p's utility u and let $F^{-1}(u; p)$ be its inverse. Similarly, we shall transform type a's utility by $G(v; a)$. Let

$$\phi^{F,G}(p, a, v) = F(\phi(p, a, G^{-1}(v; a)); p),$$

be the new frontier functions after transformations F and G are applied. We shall call $\phi^{F,G}$ a *representation* of ϕ. Its "inverse" is $\psi^{G,F}(a, p, u) = G\left(\psi\left(a, p, F^{-1}(u; p)\right); a\right)$ We have the following result:

Lemma 1. *Suppose GID holds for ϕ. Then GID holds for any other frontier function generated from ϕ by increasing transformations of types' utilities. The same result is true for GDD,*

Though this result follows from Proposition 2 along with invariance of the core under ordinal transformations of agents' utilities, a direct proof is useful.

Proof. We consider the case for GID; the proof for GDD is virtually identical. It is enough to show that the map $\phi^{F,G}(p,a,v)$ satisfies GID, that is that $\phi^{F,G}\left(p,a,\psi^{F,G}(a,p',u)\right)$ is increasing in a.

$$\begin{aligned}\phi^{F,G}\left(p,a,\psi^{F,G}\left(a,p',u\right)\right) &= F(\phi(p,a,G^{-1}(\psi^{G,F}\left(a,p',u\right);a);p) \\ &= F\left(\phi(p,a,G^{-1}(G\left(\psi\left(a,p',F^{-1}\left(u;p'\right)\right);a\right);a);p\right) \\ &= F\left(\phi(p,a,\psi\left(a,p',F^{-1}\left(u;p'\right)\right);p\right),\end{aligned}$$

since $F(\cdot;p)$ is strictly increasing, $\phi^{F,G}\left(p,a,\psi^{F,G}(a,p',u)\right)$ is increasing in a only if $\phi(p,a,\psi\left(a,p',F^{-1}(u;p')\right))$ is increasing in a, which is true since ϕ satisfies GID. ■

A suitably chosen representation of ϕ may be easier to work with than ϕ itself:

Example 8. *In the logarithmic version of the risk sharing example, transform the utility by exponentiation for all types, i.e., set $F(v;p)=e^v$ and $G(u;a)=e^u$; then $\phi_{\log}^{F,G}(p,a,v)=e^{\phi_{\log}(p,a,\log v)}=e^{\log(1-e^{\log v-\Sigma_{pa}})+\Sigma_{pa}}=e^{\Sigma_{pa}}-v$, .and $\psi_{\log}^{G,F}=e^{\Sigma_{pa}}-u$. This function may be more manageable than $\phi_{\log}$. GDD for $\phi_{\log}^{F,G}$ is satisfied when $\phi_{\log}^{F,G}(p,a,\psi_{\log}^{G,F}(a,p',v))<\phi_{\log}^{F,G}(p,a',\psi_{\log}^{G,F}(a',p',v))$ which is equivalent to $e^{\Sigma_{pa}}-(e^{\Sigma_{pa'}}-v)<e^{\Sigma_{p'a}}-(e^{\Sigma_{p'a'}}-v)$. This last inequality is simply DD of $e^{\Sigma_{pa}}$: since $\frac{\partial^2}{\partial a\partial b}e^{\Sigma_{ab}}=-e^{\Sigma_{ab}}Var(\frac{1}{w+a+b+2})$ is negative, the result follows.*

Though the generalized difference conditions are preserved for all representations of ϕ, not so the differential conditions in Proposition 4. Recall that those conditions do not hold for $\phi_{\log}$. But they do for the above transformed version $\phi_{\log}^{F,G}$: indeed, $\phi_{\log_2}^{F,G}>0$, $\phi_{\log_{12}}^{F,G}<0$, and $\phi_{\log_{13}}^{F,G}=0$. This suggests the following strengthening of Proposition 4, whose proof is an immediate consequence of the fact that the differential conditions imply GID, which in turn implies GID of any representation of ϕ.

Corollary 1. *(1) A sufficient condition for segregation (or PAM) is that there exists a representation* $\phi^{F,G}$ *of* ϕ *such that for all* $p, a \in P \times A$ *and* $v \in [0, \psi(a, p, 0))$,

$$\phi_{12}^{F,G}(p, a, v) \geq 0, \phi_{13}^{F,G}(p, a, v) \geq 0 \text{ and } \phi_{1}^{F,G}(p, a, v) \geq 0.$$

(2) A sufficient condition for NAM is that there exists a representation $\phi^{F,G}$ *of* ϕ *such that for all* $p, a \in P \times A$ *and* $v \in [0, \psi(a, p, 0))$,

$$\phi_{12}^{F,G}(p, a, v) \leq 0, \phi_{13}^{F,G}(p, a, v) \leq 0 \text{ and } \phi_{1}^{F,G}(p, a, v) \geq 0.$$

6.3.1. TU Representation

Notice that the transformation of payoffs $\phi_{\log}^{F,G}$ actually yields an expression of the frontiers in a transferable utility form. This cannot be done with all NTU models, of course (see Legros-Newman, 2003 for more on this topic), but there are some well-known-examples. For instance, the Principal-Agent model with exponential utility (Holmström-Milgrom, 1987) can be given a TU representation by looking at players' certainty equivalent incomes rather than their expected utility levels.

Start with a model $\phi(p, a, v)$ and say that it is *TU-representable* if there is a representation $\phi^{F,G}$ of ϕ and a function $h(p, a)$ such that

$$\forall p, a, v, \; F(\phi(p, a, v); p) = h(p, a) - G(v; a). \tag{6.5}$$

Then $F(\phi(p, a, v); p)$ is a TU model, since the transformed payoffs to (p, a) sum to $h(p, a)$, independently of the distribution of transformed utility between p and a. It follows from the definition that h is symmetric.[9] The main observation of this subsection is the following.

Proposition 5. *Suppose that* ϕ *has a TU representation* $\phi^{F,G}$*. Then* ϕ *satisfies GID (GDD) if and only if* h *satisfies ID (DD), where* h *is defined in (6.5).*

[9] To see this, note that $F(\phi(p, a, v); p) = h(p, a) - F(v; a)$, but $F(v; a) = F(\psi(a, p, \phi(p, a, v)); a) = h(a, p) - F(\phi(p, a, v); p)$, so that $F(\phi(p, a, v); p) = h(a, p) - F(v; a)$; hence $h(a, p) = h(p, a)$.

Proof. Take $p > p'$, $a > a'$, and u and assume GID holds. Then

$\phi(p, a, \psi(a, p', u)) \geq \phi(p, a', \psi(a', p', u))$

$\iff F(\phi(p, a, \psi(a, p', u)); p) \geq F(\phi(p, a', \psi(a', p', u)); p)$

$\iff h(p, a) - F(\psi(a, p', u); a) \geq h(p, a') - F(\psi(a', p', u); a')$

$\iff h(p, a) - h(a, p') + F(u; p') \geq h(p, a') - \psi(a', p', v) + F(u; p')$

$\iff h(p, a) - h(p', a) \geq h(p, a') - h(p', a)$,

i.e., ψ satisfies increasing differences. The proof for GDD simply reverses all the weak inequalities. ■

Example 9. *Consider the logarithmic version of the principal- agent example, in which* $V(y) = \log y$. *Then* $\phi(p, a, v) = R + a - e^v(pe^{\frac{1}{p}} + (1-p))$. *Now put* $F(u; p) = \frac{1}{pe^{\frac{1}{p}}+(1-p)} u$, *and* $G(a; v) = e^v$. *The sum* $F(\phi(p, a, v); p) + G(v; a)$ *is then* $h(p, a) = \frac{R+a}{pe^{\frac{1}{p}}+(1-p)}$ *, which satisfies ID: matching is always positive assortative.*

Example 10. *The certainty equivalent transformation yields a TU representation for the exponential version of the risk-sharing model (Example 6): put* $F(u; p) = \frac{1}{p} \log(1 - u)$, $G(v; a) = \frac{1}{a} \log(1-v)$, $h(a, p) = \frac{a+p}{ap} \log(\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}})$; *then* $\phi^{F,G}_{\text{exp}} = \frac{a+p}{ap} \log(\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}}) - \frac{1}{a} \log(1 - v)$. *We verified earlier that* $\frac{a+p}{ap} \log(\Sigma_i \pi_i e^{-\frac{apw_i}{a+p}})$ *has decreasing differences.*

6.4. Lattice Theoretic Conditions

6.4.1. Supermodularity

Proposition 4 can be weakened by considering (possibly) nondifferentiable functions that are supermodular in pairs of variables.

Proposition 6. *(1) A sufficient condition for PAM is that* $\phi(p, a, v)$ *is supermodular in* (p, a)*, increasing in* a*, and supermodular in* (p, v)*.*

(2) A sufficient condition for NAM is that $\phi(p, a, v)$ *is submodular in* (p, a)*, increasing in* a*, and submodular in* (p, v)*.*

Proof. Consider case (1); the other case is similar. Take u, $p > p'$ and $a > a'$. Clearly, ϕ increasing in a implies ψ increasing in a. Supermodularity in own type and partner's utility, along with increasing in partner's type implies $\phi(p,a,\psi(a,p',u)) + \phi(p',a,\psi(a',p',u)) \geq \phi(p,a,\psi(a',p',u)) + \phi(p',a,\psi(a,p',u))$, or $\phi(p,a,\psi(a,p',u)) - \phi(p',a,\psi(a,p',u)) \geq \phi(p,a,\psi(a',p',u)) - \phi(p',a,\psi(a',p',u))$. But the right hand side of the latter inequality weakly exceeds $\phi(p,a',\psi(a',p',u)) - \phi(p',a',\psi(a',p',u))$ by supermodularity in types. Thus $\phi(p,a,\psi(a,p',u)) - \phi(p',a,\psi(a,p',u)) \geq \phi(p,a',\psi(a',p',u)) - \phi(p',a',\psi(a',p',u))$, and since $\phi(p',a,\psi(a,p',u)) = \phi(p',a',\psi(a',p',u)) = u$, we have $\phi(p,a,\psi(a,p',u)) \geq \phi(p,a',\psi(a',p',u))$, which is GID. ■

It is evident from this proposition that a stronger sufficient condition for PAM is that ϕ itself is a supermodular function that is increasing in a, since this implies the condition in Proposition 6. The principal interest of this observation is that it enables us to offer sufficient conditions for monotone matching expressed in terms of the fundamentals of the model, rather than in terms of the frontiers.

Let U be the utility payoff of agents on side 1 and V the payoff for agents on side 2. These payoffs may depend directly on agents types in the match as well as choice variables that we represent by x. The feasible set $\Omega(p,a)$ of choice variables will typically depend on types. The frontier can be expressed fairly generally as

$$\begin{aligned} \phi(p,a,v) = \ & \max_x U(x,p,a) \\ & \text{s.t. } V(x,a;p) \geq v \\ & x \in \Omega(p,a). \end{aligned}$$

Here $\Omega(p,a)$ is a (sub)lattice of some $\mathbb{R}^n$. A sufficient condition for ϕ to be non-decreasing in a is that U is non-decreasing in a and Ω is continuous and non-decreasing (in the set inclusion order) in a. Clearly ϕ is non-decreasing in v. We also need the set

$$S = \{(p,a,v,x) | p \in P, a \in A, v \in \mathbb{R}, x \in \Omega(p,a)\}$$

to form a sublattice. Then an application of Theorem 2.7.2 of Topkis (1998) yields

Corollary 2. *If payoffs functions are supermodular (submodular), strictly increasing in choices, and increasing in type; choice sets are continuous and increasing in own type; and the set of types, payoffs and feasible choices forms a sublattice, then the economy is* *segregated in the one-sided case and positively matched in the two-sided case (negatively matched).*

Topkis's theorem tells us that under the stated hypotheses, ϕ will be supermodular (submodular); since it is also non-decreasing in a by the hypotheses on Ω and U, the result follows.

As a practical matter, the usefulness of this corollary hinges on the ease of verifying that the sets S and Ω have the required properties. In many cases it may be more straightforward to compute the frontiers and apply Propositions 1, 4, or 6. Note, for example, that since the frontier function in the logarithmic risk-sharing example is not submodular despite the fact that the objective function is, the choice-parameter set S is not a sublattice.

6.4.2. Quasi-Supermodularity

Since supermodularity is stronger than one needs, a natural question to ask is whether weaker lattice theoretical conditions would suffice for assortative matching. Such a weaker concept is quasi-supermodularity introduced in Milgrom and Shannon (1994). QSM of ϕ requires that for any $x, y \in P \times A \times \mathbb{R}_+$, we have

$$\phi(x) \geq \phi(x \wedge y) \Rightarrow \phi(x \vee y) \geq \phi(y), \tag{6.6}$$

with strict inequality on the left hand side implying strict inequality on the right hand side. We show that as long as $\phi(p, a, v)$ is increasing in a, quasi-supermodularity implies our GID condition.

Proposition 7. *Consider the set of maps $\phi : P \times A \times \mathbb{R}_+ \to \mathbb{R}_+$, decreasing in the third argument. If ϕ is increasing in its second argument, then QSM implies GID.*

Proof. Let $p > p'$, $a > a'$. There is PAM if any negative match $(p, a', w), (p', a, v)$, where w is the payoff to a' and v is the payoff to a is not stable. The negative match is not stable if and only if

$$\begin{aligned} \text{either } \phi\left(p', a', w\right) &> \phi\left(p', a, v\right) \\ \text{or } \phi\left(p, a, v\right) &> \phi\left(p, a', w\right) \end{aligned}$$

that is either p' or p prefers the other partner while paying that partner his equilibrium payoff. This condition can be written as

$$\phi\left(p', a', w\right) \leq \phi\left(p', a, v\right) \Rightarrow \phi\left(p, a, v\right) > \phi\left(p, a', w\right) \tag{6.7}$$

We assume that ϕ satisfies (6.6) and we show that ϕ satisfies (6.7). Set $x = (p', a, v)$ and $y = (p, a', w)$. Let us show (6.7). Suppose that

$$\phi\left(p', a', w\right) \leq \phi\left(p', a, v\right). \tag{6.8}$$

Suppose first that $v > w$. Then $x \wedge y = (p', a', w)$ and $x \vee y = (p, a, v)$. Then, since ϕ is decreasing in its third argument, $\phi(p', a', v) < \phi(p', a', w)$ and (6.8) is $\phi(x \wedge y) < \phi(x)$. By (6.6), $\phi(y) < \phi(x \wedge y)$, or $\phi(p, a', w) < \phi(p, a, v)$, which is the desired conclusion for (6.7).
Suppose now that $v \leq w$, then $x \wedge y = (p', a', v)$ and $x \vee y = (p, a, w)$. PAM is violated if $\phi(p, a, v) \leq \phi(p, a', w)$; however since ϕ is increasing in its second argument, and decreasing in its third argument, $\phi(p, a, v) > \phi(p, a', v) \geq \phi(p, a', w)$, and a violation of PAM is impossible in this case (note that this is independent of whether ϕ is QSM). Since PAM and GID are equivalent, this proves the lemma. ■

The inclusion is strict as the following example shows:

Example 11. *Remember that in the pure NTU case, the Pareto efficient point is* $(h(p, a), f(a, p))$. *Consider* $\{p, p'\}, \{a, a'\}$ *and define* ϕ *to be strict NTU for pairs* $(p, a'), (p', a')$ *and* (p', a) *with values*

$$\begin{aligned} h\left(p, a'\right) &= f\left(a', p\right) = w \\ h\left(p', a'\right) &= f\left(a', p'\right) = v \\ h\left(p', a\right) &= \hat{w}, f\left(a, p'\right) = v \end{aligned}$$

and let

$$\phi(p,a,u) = 2w - u.$$

Choosing $w > v$ *and* $\hat{w} \in (w, 2w - v)$, *ensures that* ϕ *is increasing in its first two arguments. Let* $x = (p, a', w), y = (p', a, v)$, *we have* $x \wedge y = (p', a', v)$ *and* $x \vee y = (p, a, w)$. *Moreover*

$$\phi(x) = w > v = \phi(x \wedge y) \text{ and } \phi(y) = \hat{w} > w = \phi(x \vee y),$$

contradicting QSM. Since $w > v$, *there is PAM, hence GID: indeed, in* (p,a) p *can get strictly more than* w *and* a *can get strictly more than* v, *which dominates any other feasible payoffs if they match with another partner.*

The condition that ϕ is increasing in a is necessary for QSM to imply GID. As the following example shows, without this assumption QSM and GID are not nested.

Example 12. *Let* $P = \{p, p'\}, A = \{a, a'\}$, $\phi(p, a, \cdot) = 0$, $\phi(p', a, v) = \phi(p, a', v) = \phi(p', a', v) = h - v$, *when* $v \leq h$ *and* $= 0$ *otherwise, where* $h > 0$ *is exogenously given. PAM is violated since* (p,a) *does not produce positive surplus and GID does not hold: for instance,* $\phi(p, a, \psi(a, p', h/2)) = 0 < \phi(p, a', \psi(a', p', h/2)) = h/2$. *For QSM, consider first* $x = (p, a, v), y = (p', a', w)$; *therefore* $\phi(y) = h - w \geq \phi(x \vee y) = 0$, *and* $\phi(x \wedge y) = h - \min(v, w) \geq \phi(x) = 0$ *and QSM holds. Consider now* $x = (p', a, v)$ *and* $y = (p, a', w)$; *then* $\phi(y) = h - w \geq \phi(x \vee y) = 0$ *and* $\phi(x \wedge y) = h - \min(v, w) \geq \phi(x) = h - v$, *and QSM is satisfied.*

7. Simple Extensions

7.1. Type-Dependent Autarchy Payoffs

Suppose that autarchy generates a payoff $\underline{u}(p)$ to type p; if p is compact and $\underline{u}(\cdot)$ continuous, without loss of generality, we can assume $\underline{u}(p) \geq 0$. Then all the propositions go through as before, since if the generalized difference or differential conditions hold for nonnegative payoffs, they hold on the restricted domain of individually rational ones.

Equilibrium will now typically entail that some types remain unmatched (even apart from excess supply issues), but *among those matched*, the pattern will be monotone if the appropriate difference condition holds.

7.2. Strict NTU

In the strict NTU case, the frontier is characterized by functions $h(p,a)$ and $f(a,p)$ such that

$$\phi(p,a,v) = \begin{cases} h(p,a) & \text{if } v \leq f(a,p) \\ 0 & \text{otherwise} \end{cases}$$

$$\psi(a,p,u) = \begin{cases} f(a,p) & \text{if } u \leq h(p,a) \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, the unique weak Pareto payoff for $\{p,a\}$ is $(h(p,a), f(a,p))$.

Becker noted that monotonicity of $h(p,a)$ in a and monotonicity of $f(a,p)$ in p are enough to insure monotone matching. For instance, if principals prefer to be matched with higher type agents and agents prefer to be matched with higher type principals, there is PAM. The argument is straighforward: any negative match $\langle p',a\rangle, \langle p,a'\rangle$ is unstable since $h(p,a) > h(p,a')$ and $f(a,p) > f(a,p')$. Note that if $h(p,a)$ is decreasing in a and $f(a,p)$ is decreasing in p, we still get PAM. This condition that all principals and all agents have the same order of preferences for partners is stronger than necessary. Following the logic of the GID condition, a weaker condition for PAM is that for any test pairs $p,p' \in P$ and $a,a' \in A$, either higher types strictly prefer to match with higher types or lower types strictly prefer to match with lower types.[10]

Condition For all $p,p' \in P$, $p > p'$, $a,a' \in A$, $a > a'$,

$$\text{either } \left[h(p,a) - h\left(p,a'\right) > 0\right] \ \& \ \left[f(a,p) - f\left(a,p'\right) > 0\right]$$

$$\text{or } \left[h\left(p',a'\right) - h\left(p',a\right) > 0\right] \ \& \ \left[f\left(a',p'\right) - f\left(a',p\right) > 0\right].$$

Proposition 8. *All equilibria $\langle \mathfrak{M}, u, V\rangle$ are payoff equivalent to PAM if and only if (7.2) is satisfied.*

[10] See also Clark (2004) who shows that the same condition generates a unique equilibrium.

Since the frontier is not strictly decreasing, equal treatment is not insured in these economies, and while (7.2) is necessary and sufficient for having *all equilibria* satisfy PAM, it is not necessary for having *an equilibrium* satisfying PAM. We provide below an example of an economy where (7.2) fails, there is an equilibrium with PAM but another that is not payoff equivalent to PAM.

Example 13. *Consider $P = \{p, p'\}$, $p > p'$ and $A = \{a, a'\}$, $a > a'$ such that*

$$h(t,a) > h(t,a') \quad \text{for } t \in P \tag{7.1}$$

$$f(a',p') > f(a',p) \tag{7.2}$$

$$f(a,p) < f(a,p') \tag{7.3}$$

that is, all members of P prefer to match with a higher type agent while a and a' have opposite preferences over partner's type. Condition (7.2) is clearly violated.
There is an equilibrium with PAM; indeed, the stability conditions for PAM are

$$h(p',a) > h(p',a') \Rightarrow f(a',p') \geq f(a',p) \tag{7.4}$$

$$h(p,a') > h(p,a) \Rightarrow f(a,p') \leq f(a,p) \tag{7.5}$$

(7.2) implies (7.4); (7.1) implies (7.5). There is another equilibrium satisfying NAM which is not payoff equivalent to PAM; the stability conditions for NAM are

$$h(p',a') > h(p',a) \Rightarrow f(a',p') \leq f(a',p) \tag{7.6}$$

$$h(p,a) > h(p,a') \Rightarrow f(a,p) \leq f(a,p') \tag{7.7}$$

(7.1) implies (7.6), (7.3) implies (7.7).

7.3. One-Sided Models

In a one sided market, agents' types are given by $T = A$, and GID takes the form $\phi(a', a, \phi(a, a'', v))$ increasing in a for all $v \leq \phi(a'', a, 0), a' > a''$.[11] If GID holds, we obtain an extreme form of assortative matching: segregation, that is matched pairs

[11] Note that $\psi(a,b,v) = \phi(a,b,v)$.

consist of identical agents. The proof mimics that of Proposition 1 in showing that any heterogenous match is unstable if GID holds. Negative matching can be defined straightforwardly in the one sided-model, and sufficiency of GDD for NAM is easy to show; we refer the reader to our working paper (Legros and Newman 2002d).

Necessity in the one-sided case is a bit more involved than in the two-sided model. It is known that in the one-sided TU model that ID is not necessary for segregation, and that this condition can be weakened to nonpositivity of a function derived from the joint payoff called the *surplus*. One-sided PAM (outside of the definition of which we have not considered here) and NAM are equivalent to something called *weak increasing differences* and *weak decreasing differences* of this derived function (Legros-Newman, 2002a).

When utility is nontransferable, a similar construction can be performed in which a surplus function is derived from the frontier ϕ; suitably weakened versions of the generalized difference conditions defined for the surplus function are then necessary as well as sufficient for monotone matching. The interested reader is referred to Legros-Newman (2002c).

8. Conclusion

Many economic situations involving nontransferable utility are naturally modeled as matching or assignment games. For these to have much use, it is necessary to characterize equilibrium matching patterns. We have presented some general sufficient and necessary conditions for monotone matching in these models. These have an intuitive basis and appear to be reasonably straightforward to apply. Specifically, if one wants to ensure PAM, it does not suffice only to have complementarity in types; one must ensure as well that there is enough type-payoff complementarity.

This paper has focused on the study of properties of the economic environment that lead to monotone matching. Implicitly motivating this analysis is the question of how changes in the environment influence changes in matching. Space, not to mention the present state of knowledge, is too limited to offer a complete answer to this question here, but the comparison of TU with NTU is no doubt an important first step. Here we simply point out that economy-wide changes to transferability may help to explain phenomena that could be characterized as mass re-assignments of partners.

For instance, mergers and divestitures involve reassignments of say, upstream and downstream divisions of firms. Transferability between divisions depends on the efficiency of credit markets, and that in turn depends on interest rates– higher ones lead to an increase in agency costs, i.e., reductions in transferability, with the magnitude of the effect dependent on characteristics of individual firms such as liquidity position or productivity. A shock to the interest rate then may lead to widespread reassignment of partnerships between upstream and downstream divisions, i.e., a "wave" of corporate reorganization (Legros-Newman, 1999).

Or consider the effects of a policy like Title IX, which requires US schools and universities receiving federal funding to spend equally on men's and women's activities (athletic programs having garnered the most public attention), or suffer penalties in the form of lost funding. If one models a college as partnership between a male and female student-athlete, identifying their types with the revenue-generating capacities of their respective sports, the policy acts to transform a TU model into an NTU one, rather like Example 1. Imposing Title IX would lead to a reshuffling of the types of males and females who match; the male wrestler (low revenue), formerly matched to the female point guard (high revenue), will now match with, say, a female rower, while the point guard now plays at a football school. There is evidence that this sort of re-assignment has taken place: the oft-noted terminations and contractions of some sports at some colleges are ameliorated by start-ups and expansions at others.

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9. Appendix

9.1. Equilibrium

The economies we study have a continuum of individuals who are designated by the sets $I = [0, 1]$ and $J = [2, 3]$ with Lebesgue measure λ. The set I is the set of principals and the set J is the set of agents. The description of a specific economy includes an assignment of individuals to types via maps $\pi : I \longrightarrow P$, and $\alpha : J \longrightarrow A$, where P and A are compact real intervals. The maps π and α are simple measurable functions.

A matching correspondence is a function $m : I \longrightarrow J$ satisfying *measure consistency:*

$$\text{for any measurable subset } S \text{ of } I, \lambda(S) = \lambda(m(S)),$$

that is the measure of principals is equal to the measure of their partners.

Definition 1. *An equilibrium specifies a matching correspondence* m *and payoff allocations* $u^* : I \longrightarrow \mathbb{R}_+$ *and* $v^* : J \longrightarrow \mathbb{R}_+$ *that are measurable and satisfy the two following conditions.*
(i) Feasibility of (u^*, v^*) with respect to m : *for almost all* $i \in I, u^*(i) \leq \phi(\pi(i), \alpha(m(i)), v^*(m(i)))$.
(ii) Stability of m with respect to (u^*, v^*) : *there do not exist* $(i, j) \in I \times J$ *and payoffs* (u, v) *such that* $u \leq \phi(\pi(i), \alpha(j), v)$, *with* $u > u^*(i)$ *and* $v > v^*(j)$.

Because $\phi(p, a, v)$ is strictly decreasing on $v \in (0, \psi(a, p, 0))$, an equilibrium satisfies equal treatment.

Lemma 2. *Let* (m, u^*, v^*) *be an equilibrium, then there is equal treatment:*

$$\begin{aligned} \forall i, i' &\in I, \pi(i) = \pi(i') \Rightarrow u^*(i) = u^*(i') \\ \forall j, j' &\in J, \alpha(j) = \alpha(j') \Rightarrow v^*(j) = v^*(i') \end{aligned}$$

Proof. Suppose that there are two agents j and j' of type a getting different equilibrium utilities $v^*(j) > v^*(j')$, and that j's partner i is of type p. Then i gets $\phi(p, a, v^*(j)) < \phi(p, a, v^*(j'))$, where the inequality follows from the fact that ϕ is strictly decreasing in v. By continuity, there exists $\epsilon > 0$ such that $\phi(p, a, v^*(j') + \epsilon) > \phi(p, a, v^*(j))$; $\{i, j'\}$ can therefore block the equilibrium, a contradiction. ■

The matching correspondence $\mathfrak{M}$ in the text can be constructed from m as

$$\mathfrak{M}(p) = \{\alpha(j), j \in J : \exists i \in I, \pi(i) = p \text{ and } j = m(i)\}.$$

Because we have equal treatment, the equilibrium payoff functions u^* and v^* depend only on the type of the individual, and it is enough to define type-dependent payoff functions $u : P \to \mathbb{R}_+$ and $v : A \to \mathbb{R}_+$ as $u(p) = u^*(\pi(i))$ when $\pi(i) = p$ for $i \in I$ and $v(a) = v^*(\alpha(j))$ when $\alpha(j) = a$ for $j \in J$.

In addition to our assumptions we need to impose for existence that ϕ is *strongly comprehensive.*

Definition 2. ϕ *is strongly comprehensive if there exists* $b > 0$ *such that whenever* $0 < u \leq \phi(p, a, v)$, $u - c \leq \phi(p, a, v + bc)$ *for any* $c \in (0, u)$.

This condition is satisfied if, for instance, the marginal utility of consumption is bounded at autarky. That equilibria exist is then a simple consequence of the existence result in Kaneko and Wooders (1996).

9.2. Proof of Proposition 1

We prove (1), the proof of (2) is similar.[12] Suppose that GID holds and that there exists an equilibrium $\langle \mathfrak{M}, u \rangle$ such that $\mathfrak{M}$ does not satisfy PAM. Hence, there exist $p > p'$, $p, p' \in P$ and $a > a'$, $a, a' \in A$ such that $a' \in \mathfrak{M}(p)$ and $a \in \mathfrak{M}(p')$. Stability requires that neither $\langle p, a \rangle$ nor $\langle p', a' \rangle$ can profitably deviate. Hence, we need

$$\begin{aligned} \psi(a, p, u(p)) &\leq \psi(a, p', u(p')) \qquad (9.1) \\ \& \; \psi(a', p', u(p')) &\leq \psi(a', p, u(p)), \qquad (9.2) \end{aligned}$$

Applying $\phi(p, a', \cdot)$ to both sides of the second inequality we obtain

$$\phi(p, a', \psi(a', p', u(p'))) \geq u(p). \qquad (9.3)$$

since by feasibility of the equilibrium $u(p) \leq \phi(p, a', 0)$. Apply now $\phi(p, a, \cdot)$ to both sides of the (9.1) ;by definition of the inverse map we have[13]

$$\begin{aligned} u(p) &\geq \phi(p, a, \psi(a, p', u(p'))) \text{ if } u(p) \leq \phi(p, a, 0) \qquad (9.4) \\ \phi(p, a, 0) &> \phi(p, a, \psi(a, p', u(p'))) \text{ if } u(p) > \phi(p, a, 0) \end{aligned}$$

and therefore combining (9.3) and (9.4) we obtain

$$\phi(p, a', \psi(a', p', u(p'))) \geq u(p) \geq \phi(p, a, \psi(a, p', u(p'))).$$

But (GID) requires that $\phi(p, a', \psi(a', p', u(p'))) \leq \phi(p, a, \psi(a, p', u(p')))$, therefore

$$u(p) = \phi(p', a, \psi(a', p', u(p'))) = \phi(p, a, \psi(a, p', u(p'))),$$

[12] For one sided matching problems, the proof of sufficiency of GDD for NAM is slightly more involved than for sufficiency of GID for PAM. See our working paper Legros and Newman (2003).

[13] By feasibility $u(p) \leq \phi(p, a', 0)$; however since we do not assume that ϕ is increasing in its second argument, we do not know whether $u(p)$ is greater or lower than $\phi(p, a, 0)$.

and therefore $u(a) = \psi(a, p, u(p))$, $u(p) = \phi(p, a, u(a))$, $u(a') = \psi(p', a', u(a'))$ and $u(p') = \phi(p', a', u(a'))$. That is the matches $\langle p, a \rangle$ and $\langle p', a' \rangle$ are stable when payoffs are u.

If in an equilibrium $\langle \mathfrak{M}, u \rangle$ matches $\langle p, a' \rangle$ and $\langle p', a \rangle$ form, GID implies that the equilibrium payoffs $(u(p), u(a))$ are on the Pareto frontier for coalition $\langle p, a \rangle$ and payoffs $(u(p'), u(a'))$ are on the Pareto frontier for coalition $\langle p', a' \rangle$.

We have showed that under (GID), if in equilibrium there are two matches that violate PAM, it is possible to reassign partners in order to obtain PAM without modifying payoffs and therefore without violating stability. This falls short however of showing that starting from an equilibrium in which $\mathfrak{M}$ does not satisfy PAM we can find a payoff equivalent equilibrium satisfying PAM.

The next part of the proof shows a simple algorithm to construct a new equilibrium $\langle \mathfrak{M}', u \rangle$ where $\mathfrak{M}'$ satisfies PAM. It is easier to develop the argument in reference to agents rather than types. Let I and J be respectively the sets of agents of types in P and in A. To simplify the exposition, assume that the sets of agents are finite and let N and M be the cardinalities of I and J. Then, we can write $I = \{i_k, k \in \{1, 2, \ldots, n\}\}$ and $J = \{j_k, k \in \{1, 2, \ldots, m\}\}$. Higher indexes correspond to lower values of the characteristic, that is agent i_k has type p and agent i_l has type p' where $p \geq p'$ if and only if $k \leq l$.

We show that under GID, any equilibrium $\langle \mathfrak{M}, u \rangle$ is payoff equivalent to the equilibrium where type payoffs are given by u and where the matching is such that for agents, the match of $i_k \in I$ is $j_k \in J$, whenever $k \leq \min(n, m)$.

Suppose that in the matching of the equilibrium $\langle \mathfrak{M}, u \rangle$ there exists an agent $i_k \in I$ who is matched with $j_l \in J$, where $l \neq k$. Let k be the first time this situation arises. Hence for all $k' < k$, the match of $i_{k'} \in I$ is $j_{k'} \in J$. Let j_l be the match of i_k and i_r be the match of j_k; by construction if i_k has type p and j_k has type a, j_l has type $a' \leq a$ and i_r has type $p' \leq p$. By our previous claim, we can match i_k with j_k and i_r with j_l, keep the same payoffs for the four agents without violating feasibility or stability. Proceeding in this fashion we can reassign the agents in such a way that i_k is matched to j_k for all $k \leq \min(n, m)$. Call $\mu : I \to J$ this matching : $\mu(i_k) = j_k$ if $k \leq \min(n, m)$,

$\mu(i_k) = \varnothing$ if $k > \min(n, m)$.

Define a subset I' of I to be *connected* if $i_k \in I'$ and $i_{k+l} \in I'$ imply that $i_t \in I'$ for all $k \leq t \leq k + l$. If I_0 and I_1 are subsets of I, write $I_0 < I_1$ if for all $i_k \in I_0$ and $i_l \in I_1$, $l > k$. Similar definitions apply to subsets of J. By construction, to the set of types P corresponds a partition $[I(p), p \in P]$ of I such that for each p, $I(p)$ is connected, similarly to the set of types A corresponds a partition $[J(a), a \in A]$ of J where each element is connected. Consider $p > p'$, by our construction, $I(p) < I(p')$ and by definition of μ, $\mu(I(p)) < \mu(I(p'))$; that is agents in $\mu(I(p))$ do not have lower type than agents in $\mu(I(p'))$. It follows that the matching correspondence on types induced by μ satisfies PAM. This proves sufficiency of GID for PAM.

9.3. Proof of Proposition 8

Note that condition (7.2) requires strict preferences. Let $p > p', a > a'$. Matches (p, a'),(p', a) are not stable under (7.2): if $h(p, a) > h(p, a')$ and $f(a, p) > f(a, p')$, p and a should form a match; if $h(p', a') > h(p', a)$ and $f(a', p') < f(a', p)$, p' and a' sould form a match instead. This shows that all equilibria satisfy PAM.

It is not possible to replace the strict inequalities by weak inequalities. For instance, suppose that $h(p, a)$ is strictly increasing in a for all values of p but that $f(a, p)$ is constant for all values of p and a. Then any matching is an equilibrium since agents are indifferent between all partners. However, equilibria are not payoff equivalent to PAM.