The Standard Parts Problem and the Complexity of Control Communication

J. Baillieul & W.S. Wong

Abstract

The objective of the standard parts optimal control problem is to find a number, m, of control inputs to a given input-output system that can be used in different combinations to achieve a certain number, n, of output objectives and to do this in such a way that a specified figure-of-merit measuring the average cost of control is minimized. The problem is especially interesting when m is significantly less than n. Distributed optimization problems of this type arise naturally in connection with recent work on *control communication complexity*. In what follows a general formulation of the standard parts optimization problem is given along with some simple illustrative examples. Control communication complexity is defined, and it is shown how one measure of this complexity naturally leads to a standard parts optimization problem. The entire circle of ideas is explored in the context of quadratic optimal control of the Heisenberg system, and recent results on computability using simple closed curve inputs are presented.

I. INTRODUCTION

A little over a quarter century ago, a number of researchers became interested in finding a lower bound on the energy required to perform a computation. The efforts sought to understand how computational limits were related to the mathematical logic employed, the way hardware was engineered, and ultimately to the fundamental bounds of thermodynamics. (See, for instance, [Ben1],[Ben2],[Tof] and the engaging monograph [Feyn].) At around the same time, other researchers worked on problems motivated by the optimal design of VLSI circuits, and they

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worked to find the best ways for components to communicate with each other in carrying out computations. A growing body of research on *communication complexity* has emerged following a seminal paper [Yao1] by Andrew C.-C. Yao, and a good introduction to the area is the monograph [K&N]. Generally speaking, papers dealing with communication complexity have been information-theoretic in character and have not directly touched upon the physical aspects of communication. Work on theoretical foundations of the complexity of implementing distributed computing by means of steering a control system has recently appeared in work of Wong, [Wong]. This work has been further developed, with potential connections to quantum computation being suggested in [WongBal1] and [WongBal2]. The essential idea behind what we have called *control communication complexity* is that multiple parties simultaneously but independently provide inputs to a control system whose state is observed to change accordingly. The problems are formulated in such a way that the state changes provide the results of computations using data encoded by the inputs of the parties. The cost of the computation is just the cost steering the control system using the chosen inputs, and in [WongBal1] and [WongBal2], this cost is given by a simple quadratic form integrated over the finite time interval on which the control system is allowed to operate.

The goal of our current work on *control communication complexity* is to understand the way in which the complexity of a computation may be understood of in terms of the physical effort required to carry it out. A very simple illustration of what is involved is provided by the following elementary optimization problems.

Problem 1. Suppose we wish to make a rectangular container of a prescribed depth and volume V. What should be length and width such that the amount of material used is minimized? This can be posed to a beginning calculus class, and the solution is easily shown to be that the length and width should be equal (to the square root of the area of the bottom of the container). $\square$

For people who have thought about elementary optimizations, the solution is intuitively obvious. A problem that is of comparable simplicity but without such an obvious solution involves the optimal design of a slightly more complex rectangular container.

Problem 2. Suppose we wish to make a rectangular container of a prescribed depth and volume and comprising two rectangular chambers, the areas of the bottoms of which are A and B. We formulate the problem of finding the minimum amount of material needed as finding length values ℓ_1, ℓ_2, ℓ_3 such that $\ell_1\ell_2 = A$, $\ell_1\ell_3 = B$, and $3\ell_1 + 2\ell_2 + 2\ell_3$ is minimized. (See Figure 1.) Again elementary calculus yields the solution:

$$\ell_1 = \sqrt{\frac{2(A+B)}{3}}, \quad \ell_2 = \frac{A\sqrt{3}}{\sqrt{2(A+B)}}, \quad \ell_3 = \frac{B\sqrt{3}}{\sqrt{2(A+B)}}.$$

It is somewhat interesting to note that this solution is close to but slightly better than what one would get by simply dividing a square base of area $2A$ in half in the case that $A = B$. $\square$

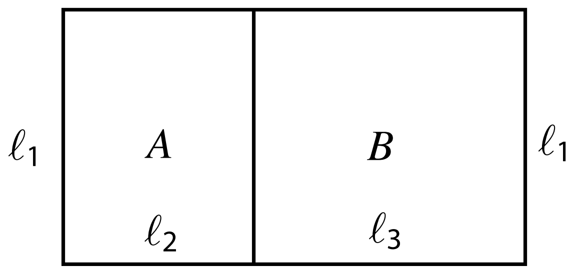

Fig. 1. A rectangular container with two chambers having areas $\ell_1 \times \ell_2$ and $\ell_1 \times \ell_3$ respectively.

Problem 1 is very classical, while Problem 2 is new and points to some essential features of finding minimum communication complexity protocols by solving constrained optimization problems. To develop the analogy, if we think of the enclosed rectangles as messages and the lengths of the sides as the cost of transmitting symbols, the single "message" (=area) of Problem 1 leaves no basis for choosing the length and width to have different lengths. The two separate areas of Problem 2 , however, cause us to choose (via mathematical optimization) shorter lengths for the sides which appear more frequently as boundary segments. If $A = B$, for instance, there are four segments of length $\ell_2 = \ell_3\sqrt{3A}/2$ and three segments of length $\ell_1 = 2\sqrt{A/3}$, and we find the optimal lengths satisfy $\ell_1 = (4/3)\ell_2$. One can also see these problems as highlighting the differences between centralized and distributed optimization. Suppose there are two agents, one of whom is assigned to select the lengths of the top and bottom segments bounding the rectangles, and the other who is assigned to select the lengths of the side segments. Problem 1 does not involve any notion of choice on the part of the agents in that only a single area must be inscribed. The agents minimize the total perimeter length by choosing the same "side-length" policy. In Problem 2, however,the two agents must select segment lengths that reflect the fact that the boundary segments must enclose two separate regions. Each agent chooses segment lengths that in an averaged sense optimally enclose the prescribed areas.

A less elementary class of optimization problems that are useful in formulating optimal information exchange protocols are what we call *distributed optimal steering problems*. In general terms, we consider control systems of the form

$$\begin{cases} \dot{x} = f(x, u), \\ y = h(x). \end{cases} \tag{1}$$

evolving on some state manifold $\mathcal{M}$ of dimension n and in which the dimension m of the input $u = (u_1, \dots, u_m)$ is greater than one. We are primarily interested in a distributed version of the problem in which the control inputs $u_i(\cdot)$ are determined by independent agents who collaborate to achieve specified control objectives. The objectives, finite in number, are expressed as terminal states $h(x_1(T)), \dots, h(x_N(T)) \in \mathcal{M}$.

We can assign a variety of meanings to these terminal states. Two different intepretations are described here.

1. *Generalized standard parts parts optimization problem.* This is a generalization of the problem mentioned in Brockett [RWB4]. The optimal control problem aims to find choices of control inputs $(u_{1,i_1}, \dots, u_{m,i_m})$ where $1 \le i_j \le k_j$, $j = 1, \dots, m$, with each m-tuple $\xi = (i_1, \dots, i_m)$ being associated with a unique control objective $h(x_\xi(T))$ from the list. Obviously, a necessary condition for the problem to be solvable is that $N \le k_1 k_2 \cdots k_m$. The case where N is equal to or close to $k_1 \cdots k_m$ is particularly desirable since the the sum of the number of possible control choices over the components of the inputs, $k_1 + \cdots + k_m$ is less than the number N of goal states. It is also of interest to note that if we represent the tuples, $(\xi, h(x_\xi(T)))$, as rows in an $m+1$ column table, the resulting table is not in the second normal form in the sense of database normalization. This implies that some of the agents do not have any effect on the control outcome. These are issues of potential interest in control design.

2. *Joint terminal state optimization problem.* This is essentially the model propounded in [WongBal1]. Each agent is allowed a finite number of choices known only to the agent. In particular, agent j has choices, $1 \le i_j \le k_j$. Each m-tuple $\xi = (i_1, \dots, i_m)$ identifies a unique utility function which is assumed to possess a unique optimal state, $x_\xi(T)$. The objective is to steer the system to reach such an optimal state.

For both classes of problems, the controls are designed so that each m-tuple $(u_{1,i_1}, \dots, u_{m,i_m})$

steers (1) from a common initial state x_0 to the goal state $x_\xi(T)$ in such a way that

$$\eta = \int_0^T \sum_{j=1}^{k_1} u_{1,j}(t)^2 + \cdots + \sum_{j=1}^{k_m} u_{m,j}(t)^2 \, dt$$

is minimized.

While the m-agent distributed optimization problem is of interest in the context of multiparty control communication complexity, the present paper will treat only the special case of $m = 2$ agents. In this special case, we seek control inputs that will drive an output function $h(\cdot)$ to a set of goal values $\mathfrak{h}_{ij}$ prescribed by an $n_1 \times n_2$ matrix **H**. We are thus concerned with the problem of finding a choice of n_1 scalar inputs u_i and n_2 inputs v_j such that together u_i and v_j steer

$$\dot{x}(t) = a(x(t), u(t), v(t)), \quad y(t) = h(x(t)) \tag{2}$$

from $x(0) = x_0 \in \mathcal{M}$ to $x(T)$ such that $h(x(T)) = \mathfrak{h}_{ij}$ and such that the collective cost

$$\int_0^T \sum_{i=1}^{n_1} u_i(t)^2 + \sum_{j=1}^{n_2} v_j(t)^2 \, dt \tag{3}$$

is minimized. In the following sections, we consider problems defined by binary matrices **H** and periodic inputs. We consider a broad class of systems (2) which have input-output maps that are bilinear in the control inputs. Before providing specifics of the problem formulation, we consider one further elementary example.

Problem 3. Consider the problem of bounding three disjoint rectangles constructed by choosing sides of up to four distinct lengths $\ell_1, \ell_2, \ell_3, \ell_4$ such that the rectangles have prescribed areas A, B, and C and such that the total length of all perimeters is minimized. If there are no restrictions on how the segments of each length can be used to construct the boundaries of the rectangles, then an optimal solution can be shown to require at most three distinct side lengths—$\ell_1 = \sqrt{A}, \ell_2 = \sqrt{B}, \ell_3 = \sqrt{C}$, and $\ell_4 = 0$. The total length of the three perimeters is then given by $4(\sqrt{A} + \sqrt{B} + \sqrt{C})$. If we impose a further constraint that the sides of the rectangles satisfy, say, $\ell_1\ell_2 = A,\ \ell_2\ell_3 = B,\ \ell_3\ell_4 = C$, then no side segment ℓ_k can be zero, and the minimum total length of the perimeters becomes $4\sqrt{(A + 2B)(C + 2B)/B}$. If we impose an additional constraint beyond this that $\ell_4 = \ell_1$, then the optimization problem becomes ill posed, and the only way to satisfy the constraints is to choose $\ell_1 = \sqrt{AB/C},\ \ell_2 = \sqrt{AC/B}$, and $\ell_3 = \sqrt{BC/A}$. The total perimeter length is $4\sqrt{ABC}(A^{-1} + B^{-1} + C^{-1})$ which is greater than $4(\sqrt{A}+\sqrt{B}+\sqrt{C})$ unless $A = B = C$, in which case the values coincide. (Cf. *Problem* 2.) □

II. Background on distributed computation using nonlinear control system dynamics

The above elementary optimization problems serve to motivate the problem in control communication complexity that we pose in terms of input-output system (2). This system is to be cooperatively controlled by two agents—Alice, who is responsible for inputs $u(\cdot)$, and Bob, who is responsible for inputs $v(\cdot)$. Special attention will be focussed on the two-input Heisenberg system (sometimes referred to as the *Brockett Integrator*)

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ vx - uy \end{pmatrix}, \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3. \tag{4}$$

The output function is defined to be

$$h((x(t), y(t), z(t)) = z(t). \tag{5}$$

The optimization problem involves giving Alice m choices $\{u_1, \ldots, u_m\}$ and Bob n choices $\{v_1, \ldots, v_n\}$ such that for any given pair of choices (u_i, v_j) the output (5) of system (4) achieves a prescribed value $z(1) = H_{ij}$ at time $t = 1$ and such that over all possible values of the pairs (u_i, v_j), the average cost

$$\eta = \frac{1}{m}\sum_{i=1}^{m}\int_0^1 u_i^2(t) + \frac{1}{n}\sum_{i=1}^{n}\int_0^1 v_i^2(t)dt \tag{6}$$

is minimized. The set of all target states are enumerated in an $m \times n$ matrix $\mathbf{H} = (H_{ij})$, and associating the cost $\eta_{\mathbf{H}}$ to **H** via (6), we thereby define the notion of *control energy complexity* for the function **H**.

It is not *a priori* assured that an arbitrary $m \times n$ matrix **H** can be computed in a distributed fashion. The problem will become solvable if Alice and Bob can exchange information related to their choices over a communication channel. In the extreme case, one agent can completely disclose the choice made to the other agent and thereby transform the distributed control problem to a centralized one. In general, the agents can control the dynamical system by means of a protocol consisting of multiple *rounds*; each round is composed of a communication phase and a control phase. This concept is adopted from classical communication complexity theory ([K&N]). We can classify distributed control protocols by the number of rounds needed in order to achieve the control target. By convention, we label protocols that do not require any communications as *zero-round protocols*. In this paper, we will focus on zero-round protocols.

III. The energy complexity of distributed computing using the Heisenberg system controlled by simple closed input curves

Consider the set of 2×2 matrices **H** all of whose entries are either $+1$ or -1. There are 16 such matrices; eight of them have rank one and eight have rank two. The rank one matrices have an even number of -1 entries while the rank two matrices have an odd number. We consider the control energy complexity of evaluating such matrices using the input-output system (4) with the inputs restricted to be sinusoids of period 1. Alice and Bob can each choose one of two loops to input to the system:

$$\begin{aligned}\text{Alice}: \quad & u_{A1}(t) = a_1 \sin(2\pi t - \varphi_1) \ 0 \le t \le 1 \text{ choice } A_1,\\ & u_{A2}(t) = a_2 \sin(2\pi t - \varphi_2) \ 0 \le t \le 1 \text{ choice } A_2,\end{aligned}$$

$$\begin{aligned}\text{Bob}: \quad & u_{B1}(t) = b_1 \sin(2\pi t - \psi_1) \ 0 \le t \le 1 \text{ choice } B_1,\\ & u_{B2}(t) = b_2 \sin(2\pi t - \psi_2) \ 0 \le t \le 1 \text{ choice } B_2.\end{aligned}$$

The goal of the optimization is to select values of the parameters $a_i, b_j, \varphi_k, \psi_\ell$, $1 \le i, j, k, \ell \le 2$ such that

$$\eta = \int_0^1 u_{A1}(t)^2 + u_{A2}(t)^2 + u_{B1}(t)^2 + u_{B2}(t)^2 \, dt \tag{7}$$

is minimized.

Remark 1: The geometry of input and state loops. When Alice chooses input u_{Ai} and Bob chooses input u_{Bj}, $(u_{Ai}(t), u_{Bj}(t))$ traces an ellipse in the (u_A, u_B)-plane whose point locus satisfies

$$u^T \cdot M(Ai, Bj) \cdot u = 1, \tag{8}$$

where

$$M(Ai, Bj) = \begin{pmatrix} \frac{1}{a_i^2 \sin^2(\varphi_i - \psi_j)} & -\frac{\cos(\varphi_i - \psi_j)}{a_i b_j \sin^2(\varphi_i - \psi_j)} \\ -\frac{\cos(\varphi_i - \psi_j)}{a_i b_j \sin^2(\varphi_i - \psi_j)} & \frac{1}{b_j^2 \sin^2(\varphi_i - \psi_j)} \end{pmatrix}$$

is a positive definite matrix provided that $a_i b_j \neq 0$ and $\varphi_i \neq \psi_j \mod \pi$. The proof of this is straightforward, noting that

$$\begin{aligned} \begin{pmatrix} u_{Ai}(t) \\ u_{Bj}(t) \end{pmatrix} &= \begin{pmatrix} a_i \sin(2\pi t - \varphi_i) \\ b_j \sin(2\pi t - \psi_j) \end{pmatrix} \\ &= U \begin{pmatrix} \sin 2\pi t \\ \cos 2\pi t \end{pmatrix}, \end{aligned}$$

where

$$U = \begin{pmatrix} a_i \cos \varphi_i & -a_i \sin \varphi_i \\ b_j \cos \psi_j & -b_j \sin \psi_j \end{pmatrix}.$$

It is easy to see, assuming as we do that $a_i b_j \neq 0$, that U is nonsingluar if and only if $\varphi_i \neq \psi_j \mod \pi$. We define $M(Ai, Bj)$ in terms of U by writing

$$M(Ai, Bj)^{-1} = UU^T.$$

This proves the statement with which we began the remark. It is useful to note several other points regarding the curves that the input choices of Alice and Bob trace.

Proposition 1: The area of the ellipse (8) is

$$\pi |a_i b_j \sin(\varphi_i - \psi_j)|. \tag{9}$$

Proof: Working through some lengthy but elementary algebra, one can show that the eigenvalues of $M(Ai, Bj)$ are

$$\frac{a_i^2 + b_j^2 + \sqrt{(a_i - b_j)^2 + 4a_i^2 b_j^2 \cos^2(\varphi_i - \psi_j)}}{2a_i^2 b_j^2 \sin^2(\varphi_i - \psi_j)},$$
$$\frac{a_i^2 + b_j^2 - \sqrt{(a_i - b_j)^2 + 4a_i^2 b_j^2 \cos^2(\varphi_i - \psi_j)}}{2a_i^2 b_j^2 \sin^2(\varphi_i - \psi_j)}.$$

Assuming $\varphi_i \neq \psi_j \mod \pi$, these are both positive real numbers which we may rewrite as $1/A^2$, $1/B^2$ respectively, where $A > 0, B > 0$. By the principal axis theorem, there is an orthogonal change of basis in the (uA, u_B)-plane such that the ellipse (8) may be represented in terms of new coordinates x, y by

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1.$$

The area of this ellipse is πAB, and another elementary but tedious calculation using the above expressions for the eigenvalues, this area is equal to $\pi |a_i b_j \sin(\varphi_i - \psi_j)|$ as claimed. ■

Explicitly integrating (4) given the input choices u_{Ai},u_{Bj}, we find that $(x(t), y(y))$ traces an ellipse in the x, y-plane. As noted by Brockett, the value of z at $t = 1$ is related to this ellipse: $z(1) = 2 \cdot Area$. As above, this area can be explicitly computed, and we find that

$$z(1) = \frac{|a_i b_j \sin(\varphi_i - \psi_j)}{2\pi}. \tag{10}$$

The solution to the problem of finding inputs u_{Ai}, u_{Bj} to compute **H** such that (7) is minimized turns out to depend on the rank of **H**. This dependency is explicitly given by considering two canonical cases

$$\mathbf{H_1} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{H_2} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

These have respectively rank 1 and rank 2. Noting how the output $z(t)$ depends explicitly on the inputs, the constraints imposed by evaluation of the two matrices are given explicitly in the following two tables.

	$B1$	$B2$
$A1$	$a_1 b_1 \sin(\varphi_1 - \psi_1) = 2\pi$	$a_1 b_2 \sin(\varphi_1 - \psi_2) = -2\pi$
$A2$	$a_2 b_1 \sin(\varphi_2 - \psi_1) = -2\pi$	$a_2 b_2 \sin(\varphi_2 - \psi_2) = 2\pi$

Table 1

	$B1$	$B2$
$A1$	$a_1 b_1 \sin(\varphi_1 - \psi_1) = 2\pi$	$a_1 b_2 \sin(\varphi_1 - \psi_2) = 2\pi$
$A2$	$a_2 b_1 \sin(\varphi_2 - \psi_1) = 2\pi$	$a_2 b_2 \sin(\varphi_2 - \psi_2) = -2\pi$

Table 2

Remark 2: On the parametric standard parts problem. That the optimization problem is formulated with respect to inputs that are *a priori* specified to be sinusoids is motivated by classical results (See, e.g. [Bal1].) in which sinusoids emerge as solutions to optimal control problems defined by quadratic cost functionals together with nonlinear evolution equations of the form (4). A more general optimization problem would be to minimize (7) over a larger class of functions—say function assumed to be piecewise analytic on $[0, 1]$. The standard parts problem at that level of generality remains open.

Remark 3: Why simple closed curves are of interest. For inputs of the form we have proposed, the output $z(1)$ is a geometric quantity—namely the area inscribed in the simple closed curve traced in one unit of time by the xy-states of (4). The geometric nature of the output (5) was noted in early work of Brockett, and it implies a certain degree of robustness in solutions to the optimization problems under study. Small amounts of noise and disturbance should have negligible effect on the output. For detailed information on the effect of phase-noise in the input to such systems, we refer to [WongBal1]. We further observe that the proposed loop-based protocol is effectively performing computations by manipulating a Berry's phase. Thus, as noted in [WongBal1], our loop-mediated computations may be useful in understanding computations using quantum spin systems.

In terms of the input loops given in the above parametric form, the problem of minimizing (7) over all loops such that $z(1)$ computes $\mathbf{H}_1$ is equivalent to minimizing $a_1^2 + a_2^2 + b_1^2 + b_2^2$ subject to the constraints in Table 1 being satisfied. The solution is given as follows.

Proposition 2: Given the parametric form of Alice and Bob's loop inputs $u_{A1}, u_{A2}, u_{B1}, u_{B2}$, the corresponding output $z(1)$ of (4)-(5) computes $\mathbf{H}_1$ while minimizing (7) if and only if $\varphi_2 = \varphi_1 + k\pi$ for some integer k and $|\varphi_i - \psi_j| = \pi/2$ for all $i, j = 1, 2$. The optimizing choices of the a_i's and b_j's have $|a_i| = |b_i| = \sqrt{2\pi}$ ($i = 1, 2$) with the choices of sign made to satisfy the constraints of Table 1. The minimizing value of $a_1^2 + a_2^2 + b_1^2 + b_2^2$ is 8π.

Proof: (Sketch) We describe a proof with three parts. *First*, we show the feasibility of the proposed solution. *Second*, we show that the phase variables φ_i and ψ_j must satisfy the stated relations. *Third*, we show that the proposed solution is indeed optimizing.

First step. For any choices of phase variables satisfying the stated conditions, the entries in the matrix $\mathbf{M}_1 = \begin{pmatrix} \sin(\varphi_1 - \psi_1) & \sin(\varphi_1 - \psi_2) \\ \sin(\varphi_2 - \psi_1) & \sin(\varphi_2 - \psi_2) \end{pmatrix}$ are all either $+1$ or -1. One can show that under the assumed relation among phase variables that the determinant of $\mathbf{M}_1$ is zero, and hence $\mathbf{M}_1$ has rank one and hence an even number of negative entries. We can solve the matrix equation

$$\mathbf{H}_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mathbf{M}_1 \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$

for appropriate values of the $a_i = \pm 1$ and $b_j = \pm 1$ to realize the matrix $\mathbf{H}_1$ with the corresponding cost $a_1^2 + a_2^2 + b_1^2 + b_2^2 = 8\pi$ as claimed.

The second and third steps of the proof are combined. First, assume without loss of generality that $\varphi_1 = 0$. Next, we note that satisfying the constraints of Table 1 imposes a relation on the phase variables: specifically $\det \mathbf{M}_1 = 0$. Using the relations of Table 1, we express a_2, b_1, and b_2 in terms of a_1 and the phase variables φ_2,ψ_1, and ψ_2 and we use these to express the optimizing value of a_1 and hence of cost function in terms of the phase variables. The values of φ_2, ψ_1 and ψ_2 that minimize the cost function are exactly those claimed in the proposition. We omit the details of these steps which are lengthy to state and quite similar to what is presented in proving the next result. ■

Proposition 3: Given the parametric form of Alice and Bob's loop inputs $u_{A1}, u_{A2}, u_{B1}, u_{B2}$, the corresponding output $z(1)$ of (4)-(5) computes $\mathbf{H}_2$ while minimizing (7) if and only if $|\varphi_2 - \varphi_1| = \pi/2$, $|\psi_2 - \psi_1| = \pi/2$, and $|\phi_1 - \psi_1| = \pi/4$ or $3\pi/4$. The optimizing choices of the coefficients have $|a_i| = |b_i| = 2^{\frac{1}{4}}\sqrt{2\pi}$ $(i = 1, 2)$ with the choices of sign made to satisfy the constraints of Table 2. The minimizing value of $a_1^2 + a_2^2 + b_1^2 + b_2^2$ is $8\pi\sqrt{2}$.

Proof: As in the previous proof, there is no loss of generality in assuming that $\varphi_1 = 0$. Again we use the relations in Table 2 to express our objective function in terms of a_1 and the phase variables:

$$a_1^2(1 + \lambda^2) + \frac{1}{a_1^2}\mu^2$$

where

$$\lambda = \frac{\sin\psi_1}{\sin(\varphi_2 - \psi_1)} \quad \text{and} \quad \mu^2 = \frac{4\pi^2}{\sin^2\psi_1} + \frac{4\pi^2}{\sin^2\psi_2}.$$

This is minimized with respect to a_1 if

$$a_1 = \left(\frac{\mu^2}{1 + \lambda^2}\right)^{\frac{1}{4}},$$

and in terms of this value, the objective function is expressed as $2\mu\sqrt{1 + \lambda^2}$. This may be written explicitly in terms of the phase variables $\varphi_1, \psi_1, \psi_2$:

$$2\mu\sqrt{1 + \lambda^2} = 4\pi\sqrt{\frac{1}{\sin^2\psi_1} + \frac{1}{\sin^2\psi_2}}\sqrt{1 + \frac{\sin^2\psi_1}{\sin^2(\varphi_2 - \psi_1)}}.$$

We seek to minimize this with respect to $\varphi_1, \psi_1, \psi_2$. The minimizing values of these variables will be the same as those that minimize

$$(\frac{1}{\sin^2 \psi_1} + \frac{1}{\sin^2 \psi_2})(1 + \frac{\sin^2 \psi_1}{\sin^2(\varphi_2 - \psi_1)}). \tag{11}$$

We note that in order for the relations in Table 2 to be satisfied, a relationship exists among the phase variables:

$$\sin \psi_1 \sin(\varphi_2 - \psi_2) + \sin \psi_2 \sin(\varphi_2 - \psi_1) = 0. \tag{12}$$

Based on this, the quantity (11) may be rewritten

$$\frac{1}{\sin^2 \psi_1} + \frac{1}{\sin^2 \psi_2} + \frac{1}{\sin^2(\varphi_2 - \psi_1)} + \frac{1}{\sin^2(\varphi_2 - \psi_2)}. \tag{13}$$

Again using (12), the variable φ_2 can be eliminated, and one can then write (13) in terms of ψ_1, ψ_2:

$$F(\psi_1, \psi_2) = \frac{(\cos(2\psi_1) + \cos(2\psi_2) - 2)^2}{2 \sin^2 \psi_1 \sin^2 \psi_2 \sin^2(\psi_1 - \psi_2)}.$$

Taking partial derivatives and simplifying the resulting expressions, we obtain

$$\frac{\partial F}{\partial \psi_1} = \frac{(-2\cos(2\psi_1) + \cos(2(\psi_1 - \psi_2)) + 1)(\cos(2\psi_1) + \cos(2\psi_2) - 2)}{\sin^3(\psi_1) \sin^3(\psi_1 - \psi_2) \sin(\psi_2)},$$

and

$$\frac{\partial F}{\partial \psi_2} = \frac{(-2\cos(2\psi_2) + \cos(2(\psi_1 - \psi_2)) + 1)(\cos(2\psi_1) + \cos(2\psi_2) - 2)}{\sin(\psi_1) \sin^3(\psi_1 - \psi_2) \sin^3(\psi_2)}.$$

Looking for minimizing values of the phase variables we set

$$\frac{\partial F}{\partial \psi_1} = \frac{\partial F}{\partial \psi_2} = 0,$$

and note that if $\cos(2\psi_1) + \cos(2\psi_2) - 2 = 0$, then it must be the case that both ψ_1 and ψ_2 are integer multiples of π, and if this is the case, it is not possible to realize the values in Table 2. Hence, the minimizing values of ψ_1, ψ_2 must satisfy the simultaneous equations

$$(-2\cos(2\psi_1) + \cos(2(\psi_1 - \psi_2)) + 1) = 0$$

and

$$(-2\cos(2\psi_2) + \cos(2(\psi_1 - \psi_2)) + 1) = 0.$$

These together imply that $\cos(2\psi_1) = \cos(2\psi_2)$, and thus we may assume that either $\psi_1 = \psi_2$ or $\psi_2 = \pi - \psi_1$. If $\psi_1 = \psi_2$, then both these variables would satisfy $-2\cos(2\psi) + 2 = 0$, and hence each would be an integer multiple of π. As note above this is not possible. Hence we must have

$\psi_2 = \pi - \psi_1$, and in this case, we obtain $-2\cos(2\psi_1) + \cos(4\psi_1) + 1 = 0$. The solutions to this equation, assuming $\psi_1 \neq k\pi$ for any integer k are $\psi_1 = \pm\pi/4$ with corresponding values $\psi_2 = 3\pi/4$, $\psi_2 = 5\pi/4$. Because ψ_1 and ψ_2 enter the problem symmetrically, there are also the solutions $\psi_1 = 3\pi/4$ and $\psi_1 = 5\pi/4$ with corresponding values $\psi_2 = \pm\pi/4$. In all cases, we see that the corresponding values of φ_2 prescribed by equation (12) are those claimed in the statement of the Proposition, completing the proof. ■

IV. Evaluating larger binary matrices using the Heisenberg system with simple closed input curves

There are currently a number of open questions regarding the control energy complexity of $n_1 \times n_2$ binary matrices **H** of the form treated in the previous section when n_1 and n_2 are greater than 2. In [WongBal2], a very general results shows this complexity to be the sum of the singular values of **H** up to a scaling. The input loops considered in [WongBal2] are generally not simple closed curves, however, and for binary matrices larger than 2×2 solutions to the control energy complexity problem in terms of simple loops are not presently available. In the present section, we shall discuss two cases.

First consider the problem of two agents (Alice and Bob) using simple loop inputs:

Alice: $u_{Ai}(t) = a_i \sin(2\pi t - \varphi_i), \quad i = 1, 2,$

Bob: $u_{Bj}(t) = b_j \sin(2\pi t - \psi_j), \quad j = 1, 2, 3.$

to evaluate a 2×3 matrix **H** whose entries are either $+1$ or -1 as in the previous section. There are 64 such matrices, 48 of which have rank 2 and 16 of which have rank 1. As in the previous section, we consider two canonical forms for the rank 1 and rank 2 cases:

$$\mathbf{H}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{H}_2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

To find the control energy complexity associated with each of these, we simplify notation by writing $s_{ij} = \sin(\varphi_i - \psi_j)$. Then the optimization problem may be stated as that of minimizing $a_1^2 + a_2^2 + b_1^2 + b_2^2 + b_3^2$ subject to $ASB = \mathbf{H}_i$ where

$$A = diag\{a_1, a_2\}, \quad S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \end{pmatrix},$$

$$B = diag\{b_1, b_2, b_3\}.$$

We omit the details of the solutions, but these can be found using essentially the same arguments as employed in the previous section. The result is:

Solution 1: For $\mathbf{H}_1$: $\varphi_2 = \varphi_1$ (mod) π. ψ_1, ψ_2, ψ_3 are chosen so that $|\varphi_i - \psi_j| = \pi/2$ for all i and j. The optimizing values of the coefficients satisfy $|a_i| = |b_j| = \sqrt{2\pi}$ for $i = 1, 2; j = 1, 2, 3$. The solution of the optimization problem for $\mathbf{H}_1$ is thus 12π.

Solution 2: For $\mathbf{H}_2$: φ_1 and φ_2 satisfy $|\varphi_2 - \varphi_1| = \pi/2$, and $|\varphi_i - \psi_j| = \pi/4 \ \text{ or } \ 3\pi/4$. The optimizing values of the coefficients satisfy $|a_i| = |b_j| = 2^{\frac{1}{4}}\sqrt{2\pi}$ for $i = 1, 2; j = 1, 2, 3$. The solution of the optimization problem for $\mathbf{H}_2$ is $12\pi\sqrt{2}$.

Finally, we come to the case of 3×3 binary matrices. There are 512 of the prescribed form, 192 of which have rank 3 and 288 which have rank 2, and 32 that have rank 1. Although a simple dimension count indicates that the optimization problem could be well-posed, the constraints on the phase variables rule out the possibility of evaluating a rank three binary matrix of the type considered above. This is expressed in the following.

Theorem 1: There is no set of simple closed-curve inputs to the Heisenberg system (4) such that the output (5) evaluates a rank 3 3×3 binary matrix.

Proof: The evaluation of a rank 3 3×3 matrix $\mathbf{H}$ whose entries are either $+1$ or -1 by two agents (Alice and Bob) using (4) with simple closed curve inputs:

Alice: $u_{Ai}(t) = a_i \sin(2\pi t - \varphi_i), \quad i = 1, 2, 3$ and

Bob: $u_{Bj}(t) = b_j \sin(2\pi t - \psi_j), \quad j = 1, 2, 3.$

is equivalent to satisfying the equation

$$ASB = \mathbf{H},$$

where

$$A = diag\{a_1, a_2, a_3\}, \quad S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \quad B = diag\{b_1, b_2, b_3\}.$$

This equation can only be satisfied if S has rank 3 (i.e. is nonsingular). It will be argued that this cannot be the case. To analyze whether S is singular, where as above the entries are $s_{ij} = \sin(\varphi_i - \psi_j)$, there is no loss of generality in assuming $\varphi_1 = 0$. The first row of S is thus $(-\sin\psi_1, -\sin\psi_2, -\sin\psi_3)$. In the second and third rows, expand $\sin(\varphi_i - \psi_j) = \sin\varphi_i \cos\psi_j - \cos\varphi_i \sin\psi_j$. Add $-\cos\varphi_2$ times the first row to the second row, and add $-\cos\varphi_3$

time the first row to the third. The resulting matrix has the same rank as S and has second row: $(\sin\varphi_2\cos\psi_1, \sin\varphi_2\cos\psi_2, \sin\varphi_2\cos\psi_3)$ and third row: $(\sin\varphi_3\cos\psi_1, \sin\varphi_3\cos\psi_2, \sin\varphi_3\cos\psi_3)$. These are scalar multiples of each other, and hence the matrix is singular. This shows S was singular to begin with. This proves no rank 3 binary matrix **H** can be realized. ■

V. Concluding remarks

We have introduced the *standard parts* optimization problem and shown how such problems arise in the study of control communication complexity. We have considered problems of this type that arise in distributed choices of closed-curve inputs that steer the Heisenberg system in such a way that the output computes prescribed binary functions. Current research continues to study the scope of classes of functions that are computable using this system with zero-round protocols. Future work will be aimed at multi-round protocols and how these compare with zero-round protocols having non-simple loop inputs (such as those defined by Fourier polynomials in [WongBal2]). Current research is also aimed at understanding the control communication complexity of motion-based communication protocols for mobile robots. (See [R&B1],[R&B2],[R&B3].)

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