What Mathematics Education Can Learn from Art: The Assumptions, Values, and Vision of Mathematics Education

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> *"It is the function of art to renew our perception. What we are familiar with we cease to see. The writer shakes up the familiar scene, and, as if by magic, we see a new meaning in it."*
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> —Anaïs Nin (as cited in Cohen-Cruz, 2005, p. 86)

ABSTRACT

Eisner (2002) proposes that educational challenges can be met by applying an artful lens. In this article, I draw from Eisner's proposal to consider the assumptions, values, and vision of mathematics education by theorizing mathematics curriculum as an art form. Specifically, I argue that by conceptualizing mathematics curriculum (both in written and enacted forms) as stories, the mathematical lessons experienced by students can be artfully crafted to inspire wonder or grab attention through surprise. An example of a mathematical story from a Grade 7 mathematics textbook is presented and discussed. By framing mathematical content in narrative terms, I seek to enable a rewriting of mathematical experiences for learners.

INTRODUCTION

The *familiar* scene in most mathematics classrooms in North America is, sadly, not a stimulating one. Although several curricular reforms have largely resulted in revised content, mathematics as commonly experienced by students continues to be uninspiring and dull. Unfortunately, there is little evidence of improvement; what was referred to as "dry as dust" over 30 years ago (Davis & Hersh, 1981, p. 169) has more recently been described as "monotonous" (Sinclair, 2005, p. 1) and "flat-lined" (Gadanidis & Hoogland, 2003, p. 489). These characteristics—and the ramifications for students—are eloquently captured by Allen-Fuller, M. Robinson, and E. Robinson (2010), who quoted an anonymous prospective teacher after observing a Grade 10 mathematics class:

> I don't know how they stand it. I couldn't stand it, and I wasn't even there all day. The lessons are so monotonous it makes you crazy. They just sit there and sort of listen or take notes or respond to the same leading, empty questions. I just thought with the *Standards* and all the new technology and everything that it would be better than when I was in school. But it's not. And yet, it absolutely has to be. *We can't just keep doing the same old thing*. (p. 231)

Clearly, as far as most mathematics courses are concerned, this scene needs rewriting. In this article, I suggest that by perceiving mathematics as a form of art, mathematics teachers and curriculum designers can "renew our perception" (Nin, as cited in Cohen-Cruz, 2005, p. 86) and avoid "doing the same old thing" (Allen-Fuller et al., p. 231).

Elliot Eisner, a pioneer in arts education, would likely agree that the arts could inform a re-imagining of mathematical experiences in classrooms. In a moving speech entitled: "What Can Education Learn from the Arts about the Practice of Education?" Eisner (2002) proposed that taking an artful approach to education could improve its quality and lead to a new vision for teaching and learning:

> I am talking about a culture of schooling in which more importance is placed on exploration than on discovery, more value is assigned to surprise than to control, more attention is devoted to what is distinctive than to what is standard, more interest is related to what is metaphorical than to what is literal. (p. 16)

In describing this vision, Eisner (2002) declares his preference for *aesthetic* ways of knowing and learning. (An explication of the term, *aesthetic*, as used in the context of mathematics, is provided in a later section of this article.) Eisner challenges traditional assumptions regarding the planning of curriculum through the establishment of fixed end goals for which instructional means are then developed. He points out that artists develop and achieve their ends through their means, exploiting emerging and unexpected opportunities along the way. His vision of education as an art inspires new questions that serve to address some of its problems including: "How can the pursuit of surprise be promoted in a classroom? What kind of classroom culture is needed?" (p. 11).

Although the arts formerly enjoyed a central role in the educational vision of the United States, beginning in the 19th century, the educational perspective became increasingly more psychological, and as a result, more reliant on the scientific. The effect, Eisner (2002) laments, was that as problems with teaching and learning were identified, science became the de facto lens through which they were addressed. In his words:

> In the process [of psychologizing perspectives on teaching and learning], science and art became estranged. Science was considered dependable; the artistic process was not. Science was cognitive; the arts emotional. Science was teaching, the arts required talent. Science was testable; the arts were

> matters of preference. Science was useful; the arts ornamental. It was clear to many then, as it is to many today, which side of the coin mattered. As I said, one relied on art when there was no science to provide guidance. Art was a fallback position. (p. 6)

Eisner (2002) connects this shift toward science as an effort to "create order, to tidy up a complex system, to harness nature, so to speak, so that our intentions can be efficiently realized" (p. 7). However, as attractive as this goal may be, part of the problem with the mathematics classes described earlier may be that they are *too* tidy. Coming to understand a mathematical idea is generally not a tidy affair, as evidenced by written accounts of mathematicians. While formal mathematical texts (such as textbooks or proofs) may communicate in tidy ways, the human process of getting to understand the mathematical ideas in order to harness them is fraught with complications, dead ends, and happy accidents, as will be further explicated later in this article.

In fact, I propose that mathematics classrooms in which the work of teachers and students is *tamed* and *harnessed* are at their core something other than mathematical. Unfortunately, the mathematical work of building an understanding of mathematical ideas through the solving of emergent and interesting problems is never *efficiently realized* in most mathematics classrooms. Thus, Eisner (2002) suggests that turning toward science, with its drive toward uniformity and tidiness, will have limited potential value in helping to achieve his vision. He wrote,

> What we can do is to generate other visions of education, other values to guide its realization, other assumptions on which a more generous conception of practice of schooling can be built. That is, although I do not think revolution is an option, ideas that inspire new visions, values, and especially new practices are. (p. 8)

Inspired by Eisner's (2002) intentions, I explore his triad of vision, values, and assumptions in relation to mathematics education. I start by challenging current *assumptions* about mathematics by describing what an artful, and thus aesthetic view of mathematics can look like. I then share my personal experience as a learner and teacher of mathematics to consider the *values* of mathematics education with respect to the current realities of teaching. Next, I draw from Eisner's artful *vision* for education to describe a new vision for mathematics curriculum by interpreting it as a form of art. Specifically, I present an interpretation of the mathematical content of curriculum through the lens of literature, that is, as a mathematical story. In the concluding section, I consider how artful interpretations of mathematics as stories can perhaps inspire new practices and thus offer a new scene for students.

Interpreting mathematics as a story repositions mathematics curriculum from an instruction manual or a collection of facts to a form of art, intentionally crafted to offer aesthetic experiences for a set of students, whether positive or negative. As Anaïs Nin (as cited in Cohen-Cruz, 2005, p. 86) suggests, this repositioning offers the possibility of renewing our perception of the familiar mathematics educational form and hopefully, allows us to recognize and describe previously hidden qualities. My goal, at least in part, is to *shake up* conventional ways of viewing mathematics curricula as conveyors of content and aims, and to invite the imagining of rich new mathematical stories. If a novel can be appreciated for engaging characters or sudden surprises, why not a mathematics lesson that is experienced by students in a classroom?

ASSUMPTIONS REGARDING THE AESTHETIC DIMENSIONS OF MATHEMATICS

The aesthetic dimensions of mathematical teaching and learning have generally been ignored in research and theory (Sierpinska, 2002), perhaps because mathematics has been taught in such a sterile manner for centuries despite curricular reform efforts. To some, the monotony may even seem appropriate; the domain of mathematics appears to be the epitome of order and is highly valued for its precise syntax and carefully deductive structure of logical claims (Leron, 1985). Yet when the work of mathematicians is carefully scrutinized, a different scene plays out (Davis & Hersh, 1981; Hofstadter, 1992; Sinclair, 2006). The *aesthetic*, in part, guides decisions, motivates the pursuit of a particular line of inquiry, and helps mathematicians sense the correctness of a result before embarking on a proof (Sinclair, 2001). Because of this disconnect between the aesthetic experienced by those who study mathematics for pleasure (i.e., mathematicians) and that experienced by most students, in the next section I discuss the potential of the mathematical aesthetic of classrooms to challenge prevailing assumptions about mathematical teaching and learning and support a new vision for learning mathematics.

Eisner (2002) draws from Dewey (1934) who describes aesthetic as an individual's response to an experience rather than an attribute of an object. Dewey's framing helps explain how an object that is viewed as aesthetically pleasing to one individual will have the opposite effect on another (or even the same individual in different circumstances). Although the term aesthetic is most often applied to works of art, and therefore associated with notions of static beauty, Dewey notes that it is not an object's attribute but the individual's perception and interaction that is the locus of aesthetic. Dewey also suggests that aesthetic is contextual; it is the effect on an individual in the particular context that makes an object or event moving (or not), and this effect is generated as the individual takes in and makes something of the experience. Framed in this way, in any experience, aesthetic is both the force compelling one to move toward or away, to push forward or against. Therefore, Dewey explains, whenever we are compelled or repelled, it is this effect that makes up the aesthetic dimensions of experience.

With Dewey's (1934) framing as a guide, what can be said of the aesthetic of mathematical experiences that will inform a new vision for the teaching of mathematics? Mathematical aesthetic can generally be understood to be an individual's response to a mathematical experience, such as a sense of fit of a possible pattern or insight into an underlying structure of a particular problem (Sinclair, 2001). In terms of exploring mathematics, then, aesthetic involves the sensing of value and truth and thus is part of what it means to think and understand mathematically (Burton, 1995, 1999). This includes being drawn toward a potential problem-solving strategy or sensing the correctness of a result. Thus, aesthetic is an essential dimension of any mathematical experience whether positive (i.e., stimulating) or negative (i.e., dull).

Throughout mathematical experiences, both professional and educative, an individual senses conditions (e.g., whether something feels complete or not) and is moved to affect them through mathematical means. These *movements* in mathematical work, while possibly very abstract, can feel embodied in the individual engaged in the mathematical experience, such as creating or maintaining balance (Lakoff & Núñez, 2000). Despite the common assumption that mathematics is a strictly rational pursuit, a mathematical experience can be a transformative, compelling enterprise of impulses and anticipation (see Hofstadter, 1992). Aesthetic is the motivating influence that can advance an individual (mathematician or student) through challenges and setbacks and dissuade the person from giving up.

How might a mathematics classroom take advantage of the potential aesthetic opportunities that mathematics offers? Sinclair (2001) defines "aesthetically-rich" learning environments as those that "enable children to wonder, to notice, to imagine alternatives, to appreciate contingencies and to experience pleasure and pride" (p. 26). This implies that "they legitimise students' expressions of innate sensibilities and subjective impressions—they 'work with' such perceptions rather than exclude or deny them" (p. 26). This aesthetically-enhanced mathematics classroom includes multiple dimensions of mathematical experiences, such as connectivity, risk-taking, engaged imagination, sensory experiences, perceptivity, and activity (Uhrmacher, 2009).

Thus, creating *aesthetically-rich* mathematical classrooms requires *untidying* the mathematical experiences of students. Responding to Eisner's (2002) call for a new vision of mathematics education will require new assumptions of the aesthetic opportunities and potential in mathematics learning. It necessitates shifting our primary focus from educational outcomes to moving experiences that have the potential to compel one toward an end. How might mathematical experiences for students be *rewritten* to allow for this possibility?

THE VALUES OF MATHEMATICS EDUCATION: A PERSONAL NARRATIVE

Eisner's (2002) call for education to embrace the art of learning and teaching resonates with my personal experience of learning mathematics and my goals for teaching mathematics. It was largely the disconnect between my love and passion for mathematics and the forces that began to broadly shape mathematics curricular decisions that helped to shift my perception of mathematics curriculum to an art form. In this section, I offer my personal narrative of the values of mathematics education in the hope that it will further ground the vision I describe in the realities of mathematics classrooms and the work of mathematics teachers.

For as long as I can remember, I have been intensely interested in mathematics and grew up seeking mathematical experiences outside the classroom. I sought out mathematical puzzle books (my favorite section of the bookstore) and spent multiple evenings on the same challenging problem or puzzle. From these experiences, I became aware that a problem or puzzle could grab my attention, raise larger questions, require me to renegotiate the way I understood a mathematical idea, and invite me to anthropomorphize mathematical objects. In a very real sense, the numbers and shapes I encountered were friends and enriched my daily experience as much as color or music.

I continued my study of mathematics in college and decided that I wanted to share this passion with students. When I began teaching mathematics, I was fortunate to find a position at an inner city high school in San Francisco that allowed me to engage with students in what I now think of as *mathematical adventures*. These adventures, which are a contrast to the typical mathematics lessons described at the beginning of this article, were often sparked by a student's question or an unexpected result, and usually no one knew how the adventure would end until we jointly arrived there. I aimed not only to help students learn content, but also stimulate their (and my own) interest and enthusiasm for the possibilities ahead of us. It was through working with these students that I recognized that mathematics could be understood in many different ways and learned that the students' experience and anticipation for what was to come were equally as important as my learning objectives.

Early on, I remember working with colleagues on curricular questions that I now interpret as trying to recognize (read) the mathematical story of our textbook as well as develop new experiences with mathematics, questions such as: How does this mathematical idea develop? How could I make this more mathematically inspiring? What if I changed the order of the parts of text, such as lessons or even chapters? At this time, the curricular focus was not only on making learning stimulating and enjoyable, but making *mathematics* stimulating and enjoyable. Curriculum choices were made in relation to overall development, and students' understanding was viewed in relation to where we were in the broader story.

My changing view and thus my use of the mathematics curriculum, required more than *following a textbook*, and soon I was critiquing the design of my lessons, asking questions such as: What could help make the focus become important to the student, so that it's the student's question and not my own? I relished designing experiences that would enable students to gasp in surprise at the mathematics and beg further exploration. Not that this would always happen, but the aim was always there. I wanted students to experience mathematics in a way that showed that it could be exhilarating, surprising, captivating, and full of wonder (see Sinclair, 2001). This commitment eventually led me to write mathematics textbooks for middle and high school, which allowed me to re-imagine what a curriculum could offer teachers and students.

I began to notice how different parts of a story could affect what came before and what came after. I also became much less interested in replicating existing curricular sequences (assuming, for example, that an algebra course had to start with a review of arithmetic) and recognized that, though not unbounded, there is great flexibility in the ways mathematical ideas can be sequenced. In the next section, I provide an example of a *mathematical story* from one of these textbooks (see Figure 4).

Between 1995 and 2005, the constraints on teaching mathematics in the U.S. underwent a dramatic change. The curricular freedom that enabled me to create inspiring mathematical experiences for my students vanished. What little public attention had been given to the aesthetic of mathematics learning and teaching evaporated. In conversations with mathematics teachers throughout the country, both colleagues and those I met at conferences and professional development sessions, it seemed that as a result of newly mandated learning expectations (e.g., state standards and district curriculum guides), content outcomes became the dominant theme. Sadly, I believe that if I had started teaching today, I do not think I would have had the curricular space and freedom to *play* with the aesthetics of my lessons.

Although discussions of teaching included pedagogical considerations, particularly with the influence of the NCTM standards (National Council of Teachers of Mathematics [NCTM], 2000), I became concerned that the content sequence and particularly its potential aesthetic opportunities for students, was rarely part of the discussion. I wondered why many of the teachers I met through professional development, who had earned my respect, rarely questioned their textbooks. There was often an ambivalence regarding a choice of sequence, as though some thought that: As long as the content is there, designed in a way I prefer, and covers the material before the high-stakes assessment, who cares if Chapter 1 starts with X instead of Y? Even when I tried to convince my colleagues that mathematical sequence is worthy of attention, I realized how difficult it was to discuss. There are few ways of conceptualizing and describing the nuances of mathematical changes throughout a lesson, let alone a unit or chapter, particularly with regard to dimensions on which I had relied so much in my textbook design work. Any attempt seemed only to scratch the surface, recognizing only *sequence* but not its *consequence*.

Through all this, I have come to suspect that the void in terminology describing how mathematical experiences compel or repel may be contributing to the current lack of attention on the aesthetic dimensions of mathematics curriculum. To address this lack of vocabulary, I assert that the mathematics education community needs to look outside our field for ways to recognize how mathematical content can inspire wonder and stimulate curiosity. To build this new vision for what mathematics curriculum can be, I turn to the arts—and specifically literature—for new language.

A NEW VISION FOR MATHEMATICS EDUCATION: MATHEMATICS AS NARRATIVE

It is reasonable to ask why I look to literature rather than other forms of artistic expression. Although mathematics is often described as the opposite of artistic expression, some casual connections can be identified. For example, geometry, with its focus on shape, can readily be connected to the visual arts such as drawing and painting (e.g., Escher, Mondrian), sculpture, and architecture. In addition, the manner in which a lesson is *composed* and *performed* by a teacher and students can be interpreted as different renditions of the same orchestral score (see Brown, 2009). The way in which the different strands of mathematics (e.g., measurement, number, shape, and algebra) are interconnected throughout curricula can be viewed as a woven art. Since mathematics exists to describe the world around us, it is likely that nearly every visual or performing art offers potential metaphorical language for structural and aesthetic qualities of mathematical experiences.

However, I suggest that stories are particularly suitable as an art form for mathematics curriculum as they are at their root pedagogic in nature. History abounds with examples of stories meant to teach important lessons of life (e.g., parables in the *Bible*). Stories integrate both logic (e.g., Does the story make sense?) and aesthetic (e.g., Does the story move me to continue reading?). While each of the other art forms offer elements of value that draw attention to particular aspects of mathematics, a narrative perspective also combines the temporality (i.e., how a story unfolds) and the message (i.e., the moral of the story) of curriculum. Stories conjure fictional worlds for which truth is self-contained, much like mathematics.

Egan (1988) argues that curriculum broadly, mathematics curriculum included, can be organized as stories and claims that this form can help students make meaning of the content. He offers an example of a mathematical story lesson, where students are told an "historical" story (a king's counselor needs to organize marbles in bowls to count the size of an army), and new content (place value) is required to solve a human problem. In this way, Egan proposes to embed mathematics content in a fictional story problem. However, Egan goes beyond the traditional argument that this strategy motivates because stories with mathematics connect students to the so-called *real world* or how the world is assumed to be. Instead, he argues that by drawing from the historical origins of the mathematical content (i.e., the problems the content was developed to solve), mathematical stories may engage students with imagining how the world once *might have been*.

However, Egan's (1988) notion of a story assumes that characters are human and the setting is the historical past in our world. In this vision, mathematics curriculum would appeal to students through an historical perspective. However, what would it mean to a reader if a mathematical story is not about fictional humans, but instead about triangles or systems of equations? How could the content of mathematics lessons, such as derivatives or arithmetic expressions, be viewed as a story?

A different approach was taken by Netz (2005), a mathematics historian, who used narrative as a way of comparing the different aesthetic dimensions of the content organized within ancient Greek mathematical texts. For example, he points out that Euclid, in *Elements*, offers an incremental sequence of layering propositions in a way that allows a reader to advance in a smooth and predictable cadence. In comparison, Netz shows that his reading of Archimedes' *Sphere and Cylinder* was anything but smooth and predictable. With puzzling turns and surprising conclusions, Archimedes offers a different mathematical experience. To explain this perspective, Netz asserts:

> The concept of "narrative" applies almost directly to mathematics, in that mathematical works—just like many other works of verbal art—tell a story: they have characters, and our information about the characters gradually evolves. (p. 262)

Therefore, the work of Netz suggests that attending to mathematics as a "verbal art" could offer insight into how mathematics curriculum might work to inspire and transform the reader/student. If mathematics curriculum is interpreted as a story, what might that look and feel like? What do we learn?

Sinclair (2005) offers an instructive strategy. In an argument that mathematics textbooks are unnecessarily devoid of inspiring and thought-provoking experiences for students, she proposes that mathematics texts can be read as drama using the effect of a Greek chorus. To demonstrate, Sinclair shares her dramatic reading of a proof of the irrationality $\sqrt{2}$ of as seen in Figure 1, offering interpretations such as those in Figure 2.

Figure 1. Sinclair's (2005) Proof of the Irrationality of $\sqrt{2}$

Suppose $\sqrt{2}$ is not irrational
Then $\exists\, p, q \in \mathrm{N}$ such that $p/q = \sqrt{2}$ and $(p, q) = 1$
So $p^2 = 2q^2$,
Then $2 \mid p^2$,
And $2 \mid p$.
Therefore, $p = 2r$, $r \in \mathrm{N}$
So $2q^2 = (2r)^2 = 4r^2$ and $q^2 = 2r^2$
Then $2 \mid q^2$,
And $2 \mid q$.
Contradiction, since p and q were supposed to be relatively prime, $\therefore$ There does not exist p and $q \in \mathrm{N}$, $(p, q) = 1$ such that $p/q = \sqrt{2}$, $\therefore \sqrt{2}$ is an irrational number.

Sinclair's (2005) interpretation is consistent with Bruner's (1996) notion of narrative thinking, which suggests that individuals construct personal narratives when trying to make sense of an event or idea. In this sense, reading a proof is construed as reading for interactions between parts, looking for what connects each statement in a coherent string. The reader is drawn to certain objects with certain properties that are revealed over time. Questions arise that are not immediately answered, but all are posed to make sense of the parts in relation to the whole, that is, what each statement tells us in relation to prior statements and how they relate together to address the larger question (e.g., Is $\sqrt{2}$ irrational or not?).

Figure 2. Sinclair's (2005) Narrative Reading of the Proof of Irrationality of $\sqrt{2}$

Well, if $\sqrt{2}$ is rational, I should be able to write is (*sic*) in a fractional form, as p/q. But if I can write it as p/q, I could also write it as $100p/100q$, and a million other ways, so let me honour my characters by introducing them in their barest form, in reduced form. (narrative reading, para. 1)

I can't get a sense of what that q^2 has to do with anything, so let's write $p^2 = 2q^2$. Aha! Now I'm getting somewhere. This looks nice: nothing is hidden in root signs or in denominators, and p has emerged as the leading actor on the left—the struggle, of p^2/q^2, now has a hero. (narrative reading, para. 2)

So far, so good. Our hero, p, big and natural and even, surefooted and strong. Now what about that dastardly q? Can I rearrange my equation again? Unmask him in his perfidy? Maybe give q a starring role? Hmmm. Even if I write $q^2 = \frac{2}{p^2}$, it doesn't give me anything useful. How can our hero help? Well! since p is even, he can disguise himself as $2r$, where r is some passing natural number. (narrative reading, para. 3)

Admittedly, mathematics curriculum is quite different in form and function from ancient mathematics texts of Euclid and Archimedes and contemporary proofs. Yet mathematics curriculum also has potential ties to narrative. For example, there is evidence that mathematical word problems can provoke the imagination, which might engage the reader in unexpected ways. Pimm (1987) offers the example of author David Roth who described becoming engaged with the details unmentioned in word problems. After being challenged with a word problem involving a discount on a coat, Roth professed wondering about unaddressed questions, such as, "To whom had the haberdasher finally sold the overcoat?" (p. 14). This demonstrates the perlocutionary effects mathematics curriculum can have on a reader, and that, though these vary by reader, they can include the raising of questions as well as the stimulation of imagination.

Mathematics Curriculum as Stories

Beyond word problems, which have an obvious connection with stories, I propose that the mathematical content of lessons found broadly in classrooms and in textbooks can also be interpreted in narrative terms, and that this offers new artistic tools for teachers and new mathematical opportunities for students. In this section, I provide an overview of this artful interpretation of mathematics curriculum before focusing on the implications of this proposal.

Drawing from the narrative theory of Bal (2009), I analyzed a variety of mathematical textbooks from the U.S. and Singapore to develop an interpretation of mathematical sequences that correspond to the dimensions of narrative (Dietiker, 2012). Through this work, I came to recognize mathematical stories as the way in which the mathematical content unfolds for a reader across written or enacted curriculum. That is, a mathematical story is an interpretation of how the mathematical content emerges and changes throughout a sequence of events. Rather than viewing mathematics as a static set of theorems and definitions, this interpretation focuses on the reader's dynamically changing mathematical understanding of the ideas under discussion.

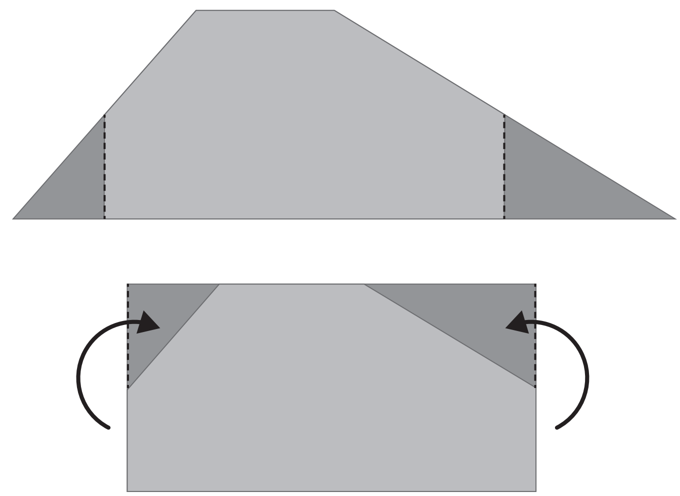
Using this perspective, the characters in mathematical stories can be recognized to be the mathematical objects that emerge explicitly and implicitly during the temporal unfolding of content. For example, an entire lesson in algebra might be about a family of quadratic functions and their relationships. Or a lesson in elementary school may focus on a set of fractions to determine how to order them from smallest to largest. In fact, when viewed this way, an entire course can often be recognized to be about a handful of mathematical characters, such as the numbers 0 and 1. This realization was behind one teacher's note to me about a geometry textbook I coauthored:

> I remember the second time I finished [teaching with a geometry textbook]. On one of the last days of school we found the equation of a circle using a right triangle. At that point I realized that the whole [geometry] book was about a triangle. The better part of this story was when I shared this realization with my students they all said, "We know." (Personal Correspondence)

But just as a literary story cannot be about characters only (for example, one that begins and ends with "Once upon a time, there were three little pigs and a wolf."), a mathematical story must have action(s) that enable(s) change in order to move the story from the beginning to the end. Mathematical action, therefore, can be thought of as what happens to the mathematical characters in the mathematical story. For example, adding 3 and 5 enables a new mathematical object to emerge in the mathematical story (the number 8). Decomposing a trapezoid into parts and rearranging into a rectangle (see Figure 2) enables its area to be calculated. Beyond the operations (i.e., addition, subtraction, multiplication, and division), mathematical actions common in K–12 classrooms include transformations (e.g., rotating, translating, and dilating functions on a coordinate plane), rewriting (e.g., combining like terms or factoring), and composing/decomposing mathematical objects (e.g., "borrowing" 10 from the hundreds place to aid in subtraction). Without mathematical action, the story has no plot.

Another dimension to stories is the setting, where the characters and actions occur in various spaces described by the narrator (Bal, 2009). In mathematical stories a setting can be thought of as the space in which the mathematical characters and actions are found. For example, the mathematical story might occur with symbols on a page or with square tiles on a desk. A linear function might be presented in a table or on a coordinate plane. A mathematical story could play out on a calculator or physically with a student hopping on a number line on the floor. Imagining mathematics curriculum as a story opens up the possibility of reimagining the mathematical activities by changing the setting. Interpreting mathematical content as narrative allows new questions to become driving forces for curriculum planning, for example:

Figure 3. Decomposing and Recomposing a Trapezoid to Find Its Area



- How might deliberate choices of representations of mathematical objects, (e.g., introducing negative numbers in the context of temperature or with two-colored tiles, where one side represents +1 and the other represents –1), affect the notion of negative numbers?
- How do these choices affect later portions of the mathematical story? What changes in the story (and thus, how we come to understand the mathematical ideas) if the mathematical setting for studying addition and subtraction is changed from a number grid (a chart in which integers are arranged in successive rows of 10) to a number line (integers represented in a single line stretching indefinitely in two directions)?

These questions and others enable the focus on key aspects of mathematical study: the mathematical objects and relationships, procedures, and representations.

Mathematical Plot: The Mathematical Impulse of Mathematical Stories

This article opened with a call for the need to rewrite the mathematical stories of our classrooms. Although the ability to reimagine the mathematical objects, procedures, and representations in new ways can open new opportunities for learning mathematics, the initial critical questions are: How might the aesthetic dimension of mathematical stories be understood and improved? How might a story that will move a student to wonder or pursue a particular line of inquiry be conceptualized?

One way a literary story can grab and hold attention is plot: how it temporally reveals and withholds information to provoke a reader to use what is known at each point of the story to ask questions about what is not known (Brooks, 1984; Nodelman & Reimer, 2003). A plot describes the temporal pull of the reader to anticipate what will happen and connects the logical (what is known) with the aesthetic (how it moves the reader). Therefore, rewriting mathematical experiences to *shake up* the familiar scene involves the *mathematical plot* of the mathematical story, creating the tension between what is known and not known by a reader throughout the temporal unfolding of the curricular sequence. In terms of a mathematical story, anticipation occurs when a reader can imagine a future result through the continuation of a pattern or structure, with a vision of closure (e.g., what a solution might look like). This is not to say that all readers will find the same mathematical story aesthetically pleasurable, only that there are mathematical impulses that can drive a reader to actively continue with the story.

However, designing mathematical lessons that allow for student anticipation for what is to come does not mean that *aesthetically-rich* mathematical plots require predictability. To the contrary, it is the *feeling* of anticipation of how the mathematical story *might* play out that matters. In fact, it is when this anticipation turns out to be false that surprise is possible. Plot twists are so named because they represent deviations from an expected path. Literary authors deliberately design sequences of events to enable particular assumptions that later will be revealed as false, and as I'll next show, mathematical stories can be designed similarly.

Consider the following example of a mathematical plot based on a lesson in a Grade 7 textbook, shown in Figure 4, for which I was a co-author (Dietiker, Kysh, Sallee, & Hoey, 2010). This lesson, adapted for length, was designed to introduce the notion of a fair game. To follow the discussion regarding the mathematical story the reader is encouraged to engage in each task in the given sequence.

This lesson can be interpreted as a mathematical story because there is a clear beginning (with a discussion about winning games with strategy) and a sequence of connected events moving the story forward:

1. Discussing what might happen in a game in Task 1
2. Playing the game with friends and analyzing the results in Task 2
3. Playing the game against the teacher with a conjectured strategy in Task 3
4. Changing the game so that it is equally likely for each color to "win"
5. A resolution (learning about the outcome of the game and how to predict the outcome)

The setting of this story is a particular game with given rules played on a given game board, and this particular setting limits all the mathematical actions and characters within the mathematical story. This is a mathematical story about colors, which mathematically can be recognized as locations along a number line from a starting position. The game focuses on the probability of landing on each of the locations with three flips of a coin. Therefore, these positions and the probabilities of landing on each are the mathematical characters of this mathematical story. The mathematical actions of this game are varied but include flipping a coin and moving a marker as well as changing the rules of the game (in Task 4) to make it equally likely that each player can win.

Figure 4. A Set of Tasks Adapted from *Making Connections Course 1*: Lesson 1.1.2

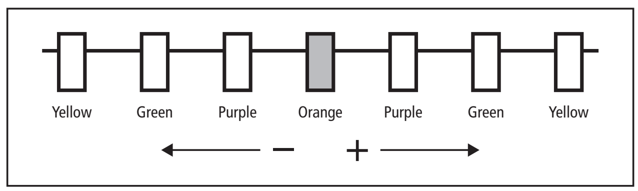
1. Have you ever watched people win a game again and again? Do you think these people just have good luck? Sometimes, winners have a strategy that increases the chance that they will win. How can you develop a winning strategy? Today you will start to answer this question.

   Your teacher will challenge your class to a game of Color-Rama! To play, a marker will be placed on the orange space on the board below. Your class will need to select *one* color for your class and *a different color* for your teacher. Then a volunteer will flip a coin three times. If the coin lands with the "+" showing, the marker will move one space to the right, and if the "–" is showing, then the marker will move one space to the left. If after three flips, the marker is on your class's color, your class wins! If it lands on a color no one picked, then no one wins. Which color should you choose?

   Before you play, discuss the questions below with your team. Then move on to task 2.

   Does it matter which color is chosen?

   Are all the colors equally likely to win? How can you decide?



2. If you want to win, is one color a better choice than the others? Is there a color that you should not pick? To decide, play the game several times and be sure to select a different color each time. What do you notice?

3. Now is the moment you have been waiting for! Play the game with your teacher. As a class, choose a color your class and a color for your teacher that will give you the best chance at beating the teacher.

4. What makes a game fair? Discuss this with your team members.
   a. Is there a way to change the rules of Color-Rama to make it a fair game? Decide on any changes to the rules that you would recommend.
   b. Play the game a few times with your new rules. Be prepared to describe to the class the changes you made and explain your reasons for making the changes.

(Dietiker, Kysh, Sallee, & Hoey, 2010, pp. 10–12; used with the permission of the publisher)

So, what is the mathematical plot of this mathematical story? To answer, the question can be rephrased: How might this mathematical story compel a student/reader in parts of the lesson to be interested in the final outcome? Although any seventh-grade student entering this lesson will bring prior knowledge unique to that individual, it can be assumed that the person recognizes that using a coin to decide if the marker moves to the left or right makes it equally likely that the marker moves in either direction. This recognition is important because it creates the opportunity for the student to make a false assumption: that all locations on the board are equally likely. Even students who take a more analytical stance at the beginning of the mathematical story are potential victims of another faulty assumption; since it is equally likely to move right or left for each flip, the "average" move can be interpreted as not moving at all. That is, each pair of flips can be assumed to have the effect of not moving as the move in one direction can be undone by a move in the opposite direction. In my experience, students with this perspective have come to one of two conclusions: either they decide that "orange" is the best guess (because it represents the center or balance point of the game board) or that "green" is the best guess (because green represents the "average" location on both the right or left of orange because it is found in the middle position between purple and yellow).

Regardless of which faulty assumption a student may make, the student is then able to be surprised when data are collected in Task 2 to show that the marker never lands on green or orange after three flips of the coin! Therefore, it can be expected that key probabilistic questions such as: How is that possible? and Why does that happen? will be generated *by the students* during the mathematical story because of the contradictory evidence that emerges during of the sequence of events. These emergent questions reflect motivation for figuring out the probabilistic nature of this game and the result when students become invested in the outcome of the story.

Therefore, it is important to note that although this mathematical story has many explicit mathematical questions in the form of curricular prompts, the implicit questions: How is this possible? Why does that happen? are not in the text. Yet the mathematical plot relies on these key implicit questions to compel the reader to dig deeper in Task 2 than just making casual observations from the data.

The implicit questions derived from the contradiction also serve other important purposes of this lesson. To answer these questions, students are compelled to reason deductively about the potential outcomes of the game. These questions encourage students to determine and compare the probabilities of landing on the colors in order to "believe" what they "see" in Task 2. The answers to these implicit questions also support reasoning about what it means for a game to be fair, an understanding which supports students when they consider ways to alter the game in Task 4 to affect the probability of winning.

In summary, the mathematical plot of this mathematical story is the increase of tension felt by a reader from Task 1 (when a deceptively simple game is offered) to that felt in Task 2 when the student readers who think they know what will happen in the game are confronted with contradictory evidence that shows them they were deceived. This increased tension between what is *known by the students* (or in this case what is known about what they don't know!) and *what they want to know* (the best color to choose) potentially motivates the student forward in reasoning about the game. This tension remains until there is resolution through deductive reasoning (e.g., proving to oneself that the marker can never land on orange or green after three flips of the coin, and that landing on purple is more likely than landing on yellow) and the game between the students and teacher is finally played in Task 3. Then, in Task 4, students gain a new perspective on the aspect of the game that created the tension, specifically the lack of fairness (i.e., that what appeared to have an equal chance, in fact, did not). The opportunity to alter the game at the end of the lesson allows students to create new potential dramas for future imaginary game players. Thus, the students are invited to *rewrite* the game.

This analysis makes no claims about *who* would read a mathematics textbook this way, but suggests that a mathematics textbook *can be read* this way. This distinction is important because there are multiple factors (e.g., the way textbooks are addressed and used in a classroom) that possibly discourage some readers (i.e., students) from recognizing a sequence of a mathematics text as a mathematical story. Even though some mathematicians and math educators recognize the possibility of mathematical plot, this does not mean that all or even most readers will do the same. It is important to emphasize that the purpose of this way of imagining curriculum is not to explain how students and teachers read mathematics textbooks, but to re-conceptualize mathematics curriculum in a way that can support the curricular design work of teachers and open new mathematical possibilities for students.

"SHAKING UP" THE FAMILIAR SCENE IN MATHEMATICS EDUCATION

Tapping what we know about literary stories offers new curricular insight. For example, consider how an unanswered question about a literary character that is sustained throughout a story can make that story compelling (e.g., Is a character in a familiar story good or evil?). How might this comparison inspire new types of mathematical stories centered on the development of mathematical characters? The framing of mathematical characters foregrounds new questions about mathematical objects within curriculum that can lead to aesthetic opportunities: What type of mathematical characters might provoke mystery? What qualities of mathematical characters could remain in question for sustained portions of the curriculum? In what ways might a mathematical object offer surprise, such as the seemingly uninteresting quadratic $x^2 - 6x + 3$, which has a central role in a mathematical story in Dietiker (in press)? How might a story be based on a group of related mathematical objects for example, a pattern of numbers (see Dietiker, 2013)?

To determine what might not be a compelling or coherent narrative in mathematics curriculum, it is instructive to think about what *doesn't* work in literary narratives. Stories that seem to have no point, offer nothing for a reader to anticipate, or are easily predictable are quickly abandoned. Serials sometimes attract a reader's interest at first but then lose it once the reader sees that all the episodes feel the same. When the mathematical stories of mathematics classrooms are analyzed, how many of these qualities are noticed?

The current lack of new ways to talk about curriculum may explain why so many U.S. textbooks offer the same scope and sequence. A survey of most algebra texts will show that the study of linear functions comes before quadratics, solving equations before inequalities, etc. But is this order necessary, or even advantageous? What advantage or disadvantage might there be in changing this order of content? In some cases, I propose, this sequence exists only because this is the way it has always been done. However, other orders may enable different mathematical connections to be recognized or may draw attention to particular mathematical relationships in a useful way. Potentially, this new conceptualization of mathematics curriculum enables the exploration of other major (and minor) changes in the mathematical sequence.

However, the potential for curricular reform goes far beyond major or even minor changes of sequence. Just as a fiction writer carefully chooses the moment in a story to introduce a character or to reveal "who done it," so too might a teacher deliberate on the point in a sequence to introduce mathematical objects or reveal important properties or relationships. Similarly, just as placing a setting of a story in a particular locale can offer particular affordances and constraints, so too can the decision to set the study of integers on a number line versus two-colored tiles. Therefore, a potential benefit of this way of interpreting mathematics curriculum is to offer a conceptual foundation on which teachers and curriculum designers make mathematical choices regarding objects and representations.

Importantly, this conceptualization offers new ways to compare stories and describe them. For example, perhaps a mathematical story is found where the resolution is the invention of a new mathematical character (e.g., $\sqrt{2}$). What other stories might also be described as a "man-hunt," the search for a character with particular qualities? Or, perhaps a story has a particular shape, which could be described as a "character study." What mathematical stories might fall into this category? Future work could examine the mathematical forms of stories and how these different forms are useful and beneficial to students for different goals and purposes.

Once possible genres of mathematical stories emerge and are articulated, new questions can be explored: If we look at the genres of mathematical stories found within a textbook, what type of variation is found? What might be the role of variation of mathematical story genres? Do certain genres lead to improvements in learning or retention by students? Are there forms or genres that are noticeably missing and could lead to the exploration of new mathematical stories? While this framework speaks to the mathematical stories found in textbooks, we might also ask: What mathematical stories are "told" in our classrooms? What mathematical stories do the students perceive?

CONCLUSION

The meaning and effect of sequential temporal experiences have been theorized and rigorously studied in terms of novels and short stories alike but have so far been ignored in regard to mathematics instruction. Although it may be unorthodox to consider mathematical objects and activity in these *novel* ways, conceptualizing the unfolding of mathematical content in a textbook as a mathematical story allows new questions to emerge. With this work, I aim to promote new and deeper curricular understanding of textbooks and inspire teachers and other curriculum designers to imagine and enact new powerful mathematical stories for students. As described in the opening of this article, the mathematical stories in most mathematics classrooms are unfortunately neither inspiring nor compelling. Therefore, paraphrasing Eisner's (2002) point, mathematics education has much to learn from art about constructing compelling mathematical experiences for students.

Herein lies the hope. With this new conceptualization of mathematics comes new opportunities. During my professional development work with practicing mathematics teachers, I am encouraged that this framing of mathematics spurs them to reflect on the types of mathematical stories they have been constructing with and for students in the past. An artful interpretation of mathematics allows both its overall structure and its aesthetic dimension, in textbooks and in the classroom, to be recognized and improved. It provides the opportunity to finally change reality for many students, who are often condemned to mathematical stories that are "the same old thing" (Allen-Fuller et al., 2010, p. 231).

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