The Hierarchy of Definability: An Extended Thesis

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Channeling Alan Turing

Alan Turing had the remarkably prescient insight that understanding the means by which we work with things can be as important as, or even equivalent to, understanding those things. Equally remarkably, he combined a deep understanding of the abstract with pragmatic good sense.

Turing produced beautiful mathematics:

- Simple and unapologetic formulation
- Direct analysis, with sophistication as needed

Examples of Turing at Work

Example

- Definition of computable via Turing Machine, universal machine, and non-computability of first order validity.
  - *As we have heard, designed a machine to simulate the discrete moments in an algorithmic calculation.*
  - *In the same paper, exhibited real numbers which are not computable:*
    - *direct diagonal construction*
    - *undecidability of validity*

Examples of Turing at Work

Example

- (1938, thesis) Investigated hierarchies of proof principles along the ordinals.
  - *Self-described motivation, to circumvent the limitations of Gödel's Incompleteness Theorem by introducing intuition in a clear and systematic way.*

Both investigations exhibited instances of transcendence, in the forms of non-computability or non-provability.

Developments since 1936

I will focus on the lines of research indicated in these two examples, particularly on the places where they intersect.

- Semantics: definability for subsets of ω.
  - Well understood hierarchy of definability for subsets of ω.
  - Characterization theorems.
- Syntax: emerging understanding of the finitary consequences of infinitary principles, especially the familiar ones.

The Turing Jump

Definition

For $X \subseteq \omega$, let X' be the set of existential sentences satisfied by $(\omega, 0, +, \times, <, X)$

Here we chose a particular formulation of the jump. There were other alternates:

- $\{e :$ the eth Turing machine relative to X halts.$\}$
- $\{\varphi : \varphi$ is provable from PA and the diagram of $X.\}$

They are equicomputable, in fact recursively isomorphic.

Transcendence

Definition

Write $X \geq_T Y$ when Y is computable from X.

Theorem

- *For all X, $X' >_T X$ (essentially by Turing's diagonal argument).*
- *For all X and Y, if $X \geq_T Y$ then $X' \geq_T Y'$.*

So, the function $X \mapsto X'$ gives a $>_T$-increasing and $\geq_T$-order-preserving function.

Transcendence

There are many types of non-recursive sets: generic, random, diagonally non-recursive, solutions to Post's problem, sets of minimal Turing degree.

Sets of these different types appear naturally: generic sets in the Baire Category Theorem, random sets in measure theory, and others in compactness arguments and in infinitary combinatorics.

Uniqueness

But there is only one Turing jump.

Theorem (Slaman and Steel)

Suppose that F is a Borel function from reals to reals with the following properties.

- Increasing: *For all X, $F(X) >_T X$.*
- Order-preserving: *For all X and Y, if $X \geq_T Y$ then $F(X) \geq_T F(Y)$.*

Then, there is a B such that for all $X \geq_T B$, $F(X) \geq_T X'$.

The Hierarchy of Arithmetic Definability

Definition

The arithmetically definable subsets of the natural numbers are those generated by iterating the Turing jump and closing under relative definability.

The arithmetically definable sets appear in a natural hierarchy based on counting the number of applications of the jump.

Uniqueness

The special role of the Turing jump in transcending the computable applies to the arithmetical hierarchy (and far beyond).

Theorem (Slaman and Steel)

Suppose that F is a Borel function which is increasing and order-preserving. Then, there is a B such that for all $X \geq_T B$, one of the following conditions holds.

- *$F(X)$ is equicomputable with $X^{(k)}$.*
- *$F(X)$ can compute X^ω, the first order theory of arithmetic relative to X.*

Martin's Conjecture

There is a conjecture, due to D. A. Martin, mathematically codifying the view that all notions of relative definability extending relative computability appear in the logical hierarchy based on first order quantification over the finite sets.

- There is substantial supporting evidence for this view.
- By results already known, there are severe limitations on possible alternate notions of "relatively definable."

The Arithmetic Hierarchy as a Calibration

The properties mentioned earlier (intractability, genericity, randomness) are naturally evaluated by criteria calibrated by the arithmetic hierarchy.

Example

- R is Martin-Löf-random iff for every uniformly recursively enumerable sequence of open sets $(O_n : n \in \omega)$ of recursively decreasing measure there is an m such that R is not an element of O_m.
- R is k-random iff R is Martin-Löf random relative to $0^{(k)}$.

The Arithmetic Hierarchy as a Calibration

The randomness of R was expressed by requiring that it avoid an arithmetically definable set of measure 0. The more complicated the definition of the set of measure 0, the more stringent is the requirement of randomness, and the more of the almost-everywhere properties of the reals will be ensured.

Example

- (Kučera-Gacs) There is a Martin-Löf random R such that $R \geq_T 0'$.
- (Sacks and others) If R is 2-random, then $R \not\geq_T 0'$.

Definability in the Wild

The hierarchy of definability applies to quantitatively and systematically describe the ingredients of mathematical investigations.

It provides the means to answer questions like the following.

- Whether there is an object, such as a real number, which can be produced using methods, principles, techniques of Type A and which satisfies Property B
- Whether principles of Type A and be used to settle questions of Type B

Remark

The the same could be said of the formal subsystems of second order arithmetic.

A Recursion Theorist's Assumption

At least with respect to familiar principles, both sorts of questions can be formulated and settled by directly considering the nature of "Type A," as Turing did with the nature of computation, with minimal reliance on the formalization of theories.

Ramsey's Theorem

Definition

For $X \subseteq \omega$, let $[X]^n$ denote the size n subsets of X. For $n, m > 0$ and $F : [\omega]^n \to \{0, \ldots, m-1\}$, $H \subseteq \omega$ is *homogeneous for* F iff F is constant on $[H]^n$.

Theorem (Ramsey, 1930)

For all $n, m > 0$ and all $F : [\omega]^n \to \{0, \ldots, m-1\}$, there is an infinite set H such that H is homogeneous for F.

If we fix n and m, then we represent that instance of Ramsey's Theorem by RT^n_m.

Recursion Theoretic Content of Ramsey's Theorem

Theorem (Jockusch)

- *There is a recursive partition of F of pairs such that there is no F-homogeneous set which is recursive in $0'$. (Recursive Comprehension is not sufficient for RT^2_2)*
- *There is a recursive partition F of triples such that $0'$ is recursive in any infinite F-homogeneous set. (RT^3_2 proves Arithmetic Comprehension.)*

Theorem (Seetapun)

There is an ideal J in the Turing degrees as follows.

- $0' \notin J$
- *For every $F : [\omega]^2 \to 2$ in J, there is an infinite F-homogeneous H in J.*

(RT^2_2 does not prove Arithmetic Comprehension.)

Challenging the Recursion Theorist's Assumption

Question

What are the finitary consequences of infinitary principles?

Here, one can ask about principles such as the existence of an infinite random source, infinite combinatorial principles such as Ramsey's Theorem, or set theoretic principles such as the existence of infinitely many cardinals or large cardinals, as studied by H. Friedman.

If the Recursion Theorist takes arithmetic on ω as given then it is not possible to semantically reason about how its theory is affected by infinitary principles.

Definability Theoretic Thinking in Non-ω-models

Our understanding of the fundamentals of definability applies perfectly well in non-standard models.

Understanding the double jump in Ramsey's Theorem for Pairs, applies to conclude Cholak-Jockusch-Slaman's theorem that any number theoretic statement provable from RT^2_2 is already provable from $P^- + I\Sigma_2$.

There are other examples of this type, including an especially important one given by Harrington about applications of compactness.

Definability Theoretic Thinking in Non-ω-models

Even so, there are mysteries and challenges here.

It is open to exactly characterize the first-order consequences of RT^2_2 or of the existence of random reals.

Missing Ingredients

We have two powerful tools with which to analyze relative definability.

- The hierarchy of definability, based on the Turing jump, provides the ability to calibrate relative definability.
- Forcing, interpreted broadly to include priority constructions and other effective implementations, provides the ability to approximate arbitrarily complicated definitions relative to G while constructing G.

Missing Ingredients

To have a widely applicable technology to answer questions about infinite/finite, we need a third set of tools.

We need tools to fine-tune the underlying structure of arithmetic so as to control the behavior of the hierarchy of definability built upon it.

Some Progress

The following is obtained by working in a highly customized nonstandard model of arithmetic, based in part on the same systems of logic arising from the iteration of consistency that appeared in Turing's thesis.

Theorem (Chong, Slaman and Yang)

- *RT^2_2 does not prove $I\Sigma_2$.*
- *RT^2_2 is not provable from SRT^2_2, its restriction to stable partitions.*

A Final Thought

We who work in Mathematical Logic are attempting to understand the interaction between the mathematical objects and the means needed to speak about them. This is as fundamental an investigation as any other in Mathematics.

No one's contribution to this investigation is greater than Alan Turing's.