LIE DERIVATIVES AND THE CAUCHY PROBLEM

IN

THE GENERAL THEORY OF RELATIVITY

by

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A DISSERTATION

Submitted to the Faculty of the Stevens Institute of Technology in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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STEVENS INSTITUTE OF TECHNOLOGY
Castle Point, Hoboken, New Jersey
1962

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ABSTRACT

The Cauchy problem in the general theory of relativity, in both a metric and a tetrad formulation, is treated with the help of the Lie derivative. A more geometrically intuitive and covariant treatment of a number of problems is thereby achieved. Expressions for the evolution of the hypersurface metric on a family of geodesically parallel hypersurfaces in an arbitrary Riemann space are given. When the field equations are imposed, the role of various components of the Riemann tensor with respect to the surface and normal directions in determining this evolution is shown. An interpretation of the constraint equations is developed which reduces their solution to the problem of embedding two hypersurfaces with arbitrary metrics into a Riemann space with vanishing Ricci tensor. The problem of continuing the hypersurface metric on an arbitrary family of hypersurfaces is solved. A Lagrangian and Hamiltonian formulation of the problem for geodesically parallel hypersurfaces is given in terms of Lie derivatives.

I. INTRODUCTION

It was recognized many years ago that, in a certain sense, the field equations of the general theory of relativity were of hyperbolic normal type. Consequently, the Cauchy problem was considered fairly early in the development of the general theory of relativity.[1] More recently, it has become clear that the Cauchy problem in relativity, as in electrodynamics, is closely bound up with the number of degrees of freedom of the pure gravitational field (i.e., the field in source-free regions), and therefore with the problems of gravitational radiation and the definition of energy.[2] These last matters, associated as they are with questions of quantization, have stimulated renewed interest in the Cauchy problem.

The aim of this paper is to give a more complete and covariant geometrical interpretation of the Cauchy problem in the general theory of relativity than has heretofore been given. This will be done with the help of the Lie derivative. But first, we shall discuss the nature of the Cauchy problem, and some ways in which this problem in general relativity differs from the usual one.

Cauchy's problem, as usually formulated, is a problem in analysis.[3] The basic space considered is that of the independent variables x^i $(i = 1,\ldots,n)$, which form an n-dimensional arithmetic space. We shall symbolize this space by A_n. A system of differential equations is given:

(1) $$\varphi^B(y^A, y^A{}_{,ijk}, \ldots, x^i) = 0 \quad (A, B = 1, \ldots, N)$$

where the y^A are a set of unknown functions of the x^i, and $y^A{}_{,ijk}$ the partial derivatives of the y^A up to some finite order. We assume that the system is determined, i.e., that the number of equations is equal to the number of unknown functions. If we choose some arbitrary hypersurface S in A_n, Cauchy's problem is to give data on S about y^A and its derivatives which is just sufficient to determine uniquely a solution to (1) with these prescribed values on S (called the initial values or initial data). The question of the types of partial differential equations for which the Cauchy problem may be correctly set, explicit construction of the solution functions, and the relation between the properties of the initial data and those of the solution function are important aspects of the Cauchy problem.[4]

In general relativity we deal with two spaces, rather than one. The fundamental space is the 4-dimensional geometrical manifold of events, which we shall symbolize by X_4. This manifold is coordinatized by placing it in one-one correspondence (at least in patches) with regions of A_4. Since we regard X_4 as the basic physical manifold, coordinate transformations — that is point transformations in A_4 — are to be looked upon as producing no more than a change in the description of fundamentally unchanged entities. In the case of general relativity, we want to consider a fixed hypersurface in X_4 (i.e., an X_3) in formulating the Cauchy problem. But this hypersurface may be represented by an arbitrary coordinate hypersurface in A_4, depending on the coordinate system used. Thus, we are led to a more geometrical formulation of the Cauchy problem in general relativity: Find geometric entities to be given on a hypersurface of an X_4, such that these entities serve to determine a unique Riemann space which obeys the Einstein field equations. Since the field variables of the theory are the $g_{\mu\nu}$, the components of the metric tensor, one would be led first to look at the $g_{\mu\nu}$ and their normal derivatives with respect to the surface as candidates for the initial data on the hypersurface.

However, a fundamental complication arises here; as is well known, the field equations of general relativity are *not* of Cauchy-Kowalewski type. In itself, the fact that a theory is expressible in an arbitrary coordinate system does not prevent it from being Cauchy-Kowalewski in character. For example, the scalar wave equation, expressed in a space with an arbitrary but fixed background metric, can be solved in any given coordinate system for the second normal derivative to an arbitrary spacelike hypersurface. Of course, if we find two solution functions in two different coordinate systems and the solutions are related by the same coordinate transformation that connects the two coordinate systems, we regard the two solutions as physically identical. In the general theory of relativity, on the other hand, no variables occur in the empty-space field equations except the metric and its derivatives; and the field equations $G_{\mu\nu}=0$ are invariant in form in every coordinate system. The group of coordinate transformations act as a gauge group, and it is well known that a theory with a gauge group cannot be of Cauchy-Kowalewski type.

In general relativity, this can be seen from the following considerations: Take any solution to the field equations, and consider the set of values on some hypersurface of the metric tensor and all its normal derivatives to that hypersurface. If we make a coordinate transformation which reduces to the identity on and some distance off the initial hypersurface, we can turn our solution into a formally different one which agrees with the original solution on the hypersurface up to any finite order of derivative that we choose. Thus, no set of values of the $g_{\mu\nu}$ and their normal derivatives on an initial hypersurface can determine a unique formal solution to the field equations. Of course, all these solutions are to be regarded as different descriptions of the same physical solution in different coordinate systems, but the fact remains that field equations compatible with such behavior cannot be of Cauchy-Kowalewski type.[5]

Formally, this can be seen from the fact that, given any hypersurface of X_4, there exist four linear combinations of the field equations which are free of second normal derivatives with respect to that hypersurface. These are called the constraint equations of the theory. Thus, not all of the field variables can be determined by the field equations, but only certain combinations of them. This does not imply any lack of causality in the theory, but merely that less than the total number of field variables are causally determined by the field equations since less than the total number are needed to characterize physically distinct solutions. The geometrical significance of this fact in the case of the general theory of relativity will be seen in Sections III and IV.

There are several ways of handling the problem of the gauge group. One can, for example, modify the field equations so that they are no longer invariant under the gauge group. The resulting equations must be of such a nature as to continue to admit at least one solution from each equivalence class of numerical solutions which correspond to a physical solution. However, they will then usually admit additional solutions, not corresponding to any physical solution of the original equations, and a method for eliminating these spurious solutions must be given. Examples of this method are the use of the Lorentz condition in electrodynamics, and the De Donder or harmonic coordinate conditions in general relativity, which make the modified field equations of Cauchy-Kowalewski type, at the expense of admitting spurious solutions; the latter are then eliminated by imposing the Lorentz condition or De Donder condition respectively on the solution functions. This approach to the Cauchy problem was among the earliest used in general relativity, and has since proved useful in a number of investigations, particularly those concerned with questions of existence proofs, domains of influence, etc.[6]

Another approach is based upon the idea of splitting the field variables into two groups — one group whose evolution is determined by the field equations, and another group whose evolution is not. This can be done, for example, by using the coordinate condition $g_{0\nu} = \delta_{0\nu}$, and solving the Cauchy problem for the g_{ab} $(a,b=1,2,3)$ starting with the surface x^0 = constant in such a coordinate system. This approach was also used at an early stage in the discussion of the Cauchy problem in general relativity.[7] The field equations then break up into two sets. One set, composed of the four $G_{0\nu}$ equations, restricting the initial values of the g_{ab} and their first derivatives with respect to x^0; and the other set, composed of the remaining six R_{ab} equations, which determine the evolution of the g_{ab} off the initial surface x^0 = constant. However, the use of coordinate conditions, as well as the ordinary derivative, obscure the geometric content of this method, and restrict it to the treatment of geodesically parallel families of surfaces.

More recently, it was shown by Dirac and Anderson that a Hamiltonian formulation of the general theory of relativity could be given, in which the $g_{0\nu}$ have zero canonical momenta, so that they may be eliminated from the theory, if we choose to look at the evolution of the system only along the family of surfaces x^0 = constant.[8] Again, the use of a particular coordinate system tends to obscure the geometrical significance of the results. It is clear from this approach, as well as that of Arnowitt, Deser and Misner, that the geometrical entities needed on the initial hypersurface of X_4 to uniquely characterize a Riemann space satisfying the field equations are the first and second fundamental forms of the surface.[9] These quantities are not freely specifiable, of course, but are subject to certain constraint equations, which will be discussed in Sections III and IV. But the relationship of these quantities to the metric tensor of the X_4 is not clear, except in a special coordinate system.

However, if we want to proceed covariantly and in such a way as to gain the maximum geometrical insight, it is best to put the coordinate system into the background, and concentrate on the geometrical manifold. First, we want to know what data must be given on the initial hypersurface of X_4 to determine a unique Riemann space satisfying the Einstein field equations. As we indicated above, the answer to this has become clear: it is the first and second fundamental forms on the surface. We show in Section III that the choice of a hypersurface in a Riemann space induces a breakup of the metric tensor into surface and normal components; and that the first fundamental form of the surface is just the surface components of the metric tensor written in terms of the coordinate system adopted on the hypersurface, while the second fundamental form is the Lie derivative in the surface unit normal direction of the first fundamental form. The choice of coordinate systems in which to express these relationships, both for the X_4 and the initial hypersurface, is completely arbitrary. Next we are interested in the way in which the metric evolves off the initial hypersurface. To make this question determinate, we must give some family of hypersurfaces in the X_4, one of which is our initial hypersurface, and indicate what geometric character we want for this family of surfaces in our Riemann space. To get us from one surface of this family to the next we need a family of point transformations. Since the Lie derivative is the covariant expression which tells us how any geometricalquantity changes under an infinitesimal point transformation,this proves to be the tool which enables us to compute how the surface component of the metric tensor evolves from surface to surface.

Section II is primarily expository. The concept of the Lie derivative is discussed, together with a number of rules for its use which will be needed.

In Sections III and IV we treat the Cauchy problem for empty space (i.e., $G_{\mu\nu}=0$) by a new method. Instead of starting from the field equations, we calculate the higher order Lie derivatives of the first fundamental form of a surface for an arbitrary Riemann space, and then show how the field equations serve to determine the second and higher order derivatives. This might be called a Newtonian approach to the problem. In Section III we carry out this approach for a family of surfaces geodesically parallel to our initial hypersurface; this family turns out to be the simplest and most natural family for which to formulate the problem. The constraint equations are discussed in this section.

In Section IV, the Newtonian treatment is extended to an arbitrary family of hypersurfaces which includes the initial hypersurface. The four arbitrary functions which occur in the Dirac formalism are seen to be interpretable as the components of an arbitrary vector field with respect to the geodesic normal vector field at that point. The constraint equations are rewritten in such a way that they may be regarded as conditions upon the orientation of this vector field.

Section V contains a discussion of the interior case, in which $G_{\mu\nu} = T_{\mu\nu}$, where $T_{\mu\nu}$ is the stress-energy tensor corresponding to some arbitrary sources. Section VI is a reformulation of the Cauchy problem in terms of the tetrad formalism. The field equations are shown to determine the evolution of certain of the rotation coefficients. If the spacelike triad is parallel propagated along the timelike vector of the tetrad which is tangent to a geodesic normal field, these rotation coefficients are the physical components of the second fundamental form of the hypersurfaces.

In Section VII we discuss the Lagrangian and Hamiltonian approaches to the Cauchy problem in terms of Lie derivatives. The variational principle for the Einstein field equations, $\delta\int(-g)^{1/2} R\, d^4x$, is reformulated in terms of Lie derivatives, and is shown to lead to the correct decomposition of the field equations. Subtraction of a total Lie derivative results in a first order Lagrangian. When the usual method of passing from the Lagrangian to the Hamiltonian is applied, using Lie derivatives, the Dirac form of the Hamiltonian is shown to result.

A summary, mentioning some of the remaining problems to which we hope to apply this method, and two appendices complete this paper.

All of our considerations will be essentially local. Thus, we shall not, at this stage, discuss such important questions as the role of boundary conditions in solving the constraint equations on the initial hypersurface; or how far the construction of the geodesic normal field may proceed off a given hypersurface before caustic points develop; or whether the imposition of Minkowskian boundary conditions at spatial infinity would eliminate the possibility of any source-free singularity free solutions to the field equations. We hope, with the aid of the method developed here, to gain more insight into the geometry of those Riemann spaces which satisfy the Einstein field equations. With the help of these insights, it may be possible to formulate these global questions as geometrical problems, as well as the many local questions still unsolved (notably the constraint equations). If "physics is geometry," which many a relativist has taken as his watchword, then perhaps geometrical insights may point the way forward to the solution of some of the difficult physical questions of energy and radiation raised by the general theory of relativity.

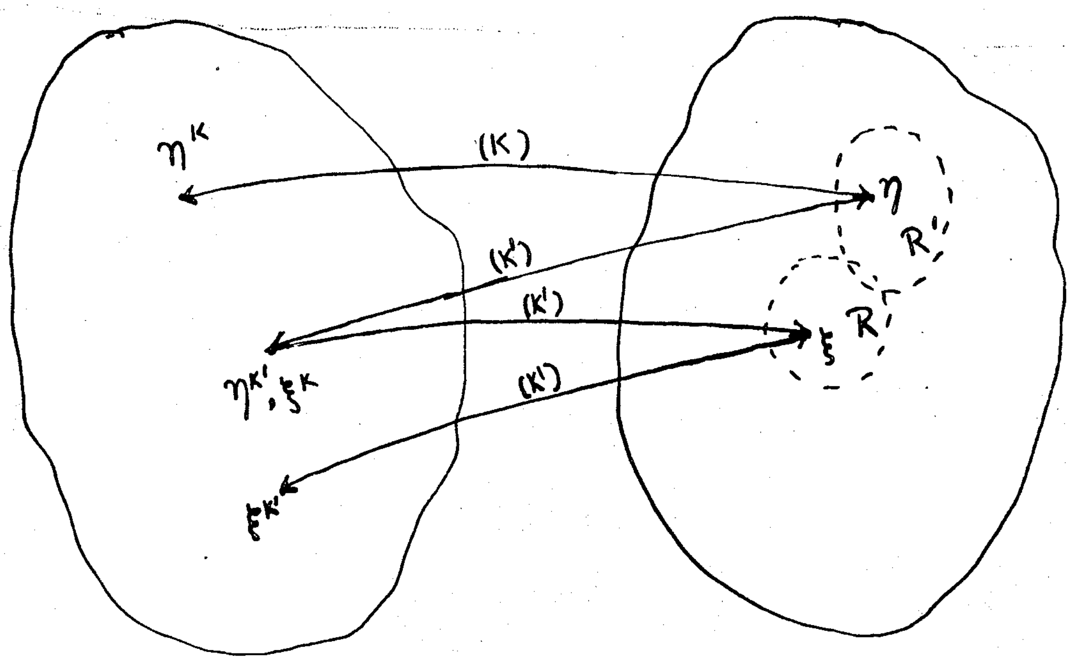
We conclude this introduction with some remarks on the notation to be used. In the main, we follow Schouten's notation.[10] Although requiring some initial effort to learn, it greatly facilitates many intricate calculations. The kernel-index notation is used consistently. Any geometrical object, such as a tensor, is denoted by a fixed kernel symbol, which never changes. The indices of its components will change, however, every time the coordinate system changes. A given vector, for example, might be represented by the symbol "ξ." "ξ^{κ}" and "$\xi^{\kappa'}$" will then represent the same vector in two different coordinate systems. "${}'\xi^{\kappa}$," however, since the kernel symbol "${}'\xi$" is different, will represent a different vector in the same coordinate system as "ξ^{κ}." Equations which hold in all coordinate systems are written with the usual equality sign " = ." Equations which hold in only particular coordinate systems are written with an asterisk over the equality sign: " $\overset{*}{=}$." Thus, $\xi^{\kappa} = \eta^{\kappa}$ asserts that the vectors ξ and η are the same vector; while $\xi^{\kappa} \overset{*}{=} \delta^{\kappa}_{\kappa'} \eta^{\kappa'}$, or more explicitly $\xi^{\kappa}(x^{\lambda}) \overset{*}{=} \delta^{\kappa}_{\kappa'} \eta^{\kappa'}(x^{\lambda'})$ asserts only that the components of the vector ξ in one coordinate system (the x^{κ} one) are numerically equal to the components of a different vector η in another coordinate system (the $x^{\kappa'}$ one).

After Section II, all Greek indices will refer to the four-dimensional manifold X_4 and all Latin indices to a three-dimensional submanifold. The metric tensor $g_{\mu\nu}$ will be taken with signature -2, while the metric tensor $'g_{ab}$ will be negative definite with signature -3.

II. THE LIE DERIVATIVE

In our formulation of the Cauchy problem, we wish to relate the values of certain geometrical quantities at different points of a four-dimensional manifold X_4. It will be sufficient, of course, to have some method of comparing values at neighboring points, so long as this differential operation may be iterated in order to compare values at points with a finite separation. Ordinarily, this is done by adopting some coordinate system, and using ordinary derivatives. But ordinary derivatives of quantities like tensors are not geometrical objects, in the sense that, given only $A^{\lambda}_{\mu,\nu}$ in some coordinate system (where A^{λ}_{μ} is a tensor), one cannot determine $A^{\lambda'}_{\mu',\nu'}$ in another coordinate system. If we were going to treat some generally covariant field theory involving field variables other than the metric in a fixed background metric space, we should naturally think of using the covariant derivative in order to relate quantities at different points. However, since it is the metric field itself which is to be determined by the field equations of general relativity, we have no covariant differentiation available until our Riemann space is constructed. What we need, then, is some intrinsic or geometrical method of comparing the values of geometrical quantities at different points of a manifold; some generalization of the ordinary derivative, which is applicable to a bare manifold. The Lie derivative provides just such a generalization, which we shall now discuss.

Let us consider some region of an n-dimensional manifold X_n, which we shall call $\underline{R}$. Let P: $\xi^\kappa \rightarrow \eta^\kappa = f^\kappa(\xi)$ be a point transformation which takes the point ξ^κ of $\underline{R}$ into the point η^κ, which may or may not lie in $\underline{R}$, but at any rate lies in some sub-region $\underline{R}'$ of X_n. Now, with any such point transformation there is uniquely correlated a coordinate transformation: namely, the coordinate transformation such that the point $\eta^{\kappa'}$ in the new coordinate system has the same coordinates as the point ξ^κ in the old coordinate system (as shown below):



A_n
Arithmetical Space

X_n
Manifold

Figure 1

Analytically this coordinate transformation is given by

(ii-1) $$\eta^{\kappa'} = \delta^{\kappa'}_{\kappa} f^{-1\,\kappa}(\eta) \quad .$$

Let Φ_Λ be a field of some set of quantities defined in the region R. We can correlate a value of the field at ξ^κ with a new value at the point η^κ as follows: let the new value at $\eta^{\kappa'}$ be such that in the new coordinate system it is numerically equal to the original value of at the point ξ^κ in the old coordinate system. We called this value the dragged-along value of the field, and symbolize it by $\overset{m}{\Phi}_\Lambda$ (the "m" standing for the German word, mitgeschleppt). Symbolically, $\overset{m}{\Phi}_{\Lambda'}\{\eta^{\kappa'}\} \overset{*}{=} \delta^{\Lambda}_{\Lambda'}\Phi_\Lambda\{\xi^\kappa\}$. Clearly, the difference between Φ_Λ and $\overset{m}{\Phi}_\Lambda$, taken at any point where both are defined, will be a quantity of the same kind as Φ_Λ if Φ_Λ is a quantity which obeys a homogeneous linear transformation law.

Now there is a one-to-one correspondence between infinitesimal point transformations and contravariant vector fields given by

(ii-3) $$\eta^\kappa = \xi^\kappa + v^\kappa dt \quad ,$$

where v^κ is any contravariant vector field, and dt is any scalar infinitesimal. Thus, with any vector field v^κ , we may associate an infinitesimally dragged-along field $\overset{m}{\Phi}_\Lambda$; and we may write the difference between the original value of the field and the dragged-along value at any point, which is clearly proportional to dt, as

(ii-4) $$\Phi_\Lambda - \overset{m}{\Phi}_\Lambda = (\underset{v}{\pounds}\,\Phi_\Lambda)\,dt \quad ,$$

where $\underset{v}{\pounds}\,\Phi_\Lambda$ is the symbol for the Lie derivative of Φ_Λ in the direction of v^κ. It is clear from this definition that if we know the Lie derivative of any quantity Φ_Λ at some point ξ^κ in a certain direction v^κ, we can compute the value of the quantity at the point $\xi^\kappa - v^\kappa dt$.

$$\Phi_\Lambda\{\xi^\kappa\} = \overset{m}{\Phi}_\Lambda\{\xi^\kappa\} + (\underset{v}{\mathcal{L}}\Phi_\Lambda)\,dt$$

$$\text{(ii-5)} \qquad \overset{*}{=} \delta^{\Lambda'}_\Lambda \Phi_{\Lambda'}\{\xi^{\kappa'} - v^{\kappa'}dt\} + \underset{v}{\mathcal{L}}\Phi_\Lambda\,dt.$$

Now we shall show that if we are given some vector field v^κ, and the value of a quantity Φ_Λ, as well as the values of its Lie derivatives of all orders with respect to the field v^κ at some point P, we can find the values of the quantity Φ_Λ at all points of that trajectory of the v^κ field which passes through the point P.

In order to show this, we first show that any vector field v^κ gives rise to a one-parameter family of finite point transformations, through iteration of the infinitesimal point transformation $\xi^\kappa \to \xi^\kappa + v^\kappa dt$. What we are looking for is a family of point transformations $\xi^\kappa \to \eta^\kappa(t)$, where t is a parameter, such that for t=0, say, the transformation reduces to the identity: $\xi^\kappa = \eta^\kappa(0)$; and such that $v^\kappa\,dt$ shall be the infinitesimal element of the group. This last condition implies that

$$\text{(ii-6)} \qquad \eta^\kappa(t + dt) = \eta^\kappa(t) + v^\kappa[\eta(t)].$$

But the left-hand side of (ii-6), when expanded to first order in dt is equal to $\eta^\kappa(t) + (d\eta^\kappa/dt)\,dt$. Therefore, the differential equation for such a family of point transformations is

$$\text{(ii-6')} \qquad d\eta^\kappa/dt = v^\kappa[\eta(t)],$$

with the initial condition that $\eta^\kappa(0) = \xi^\kappa$. The solution to this differential equation is easily seen to be

$$\eta^\kappa = \xi^\kappa + v^\kappa t + v^\mu \partial_\mu v^\kappa \frac{t^2}{2!} + \ldots = \left(1 + \sum_{n=1}^{\infty} (v^\mu \partial_\mu)^n \frac{t^n}{n!}\right) \xi^\kappa \tag{ii-7}$$

or, symbolically,

$$\eta^\kappa = e^{t(v^\mu \partial_\mu)} \xi^\kappa \quad . \tag{ii-8}$$

This one parameter family of point transformations may be symbolized by $\eta^\kappa = \overset{t\,v}{T} \xi^\kappa$. If we choose some initial hypersurface, which we shall take to be a submanifold symbolized by $\overset{0}{X}_{n-1}$, then this family of point transformations generates a one-parameter family of hypersurfaces. For each transformation $\xi^\kappa \to \eta^\kappa(t)$ takes the points of our hypersurface into a new hypersurface. This new hypersurface will clearly be a submanifold, since by dragging along the original coordinate systems of both the X_n and the $\overset{0}{X}_{n-1}$, we see that the same parametrization will do, for each succeeding hypersurface, in the new coordinate system. We symbolize that hypersurface in the family corresponding to the value $\underline{t}$ of the parameter by $\overset{t}{X}_{n-1}$.

We can always choose a coordinate system in which the vector v^κ has the components δ^κ_1 . In this coordinate system, we see from (ii-7) that $\eta^\kappa \overset{*}{=} \xi^\kappa + v^\kappa t$. From the definition of the dragged-along field, it also follows that in this coordinate system

$$\overset{m}{\Phi}_\Lambda \{\xi\} \overset{*}{=} \Phi_\Lambda \{\xi - vt\} \quad ; \tag{ii-9}$$

and, expanding the right-hand side in a power series, we get

$$\overset{m}{\Phi}_\Lambda \{\xi\} \overset{*}{=} \Phi_\Lambda \{\xi\} - \partial_1 \Phi_\Lambda \{\xi\} t + \ldots \tag{ii-10}$$

But in this coordinate system the Lie derivative reduces to the ordinary derivative:

(ii-11) $$\mathop{£}_{v} \Phi_{\Lambda}\{\xi\} \overset{*}{=} \partial_{1} \Phi_{\Lambda}\{\xi\} \;;$$

so that we may rewrite (ii-10) as follows:

(ii-12) $$\tilde{\Phi}_{\Lambda}\{\xi\} \overset{*}{=} \Phi_{\Lambda}\{\xi\} - \mathop{£}_{v} \Phi_{\Lambda}\{\xi\}\, t + \mathop{£}_{v}^{2} \Phi_{\Lambda}\{\xi\} \frac{t^{2}}{2!} + \dots \; .$$

But this equation is invariant term by term, so that we can drop the asterisk, and write it symbolically as

(ii-13) $$\tilde{\Phi}_{\Lambda}\{\xi\} = e^{-t \mathop{£}_{v}} \Phi_{\Lambda}\{\xi\} \; .$$

Suppose we now apply the inverse family of transformations generated by $-v^{\kappa}$. Then $\tilde{\Phi}_{\Lambda}\{\xi\}$ becomes the value of the Φ_{Λ} field in the dragged-along coordinate system, at succeeding points of the trajectory of the v^{κ} field. Equation (ii-13) when applied to the $-v^{\kappa}$ field becomes

(ii-14) $$\tilde{\Phi}_{\Lambda} = e^{-t \mathop{£}_{-v}} \Phi_{\Lambda} = e^{t \mathop{£}_{v}} \Phi_{\Lambda} \; .$$

Thus, if we know the value of the Φ_{Λ} field and of all its Lie derivatives with respect to some vector field at one point, we can find the values of the field inthe dragged-along coordinate system at all points of that trajectory of the vector field which passes through our original point. The values at the succeeding points (in the sense of increasing t) are found from (ii-14), and of preceding points (in the sense of decreasing t) from (ii-13).

If we know all the values of some field Φ_{Λ} on a hypersurface of a manifold, as well as all its Lie derivatives in the direction of some vector field which does not lie in the hypersurface, or as we shall say, which transvects the hypersurface, then we can calculate the values of the field Φ_Λ on all hypersurfaces defined by the family of point transformations associated with v^κ; and thus calculate the values of Φ_Λ throughout some region $\underline{R}$. It is this property of the Lie derivative of which we shall make extensive use in the treatment of the Cauchy problem. We shall symbolize this procedure by (ii-15) .

(ii-15) $$\Phi_\Lambda(\overset{+}{X}_{n-1}) = e^{+\underset{v}{\xi}}\,\Phi_\Lambda(\overset{\circ}{X}_{n-1}) \quad ,$$

using our previous notation for the hypersurfaces.

As an example, we shall compute the Lie derivative of a covariant vector field w_κ with respect to v^κ. By definition, the value of the dragged-along field at the point ξ^κ in the dragged-along coordinate system equals the value of the origin field at the point $\xi^\kappa - v^\kappa dt$ in the original coordinate system:

$$\overset{m}{w}_{\kappa'}\{\xi\} \overset{*}{=} \delta^\kappa_{\kappa'}\, w_\kappa\{\xi - v\,dt\}$$

$$\overset{*}{=} \delta^\kappa_{\kappa'}\left[w_\kappa\{\xi\} - v^\mu\partial_\mu w_\kappa\{\xi\}\,dt\right]$$

to first order in dt. But $\overset{m}{w}_\lambda\{\xi\} = A^{\kappa'}_\lambda \overset{m}{w}_{\kappa'}\{\xi\}$, where $A^{\kappa'}_\lambda = \dfrac{\partial\xi^{\kappa'}}{\partial\xi^\lambda}$.

Thus

(ii-16) $$\overset{m}{w}_{\lambda}\{\xi\} = A^{\kappa'}_\lambda\,\delta^\kappa_{\kappa'}\left[w_\kappa\{\xi\} - v^\mu\partial_\mu w_\kappa\{\xi\}\,dt\right] .$$

The coordinates of the new point $\xi^{\kappa'}$ have the same values in the dragged-along coordinate system as the old point in the old coordinate system:

$$\xi^{\kappa'} \overset{*}{=} \delta^{\kappa'}_{\sigma}(\xi^{\sigma} - v^{\sigma}\,dt)$$

so that

$$A^{\kappa'}_{\lambda} = \delta^{\kappa'}_{\sigma}\left(\frac{\partial \xi^{\sigma}}{\partial \xi^{\lambda}} - \partial_{\lambda} v^{\sigma}\,dt\right)$$

(ii-17) $$= \delta^{\kappa'}_{\sigma}\left(A^{\sigma}_{\lambda} - \partial_{\lambda} v^{\sigma}\,dt\right),$$

where $A^{\sigma}_{\lambda} = \frac{\partial \xi^{\sigma}}{\partial \xi^{\lambda}}$. Combining equations (ii-16) and (ii-17), we get

$$\tilde{w}_{\lambda}\{\xi\} = \delta^{\kappa'}_{\sigma}[A^{\sigma}_{\lambda} - \partial_{\lambda} v^{\sigma}\,dt]\,\delta^{\kappa}_{\kappa'}[w_{\kappa}\{\xi\} - v^{\mu}\partial_{\mu} w_{\kappa}\{\xi\}\,dt]$$

(ii-18) $$= w_{\lambda} - (w_{\kappa}\partial_{\lambda} v^{\kappa} + v^{\mu}\partial_{\mu} w_{\lambda})\,dt.$$

All quantities in (ii-17) are taken at the same point ξ, and we have dropped the asterisk, since the equation now involves quantities in only one coordinate system. From equation (ii-4), it then follows immediately that

$$\underset{v}{\pounds} w_{\lambda} = v^{\mu}\partial_{\mu} w_{\lambda} + w_{\mu}\partial_{\lambda} v^{\mu}$$

Further examples of the computing of the Lie derivative may be found in Schouten.[11]

We shall be concerned not only with tensors of the X_n, but also with tensors of an arbitrary X_{n-1} of the manifold (in all our applications, of course, $n = 4$). Suppose x^{μ} are the coordinates of X_n ($\mu=1,\ldots,n$), while the $'x^a$ are the coordinates of the submanifold ($a=1,,,,n-1$). The relationship between the two manifolds can be described in either of two ways. We may give an equation $C(x\ \ ) = 0$, for the submanifold; or we may give the parametric representation $x^{\mu} = B^{\mu}('x^a)$. The contravariant connecting quantities

$B_a^\mu = \partial_a X^\mu$, which are contravariant vectors of the X_n and covariant vector of the X_{n-1}, serve as projection operators. For example, if v_μ is a covariant vector of X_n, $B_a^\mu v_\mu$ gives the covariant components in the ${}'x_a$ system of the projection of v_μ into the hypersurface. Similarly, if s^a is a contravariant vector of the X_{n-1}, $B_a^\mu s^a$ gives the contravariant components in the x^μ coordinate system of this vector, considered as a vector of the X_n. In a manifold, given nothing but a hypersurface, we can form no more than these. In order to define the projection of a contravariant vector of X_n on the hypersurface, or the vector of X_n corresponding to a covariant vector of the X_{n-1}, we must rig the submanifold. In the case of a hypersurface, rigging means giving a direction at each point of the hypersurface which does not lie in the hypersurface. For a metric space, the normal to the surface provides a natural rigging field (so long as it is not a null surface, in which case the normal lies in the surface). Once the surface is rigged, covariant connecting quantities B_μ^a can be defined which allow us to find the hitherto undefined projections and components. The connecting quantities B_a^μ and B_μ^b obey the following relations:

$$B_\mu^a B_b^\mu = B_b^a \; ; \; B_a^\mu C_\mu = 0 \; ; \; B_\mu^a V^\mu = 0 ,$$

where B_b^a is the unity tensor of the X_{n-1}, $C_\mu = \partial_\mu C$, and v^μ is any contravariant vector in the direction of the rigging.

For tensor fields defined on the X_{n-1} the Lie derivative is not yet defined, for we do not have a unique prescription for the behavior of the connecting quantities B_a^μ and B_μ^a, under the dragging process. However, if we project the surface tensor onto the full manifold, we have a tensor of the X_n, whose Lie derivative can be computed just like that of any tensor. We then can define the Lie derivative of the surface tensor as the components of the Lie derivative of the corresponding tensor of X_n which lie in the submanifold.[12]

For a vector $'p_a$ of X_m, for example, we first form the vector

$$'p_\alpha = B^a_\alpha\, 'p_a \tag{ii-19}$$

of X_n, take the Lie derivative of $'p_\alpha$ with respect to any vector field v^μ, and define the Lie derivative of $'p_a$ as

$$\underset{v}{\mathcal{L}}\, 'p_a = B^\alpha_a \underset{v}{\mathcal{L}}\, 'p_\alpha \quad . \tag{ii-20}$$

With this definition, should $'v^\mu$ be a vector field lying in the submanifold, then the Lie derivative of a tensor field on the X_{n-1} will be the same whether computed from the above definition, or by carrying out the operation of taking the Lie derivative entirely on the surface, as is clearly possible in this case. For example,

$$\begin{aligned} \underset{'v^\mu}{\mathcal{L}}\, 'p_a &= B^\alpha_a \underset{'v}{\mathcal{L}}\, 'p_\alpha = B^\alpha_a\left[(\partial_\kappa 'p_\alpha)'v^\kappa + 'p_\kappa \partial_\alpha 'v^\kappa\right] \\ &= B^{\alpha k}_{a \kappa}\, 'v^k \partial_\kappa 'p_\alpha + B^{\alpha k}_{a \kappa}\, 'p_\kappa \partial_\alpha 'v^\kappa \\ &= 'v^k \partial_k 'p_a + 'p_k \partial_a 'v^k = \underset{'v^\mu}{\mathcal{L}}\, 'p_a \end{aligned} \tag{ii-21}$$

where $B^{\alpha \kappa}_{a k} = B^\alpha_a B^\kappa_k$.

If we should be given some tensor field over a hypersurface, together with all its Lie derivatives as defined above with respect to some vector field v^{μ}, then we can build up the tensor field on the family of hypersurfaces generated by v^{μ} by a formula analogous to (ii-14) if we carry along the coordinate system $'x^a$ of the original hypersurface to each succeeding hypersurface.

We shall end this section with a few useful, easily proved rules for the manipulation of the Lie derivative, as well as a list of Lie derivatives for some tensorial entities.

A. The Lie derivative of a sum of quantities equals the sum of the Lie derivatives.

B. The Lie derivative of a product is given by the rule of Leibnitz

$$\underset{v}{\mathfrak{L}}(\Phi\Psi) = \Phi\underset{v}{\mathfrak{L}}\Psi + \Psi\underset{v}{\mathfrak{L}}\Phi .$$

C. The Lie derivative with respect to the sum of two vector fields of any tensorial entity is the sum of the Lie derivatives with respect to each of the fields separately.

D. The rule for Lie derivation of a tensor is as follows: Take the ordinary derivative of the tensor and contract it with the vector field. Then,

1. for every covariant index, replace it by a dummy index, take the derivative of the vector field with respect to the replaced index, sum over the dummy index, and add these terms.

2. for every contravariant index, replace it by a dummy index, put this index on the vector field and differentiate it with respect to the dummy index;

sum over the dummy index, and add these terms.

E. In a space with a linear connection, all ordinary derivatives may be replaced with covariant derivatives in the formula for the Lie derivative of a tensor.

F. The Lie derivative commutes with ordinary differentiation.

A few examples of the Lie derivatives of tensor fields of various ranks which we shall use are:

Scalar:

$$\underset{V}{\pounds}\rho = (\partial_\mu \rho)V^\mu$$

Vector:

Contravariant $\underset{V}{\pounds} n^\kappa = (\partial_\mu n^\kappa)V^\mu - n^\mu(\partial_\mu V^\kappa)$

Covariant $\underset{V}{\pounds} m_\kappa = (\partial_\mu m^\kappa)V^\mu + m_\mu(\partial_\kappa V^\mu)$

Tensor, rank two:

Two covariant indices $\underset{V}{\pounds} t_{\alpha\beta} = (\partial_\mu t_{\alpha\beta})V^\mu + t_{\mu\beta}\partial_\alpha V^\mu + t_{\alpha\mu}\partial_\beta V^\mu$

Two contravariant indices $\underset{V}{\pounds} t^{\alpha\beta} = (\partial_\mu t^{\alpha\beta})V^\mu - t^{\mu\beta}\partial_\mu V^\alpha - t^{\alpha\mu}\partial_\mu V^\beta$

Mixed indices $\underset{V}{\pounds} t^{\alpha}_{\cdot\beta} = (\partial_\mu t^{\alpha}_{\cdot\beta})V^\mu - t^{\mu}_{\cdot\beta}\partial_\mu V^\alpha + t^{\alpha}_{\cdot\mu}\partial_\beta V^\mu$.

As follows from rule D, Lie derivatives of higher rank tensors can easily be taken by noting that each covariant or contravariant index is to be treated in the same way as the covariant or contravariant index of the mixed rank two tensor, keeping all other indices fixed. In all the above formulae, as rule E above indicates, the ordinary derivative may be replaced by the covariant derivative if the space has an affine connection.

III. THE CAUCHY PROBLEM — GEODESIC NORMAL FIELD

Our problem can be stated as follows: given a manifold of four dimensions X_4 and an arbitrary submanifold of three dimensions $\overset{o}{X}_3$, to give such geometrical quantities on $\overset{o}{X}_3$ as will enable us to construct a metric which makes X_4 into a Riemann space of signature -2 in which $G_{\mu\nu}=0$. We shall assume X_4 to be of some simple topological structure, for example, the product of a Euclidean three-space and a line, so that there will always exist at least one contravariant vector field v^{μ} which transvects our original surface, and whose trajectories fill up the space. Then we shall demonstrate by construction that (locally at least) given a negative definite metric $'g_{ab}$ on $\overset{o}{X}_3$, the initial hypersurface, and the first Lie derivative of $'g_{ab}$ with respect to the vector field v^{μ} (both of which obey certain constraint equations which we shall discuss later) it is always possible to construct such a Riemann space. Further, it will be seen that the initial data on the $\overset{o}{X}_3$ is equivalent to the surface components of the metric tensor (which we symbolise by $'g_{\mu\nu}$) on the hypersurface and the first Lie derivative of $'g_{\mu\nu}$ with respect to v^{κ}.

We shall proceed in two steps. In this section we consider the problem of constructing the metric in such a way that v^{κ}, after our construction of the Riemann space, will be the time-like geodesic normal field to our initial hypersurface. It will be seen that this constitutes a determinate geometrical problem, with a geometrical construction leading to its solution. In the next section, we take up the case in which v^{μ} becomes an arbitrary vector field, i.e., one whose relationship to the geodesic normal field is arbitrary. Here, of course, an additional element of arbitrariness enters the problem; the geometric arbitrariness of the relationship between the two vector fields. However, this element of arbitrariness does not lead to the construction of a different Riemann space; for with the same initial data on the initial hypersurface, it merely results in a new geometrical construction of the same metric space.

To facilitate comprehension, we start our analysis in reverse, i.e., we first analyze a given Riemann space. In such a space for which $G_{\mu\nu}=0$, we develop certain relationships which involve only the initial data on a hypersurface of the manifold on which our Riemann space is built. Then we shall easily be able to reverse our reasoning, and see that starting from a bare manifold, upon a hypersurface of which we impose certain initial data, we can build up a Riemann space satisfying the Einstein field equations.

We assume we have a V_4 of signature -2, for which $G_{\mu\nu}=0$, and freely use all metrical concepts in our reasoning. Let $g_{\mu\nu}$ be the metric of the V_4. Then the metric of our initial hypersurface $\overset{o}{V}_3$ (which will be our $\overset{o}{X}_3$) is given by

$$(\text{iii-1}) \qquad {}'g_{ab} = B^{\mu\nu}_{ab}\, g_{\mu\nu}$$

Let us symbolize the unit geodesic normal field to $\overset{o}{V}_3$ by n^{μ}, which shall be our v^{μ} (we shall always use n^{μ} when v^{μ} is intended to be a unit geodesic normal field). If $C_\mu n^\mu = \lambda^{-1}$ (where $C(x^\mu) = 0$ is the equation of the $\overset{o}{V}_3$), then we shall define $\bar{C} = \lambda C$, so that $\bar{C}_\mu n^\mu = 1$, i.e., $\bar{\underline{C}}^\mu$ is the covariant normal field. Now let us compute $\underset{n}{\mathcal{L}}\,{}'g_{ab}$, remembering our definition in Section II:

$$\underset{n}{\mathcal{L}}\,{}'g_{ab} = B^{\mu\nu}_{ab}\,\underset{n}{\mathcal{L}}\,{}'g_{\mu\nu} = B^{\mu\nu}_{ab}\,\underset{n}{\mathcal{L}}(g_{\mu\nu} - n_\mu n_\nu)$$

(iii-2)
$$= B^{\mu\nu}_{ab}(\nabla_\mu n_\nu + \nabla_\nu n_\mu) = 2B^{\mu\nu}_{ab}\nabla_\mu n_\nu \,;$$

the order of differentiation of $\nabla_\mu n_\nu$ may be reversed since n^μ is a gradient field. But $B^{\mu\nu}_{ab}\nabla_\mu n_\nu$ is equal to minus the second fundamental form of the hypersurface, which we symbolize by h_{ab}. Thus,

(iii-3)
$$\underset{n}{\mathcal{L}}\,{}'g_{ab} = -2\,h_{ab}.$$

In other words, giving the Lie derivatives of the surface metric in the direction of the unit normal is equivalent to giving the second fundamental form of the surface -- a knowledge of the first and secone fundamental forms is equivalent to a knowledge of the metric on an infinitesimally close parallel hypersurface to the initial one. Now let us compute the second Lie derivative of $'g_{ab}$ on the initial hypersurface:

$$\underset{n}{\mathcal{L}}\,{}'g_{ab} = -2\underset{n}{\mathcal{L}} h_{ab} = 2B^{\mu\nu}_{ab}\underset{n}{\mathcal{L}}\nabla_\mu n_\nu$$

$$= 2B^{\mu\nu}_{ab}\left[\nabla_\kappa(\nabla_\mu n_\nu)n^\kappa + (\nabla_\kappa n_\nu)(\nabla_\mu n^\kappa) + (\nabla_\mu n_\kappa)(\nabla_\nu n^\kappa)\right]$$

$$= 2B^{\mu\nu}_{ab}\left[n^\kappa\nabla_\mu\nabla_\kappa n_\nu - R_{\kappa\mu\nu}{}^{\rho}n_\rho n^\kappa + (\nabla_\kappa n_\nu)(\nabla_\mu n^\kappa) + (\nabla_\mu n_\kappa)(\nabla_\nu n^\kappa)\right]$$

$$= 2B^{\mu\nu}_{ab}\left[-R_{\kappa\mu\nu\rho}n^\rho n^\kappa + (\nabla_\mu n_\kappa)(\nabla_\nu n^\kappa)\right].$$

since $n^\kappa \nabla_\kappa n_\nu = 0$, because n^μ is tangent to a geodesic;

$$= 2B^{\mu\nu}_{ab}(\nabla_\mu n_\kappa)(\nabla_\nu n_\rho)\, g^{\kappa\rho} - 2B^{\mu\nu}_{ab} R_{\kappa\mu\nu\rho}\, n^\rho n^\kappa$$

$$= 2B^{\mu\nu}_{ab}(\nabla_\mu n_\kappa)(\nabla_\nu n_\rho)({}'g^{\kappa\rho} + n^\kappa n^\rho) - 2B^{\mu\nu}_{ab} R_{\kappa\mu\nu\rho}\, n^\rho n^\kappa$$

$$= 2B^{\mu\nu}_{ab}(\nabla_\mu n_\kappa)(\nabla_\nu n^\rho)\,{}'g^{\kappa\rho} - 2B^{\mu\nu}_{ab} R_{\kappa\mu\nu\rho}\, n^\rho n^\kappa,$$

since $n^\kappa \nabla_\mu n_\kappa = 0$ because n^κ is a unit vector;

$$= 2B^{\mu\nu\kappa\rho}_{abkr}\,{}'g^{kr}(\nabla_\mu n_\kappa)(\nabla_\nu n_\rho) - 2B^{\mu\nu}_{ab} R_{\kappa\mu\nu\rho}\, n^\rho n^\kappa$$

$$= 2\,{}'g^{kr}(-h_{ak})(-h_{br}) - 2B^{\mu\nu}_{ab} R_{\kappa\mu\nu\rho}\, n^\rho n^\kappa$$

(iii-4) $$\mathop{\pounds}_{n}^{2}\,{}'g_{ab} = 2\left(h^r_a h_{br} - B^{\mu\nu}_{ab} R_{\kappa\mu\nu\rho}\, n^\rho n^\kappa\right).$$

Thus, we see that in order to know the second Lie derivatives of the surface metric, we must know certain projections of the Riemann tensor onto the surface and normal directions. First of all, let us see what types of projections are possible. We can project four times onto the surface by means of the operator $B^{\kappa\lambda\mu\nu}_{klmn}$, getting $B^{\kappa\lambda\mu\nu}_{klmn}R_{\kappa\lambda\mu\nu}$. We can project the Riemann tensor $R_{\kappa\lambda\mu\nu}$ on to the normal direction n^ν once, and three times on to the surface by means of the operator $B^{\kappa\lambda\mu}_{klm}\,n^\nu$, getting $B^{\kappa\lambda\mu}_{klm}\,n^\nu R_{\kappa\lambda\mu\nu}$. We can project twice on the normal direction and twice on the surface by means of $B^{\kappa\lambda}_{kl}\,n^\mu n^\nu$, getting $B^{\kappa\lambda}_{kl}n^\mu n^\nu R_{\kappa\mu\lambda\nu}$; and that is all. For, if we project three times on the normal by means of $n^\kappa n^\lambda n^\nu$, we shall get zero, because of the symmetries of the Riemann tensor.

Now the projections of the Riemann tensor four times on the surface, and three times on the surface and once in the normal direction are completely determined by the first and second fundamental forms of the hypersurface. Indeed this is the content of the generalized Gauss-Codazzi equations, which state that,[13]

(iii-5) $$B^{\kappa\lambda\mu\nu}_{klmn} R_{\kappa\lambda\mu\nu} = {}'R_{klmn} + h_{km} h_{ln} - h_{kn} h_{lm}$$

(iii-6) $$B^{\kappa\lambda\mu}_{klm} n^{\nu} R_{\kappa\lambda\mu\nu} = {}'\nabla_k h_{lm} - {}'\nabla h_{km}$$

Thus a knowledge of the first and second fundamental forms on the hypersurface determines all the projections of the Riemann tensor except those twice in the normal direction and twice on the surface, which are left undetermined by a knowledge of the intrinsic and extrinsic curvature properties of the surface. Indeed, equation (iii-4) shows us why this must be so in general. A knowledge of the projection of the Riemann tensor in the double normal direction (as we shall call the $B^{\kappa\lambda}_{kl} n^{\mu} n^{\nu} R_{\kappa\mu\nu\lambda}$ components), together with the second fundamental form, suffices to determine the metric on the second infinitesimally close parallel surface, and knowledge of the higher Lie derivatives of $B^{\kappa\lambda}_{kl} n^{\mu} n^{\nu} R_{\kappa\mu\nu\lambda}$ would suffice to determine the metric on further and further distant geodesically parallel surfaces, by the method discussed in Section II. And it must certainly be true that in an arbitrary Riemann space, with no field equations imposed, we have the freedom to specify the metric arbitrarily (perhaps subject to some continuity requirements), on such succeeding surfaces.

The field equations eliminate that freedom, by fixing the $B^{\mu\nu}_{mn} n^{\kappa} n^{\lambda} R_{\kappa\mu\nu\lambda}$ in terms of the first and second fundamental form on the surface. In addition, they also impose certain restrictions on the initial data as well, through the Gauss-Codazzi equations, but we shall return to this point later.

Since $R_{\mu\nu} = g^{\kappa\lambda} R_{\kappa\mu\nu\lambda} = {}'g^{\kappa\lambda} R_{\kappa\mu\nu\lambda} + n^\kappa n^\lambda R_{\kappa\mu\nu\lambda}$,

$$B^{\mu\nu}_{mn} R_{\mu\nu} = B^{\mu\nu}_{mn} R_{\kappa\mu\nu\lambda} {}'g^{\kappa\lambda} + B^{\mu\nu}_{mn} n^\kappa n^\lambda R_{\kappa\mu\nu\lambda}$$

$$= B^{\mu\nu\kappa\lambda}_{mnkl} R_{\kappa\mu\nu\lambda} {}'g^{kl} + B^{\mu\nu}_{mn} n^\kappa n^\lambda R_{\kappa\mu\nu\lambda}$$

But from Gauss' equation (iii-5), we have

$$B^{\kappa\mu\nu\lambda}_{kmnl} R_{\kappa\mu\nu\lambda} = {}'R_{kmnl} + h_{kn} h_{ml} - h_{kl} h_{mn} .$$

Thus,

(iii-7) $$B^{\mu\nu}_{mn} R_{\mu\nu} = {}'R_{mn} + h^l_n h_{ml} - (h^l_l) h_{mn} + B^{\mu\nu}_{mn} n^\kappa n^\lambda R_{\kappa\mu\nu\lambda}$$

and if $R_{\mu\nu} = 0$ (and indeed for any surface for which $B^{\mu\nu}_{mn} R_{\mu\nu} = 0$),

(iii-8)

Thus, with the help of the field equations $R_{\mu\nu} = 0$ on the surface $\overset{o}{V}_3$ (which are of course equivalent to $G_{\mu\nu} = 0$), we can now determine the second Lie derivatives of the surface metric:

(iii-9) $$\underset{n}{\mathcal{L}}^2 {}'g_{ab} = 2\left({}'R_{ab} + 2h^r_a h_{br} - (h^r_r) h_{ab}\right)$$

It is clear that this has been the crucial step in our problem; for successive Lie derivatives of the surface metric can now be found by using the fact that all successive Lie derivatives of the surface projections of the field equations, $B^{\mu\nu}_{mn} R_{\mu\nu}$, vanish on the initial hypersurface. Equation (iii-4) always holds, and therefore,

$$(iii\text{-}10)\quad \underset{n}{\mathcal{L}}^{(k)}\,{}'g_{ab} = 2\underset{n}{\mathcal{L}}^{(k-2)} h^{r}_{a}\, h_{br} - 2B^{\mu\nu}_{mn}\underset{n}{\mathcal{L}}^{(k-2)} R_{\kappa\mu\nu\lambda}n^{\kappa}n^{\lambda},\ k \geq 2.$$

But $\underset{n}{\mathcal{L}}^{(k-2)} h_{ab}$ are known, since they only depend on a knowledge of $\underset{n}{\mathcal{L}}^{(k-1)}\,{}'g_{ab}$ and we assume all lower order Lie derivatives known; and if we assume $\underset{n}{\mathcal{L}}^{(k-2)} R_{\mu\nu}=0$, then equation (iii-8) gives us $\underset{n}{\mathcal{L}}^{(k-2)} B^{\mu\nu}_{ab} R_{\kappa\mu\nu\lambda}n^{\kappa}n^{\lambda}$ in terms of $\underset{n}{\mathcal{L}}^{(k-2)} h_{ab}$ and $\underset{n}{\mathcal{L}}^{(k-2)}\,{}'R_{ab}$. The former we have seen is known; as far as the latter goes, i.e., $\underset{n}{\mathcal{L}}^{(k-2)}\,{}'R_{ab}$, it is clear geometrically that this is known, since it depends on no more than a knowledge of the metric on the (k-2)nd infinitesimally close geodesically parallel surface while we actually know the (k-1)st metric. Analytically it is clear that $\underset{n}{\mathcal{L}}^{(k)}\,{}'R_{ab}$ can be computed from $\underset{n}{\mathcal{L}}^{(k)}\,{}'g_{ab}$ and property F of the Lie derivative, i.e., the fact that it commutes with ordinary differentiation.[14]

Thus it is clear that all succeeding Lie derivatives of the surface metric in the direction of the geodesic normal field to $\overset{o}{V}_3$ are determined by the first fundamental form and its first Lie derivative in the normal direction (equivalent to the second fundamental form) and the field equations on the initial surface, together with all its Lie derivatives on the surface (which is equivalent to the field equations throughout V_4).

Using the argument outlined in Section II, it now follows that the metric on any surface geodesically parallel to the initial surface can be written in the form:

$$(iii\text{-}11)\qquad {}'g_{ab}(\overset{+}{V}_3) = e^{t\underset{n}{\mathcal{L}}}\left({}'g_{ab}(\overset{\circ}{V}_3)\right)\ .$$

We can, of course, project $'g_{ab}$ into the V_4 with the help of the connecting quantities B^{a}_{μ} to get the tensor $'g_{\mu\nu}$

$$'g_{\mu\nu} = B^{ab}_{\mu\nu}\,'g_{ab} \quad .$$

This tensor is the surface component of the metric tensor. Indeed $g_{\mu\nu} = {}'g_{\mu\nu} + n_\mu n_\nu$. Similarly, the Lie derivative of $'g_{\mu\nu}$ with respect to n^μ is a surface tensor, whose components in the surface coordinate system are $-2h_{ab}$. Thus, it is the surface components of the metric tensor which are determined by the field equations, and we may write $'g_{\mu\nu}(\overset{+}{V}_3) = e^{+\pounds_n}\,'g_{\mu\nu}(\overset{\circ}{V}_3)$.

Before proceeding with our original problem, of constructing a metric with vanishing $G_{\mu\nu}$ for a manifold X_4 by giving initial data on an initial hypersurface $\overset{\circ}{X}_3$, it is necessary to discuss the conditions which the $'g_{ab}$ and $\pounds_n\,'g_{ab}$ satisfy on any hypersurface on which the Ricci tensor (and therefore $G_{\mu\nu}$) vanishes. To do this, let us evaluate the components of G projected twice in the normal direction $G_{\mu\nu}n^\mu n^\nu$, and projected once in the normal direction and once on the surface $G_{\mu\nu}B^\mu_m n^\nu$.

For this purpose, we need the following expression for R, the curvature scalar:

$$R = g^{\mu\lambda}R_{\mu\lambda} = g^{\mu\lambda}g^{\kappa\nu}R_{\kappa\mu\lambda\nu} = g^{\mu\lambda}('g^{\kappa\nu}R_{\kappa\mu\lambda\nu} + n^\kappa n^\nu R_{\kappa\mu\lambda\nu})$$
$$= {}'g^{\mu\lambda}\,'g^{\kappa\nu}R_{\kappa\mu\lambda\nu} + 2\,'g^{\mu\lambda}n^\kappa n^\nu R_{\kappa\mu\lambda\nu}$$
$$= B^{\kappa\mu\lambda\nu}_{kmln}\,'g^{ml}\,'g^{kn}R_{\kappa\mu\lambda\nu} + 2R_{\kappa\nu}n^\kappa n^\nu$$
$$= B^{\kappa\mu\lambda\nu}_{kmln}R_{\kappa\mu\lambda\nu}\,'g^{ml}\,'g^{kn} + 2R_{\kappa\nu}n^\kappa n^\nu$$

(iii-12) $$R = (B^{\kappa\mu\lambda\nu}_{kmln}R_{\kappa\mu\lambda\nu})\,'g^{ml}\,'g^{kn} + 2R_{\kappa\nu}n^\kappa n^\nu .$$

Using Gauss' equation (iii-5), we get

$$R = {}'R_{abcd}\,{}'g^{ab}\,{}'g^{cd} + 2h_{[b[d}h_{c]a]}\,{}'g^{ab}\,{}'g^{cd} + 2R_{\kappa\nu}n^{\kappa}n^{\nu}. \tag{iii-13}$$

Then we easily find that

$$n^{\mu}n^{\lambda}G_{\mu\lambda} = -\frac{1}{2}B^{\nu\rho\sigma\kappa}_{bcda}R_{\nu\rho\sigma\kappa}\,{}'g^{ab}\,{}'g^{cd},$$

and using Gauss' equation again

$$n^{\mu}n^{\lambda}G_{\mu\lambda} = -\frac{1}{2}({}'R + h_{ab}h^{ab} - h^{a}_{a}h^{b}_{b})\,; \tag{iii-14}$$

and since $\mathcal{L}_{n}\,{}'g_{ab} = -2\,h_{ab}$, the vanishing of $n^{\mu}n^{\lambda}G_{\mu\lambda}$ has as a consequence that

$${}'R + h_{ab}h^{ab} - h^{2} = 0 \quad , \tag{iii-15}$$

for any spacelike hypersurface in a manifold on which $G_{\mu\nu}$ vanishes. We use here the abbreviation $\underline{h}$ for h^{a}_{a}, the trace of h_{ab}.

Similarly, it is easily seen that

$$B^{\mu}_{m}n^{\lambda}G_{\mu\lambda} = B^{\kappa\nu\mu}_{abm}R_{\nu\mu\lambda\kappa}n^{\lambda}\,{}'g^{ab},$$

and by the use of Codazzi's equations (iii-6) this reduces to

$$B^{\mu}_{m}n^{\lambda}G_{\mu\lambda} = {}'\nabla_{b}h^{b}_{m} - {}'\nabla_{m}h \tag{iii-16}$$

Again, for any surface for which $B^{\mu}_{m}n^{\lambda}G_{\mu\lambda}$ vanishes, we must therefore have

$${}'\nabla_{b}h^{b}_{m} - {}'\nabla_{m}h = 0 \quad . \tag{iii-17}$$

We shall refer to equations (iii-15) and (iii-17) as the constraint equations on the initial data.

Now we are ready to return to our original problem. Given a manifold X_4, we pick a hypersurface $\overset{o}{X}_3$ in it, and a vector field n^μ. We choose a negative definite metric $'g_{ab}$ for the initial surface (negative because of our sign convention, definite, to make the initial surface spacelike) together with the first Lie derivatives of the surface metric $\underset{n}{\mathcal{L}}\, 'g_{ab}$, with respect to the vector field n^μ. We impose the condition that n^μ is to be the timelike unit geodesic normal field to our initial surface after we finish constructing our V_4 out of the X_4. The n^μ field rigs the surface $\overset{o}{X}_3$ (which we have already made into a V_3 by choice of a surface metric), so that we now know B^m_μ as well as B^μ_m. We can use the procedure explained above to normalize ζ, so that $n_\mu = \zeta_\mu$ becomes the unit covariant normal: $n_\mu n^\mu = 1$ (positive to make the n^μ field timelike). Then we define the full metric $g_{\mu\nu}$ on the initial hypersurface as follows:

$$g_{\mu\nu} = B^{mn}_{\mu\nu}\, 'g_{mn} + n_\mu n_\nu = 'g_{\mu\nu} + n_\mu n_\nu \,. \qquad \text{(iii-18)}$$

Now we define the higher Lie derivatives of $'g_{ab}$ by equation (iii-9) and its successive Lie derivatives; and use (iii-18) to define the full metric $g_{\mu\nu}$ on each surface we reach by the point transformations generated by n^μ: $\eta^\mu = e^{t\underset{n}{\mathcal{L}}}\xi^\mu$, as discussed in Section II. This family of surfaces, of course, will be geodesically parallel, by virtue of this definition of the metric. It is clear from our discussion of the constraint equations above, that unless our initial data satisfies equations (iii-15) and (iii-17), we cannot have the components of $G_{\mu\nu}$ in the normal-normal and normal-surface directions vanish, with the definition of the full metric given by (iii-18). Thus initial data must be given which satisfies the constraint equations. Now, if this data evolves according to (iii-9) and its Lie derivatives, it is clear from the identity,

$$\text{(iii-19)} \qquad \underset{n}{\mathcal{L}}^{2}\, {}'g_{ab} = 2({}'R_{ab} + 2h_{a}^{r}h_{br} - h\, h_{ab}) - B_{ab}^{\mu\nu}R_{\mu\nu} \quad ,$$

which follows by substituting (iii-7) into (iii-4), that $B_{ab}^{\mu\nu}R_{\mu\nu}$, and its successive Lie derivatives must vanish. Thus, on the initial surface $B_{ab}^{\mu\nu}R_{\mu\nu}$ and $n^{\mu}n^{\nu}G_{\mu\nu}$ and $B_{m}^{\mu}n^{\nu}G_{\mu\nu}$ all vanish. It is easily seen that this implies that $B_{ab}^{\mu\nu}G_{\mu\nu}$ also vanishes on the initial surface. An analysis of the contracted Bianchi identities $\nabla_{\kappa}G_{\mu}^{\kappa} \equiv 0$ (see Appendix I) shows they can be rewritten in the following form:

$$\text{(iii-20)} \qquad \begin{aligned} &{}'g^{km}\,{}'\nabla_{k}(G_{\mu\nu}B_{m}^{\mu}n^{\nu}) - {}'g^{km}B_{1}^{\kappa}(\nabla_{\kappa}B_{m}^{\nu})[(G_{\mu\lambda}n^{\mu}B_{1}^{\lambda})B_{\nu}^{1} + \\ &+ (G_{\mu\lambda}n^{\mu}n^{\lambda})n_{\nu}] - {}'g^{kn}B_{k}^{\kappa}[(G_{\lambda\nu}B_{1n}^{\lambda\nu})B_{\mu}^{1} + \\ &+ (G_{\lambda\nu}B_{n}^{\nu}n^{\lambda})n_{\mu}]\nabla_{\kappa}n^{\mu} + \underset{n}{\mathcal{L}}(G_{\mu\nu}n^{\mu}n^{\nu}) = 0; \end{aligned}$$

$$\text{(iii-21)} \qquad \begin{aligned} &{}'g^{kn}\,{}'\nabla_{k}(G_{\mu\nu}B_{mn}^{\mu\nu}) + {}'g^{kn}B_{k}^{\kappa}[n_{\nu}(G_{\mu\lambda}B_{m}^{\mu}n^{\lambda}) + B_{\nu}^{1}(G_{\mu\lambda}B_{m1}^{\mu\lambda})]\nabla_{\kappa}B_{n}^{\nu} + \\ &- {}'g^{kn}B_{k}^{\kappa}[n_{\mu}(G_{\lambda\nu}n^{\lambda}B_{n}^{\nu}) + (G_{\lambda\nu}B_{1n}^{\lambda\nu})B_{\mu}^{1}]\nabla_{\kappa}B_{m}^{\mu} - \\ &- B_{m}^{\mu}\nabla_{\mu}n^{\kappa}[B_{\kappa}^{1}(G_{\lambda\nu}n^{\nu}B_{1}^{\lambda}) + (G_{\lambda\nu}n^{\lambda}n^{\nu})n_{\kappa}] + \underset{n}{\mathcal{L}}(B_{m}^{\mu}n^{\nu}G_{\mu\nu}) = 0. \end{aligned}$$

Thus, if $B_{mn}^{\mu\nu}G_{\mu\nu}$, $B_{m}^{\mu}n^{\nu}G_{\mu\nu}$ and $n^{\mu}n^{\nu}G_{\mu\nu}$ all vanish on the initial hypersurface, the Lie derivatives of the constraint equations will vanish as well. The iterated Lie derivatives of these equations show that if the constraint equations hold initially, and the Lie derivatives of $B_{mn}^{\mu\nu}R_{\mu\nu}$ vanish to all orders, then the Lie derivatives of the constraint equations will vanish to all orders as well. Thus, the use of (ii-9) and its Lie derivatives with initial data satisfying the constraint equations, together with the definition of the metric (iii-18) does yield a Riemann space in which the Ricci tensor, and therefore $G_{\mu\nu}$ vanish everywhere.

Several elements of arbitrariness seem to be inherent in this argument. First of all, if we had used another coordinate system with which to describe the manifold, it is clear that the analytic form of the $g_{\mu\nu}$ as functions of the new coordinates would be different from their form as functions of the old coordinates. But a coordinate transformation would connect these forms, so that the same Riemann space would be described by both.

Secondly, we could just as well have used the same coordinate system, but picked a different surface as our initial surface, and a different vector field as our geodesic normal field. Here again, the analytic form of the results would then be different. But there will always exist a point transformation carrying the first initial surface, and all the surfaces gotten by dragging it along the first vector field, into the second initial surface and all the surfaces gotten by dragging it along the second vector field. If we carry out the coordinate transformation corresponding to such a point transformation, then the coordinates of the new surfaces in the new coordinate system will be the same as those of the old surfaces in the old coordinate system. If we place the initial data on corresponding surfaces, then in this new coordinate system, the metric tensor resulting from our construction will be the same function of the new coordinates as the old metric tensor was of the old coordinates. Choosing a new initial surface and vector field result in the same Riemann space. Stated another way, since there is no such thing as absolute orientation or inherent metric in a manifold, we have done nothing but change the way in which the metric is attached to points of the manifold. Such a change does not influence the resulting Riemann space.

IV. THE CAUCHY PROBLEM — ARBITRARY FIELD

Now that we have seen how to solve the Cauchy problem for a geodesic normal field, we shall generalize the results to an arbitrary vector field. This is not only of formal interest, showing how the metric tensor field may be built up using an arbitrary family of surfaces, but proves useful in treating other problems, such as the case where a time-like Killing vector field exists (stationary solution). We start again with the manifold X_4, an initial hypersurface $\overset{o}{X}_3$, and a vector field v^μ transvecting the initial hypersurface. We now give a surface metric tensor $'g_{ab}$, which is negative definite, for the initial hypersurface, together with its Lie derivative with respect to the vector field v^μ, $\underset{v}{\mathcal{L}}\, 'g_{ab}$. Our problem is to build up a metric for the X_4, such that the resulting V_4 will have a vanishing $G_{\mu\nu}$, and will give results on the initial surface coinciding with the given initial data. As before it will prove easier to first imagine such a V_4 given, and work out the necessary relationships.

In this section, we consider the field v^μ to be entirely arbitrary. Its relationship with the geodesic normal field passing through our initial surface must be of the form

(iv-1) $$v^\mu = \rho n^\mu + \sigma^\mu \quad , \quad n_\mu \sigma^\mu = 0 ,$$

where n^μ is, as usual, the geodesic normal field; the v^μ field at each point has components parallel and normal to the n^μ field. Thus ρ and σ^μ may be any scalar field and tensor field respectively, subject only to σ^μ being orthogonal to n^μ.

We can reach any arbitrary first (infinitesimally distant) neighboring surface by proceeding a variable distance ρdt along the normal direction. Thus ρn^μ takes us to an arbitrary neighboring surface; while σ^μ serves to drag points along the original surface. Together, they take us from a point on the initial surface to any other point on an arbitrary neighboring surface.

Let us express the Lie derivative of $'g_{ab}$ with respect to v^μ in terms of its Lie derivatives with respect to n^μ and σ^μ:

$$\underset{v}{\mathcal{L}}\, 'g_{ab} = B^{\alpha\beta}_{ab} \underset{v}{\mathcal{L}}\, 'g_{\alpha\beta} = B^{\alpha\beta}_{ab} \underset{v}{\mathcal{L}} (g_{\alpha\beta} - n_\alpha n_\beta) = B^{\alpha\beta}_{ab} \underset{v}{\mathcal{L}} g_{\alpha\beta}$$

(iv-2) $$= B^{\alpha\beta}_{ab} \underset{\rho n}{\mathcal{L}} g_{\alpha\beta} + B^{\alpha\beta}_{ab} \underset{\sigma}{\mathcal{L}} g_{\alpha\beta} \quad ,$$

using rule C for Lie derivatives (p.22). We need to evaluate both $B^{\alpha\beta}_{ab} \underset{\rho n}{\mathcal{L}} g_{\alpha\beta}$ and $B^{\alpha\beta}_{ab} \underset{\sigma}{\mathcal{L}} g_{\alpha\beta}$, which are equivalent to $\underset{\rho n}{\mathcal{L}}\, 'g_{ab}$ and $\underset{\sigma}{\mathcal{L}}\, 'g_{ab}$. Thus, our calculation breaks into two parts. Now we have

$$\underset{\rho n}{\mathcal{L}}\, 'g_{ab} = B^{\alpha\beta}_{ab} \underset{\rho n}{\mathcal{L}} g_{\alpha\beta} = B^{\alpha\beta}_{ab} \left[\nabla_\alpha(\rho n_\beta) + \nabla_\beta(\rho n_\alpha) \right]$$

$$= \rho B^{\alpha\beta}_{ab} (\nabla_\alpha n_\beta + \nabla_\beta n_\alpha) + B^{\alpha\beta}_{ab} (n_\beta \partial_\alpha \rho + n_\alpha \partial_\beta \rho)$$

(iv-3) $$\underset{\rho n}{\mathcal{L}}\, 'g_{ab} = -2\rho h_{ab} \quad .$$

On the other hand

(iv-4) $$\underset{\sigma}{\mathcal{L}}\, 'g_{ab} = B^{\alpha\beta}_{ab} \underset{\sigma}{\mathcal{L}} g_{\alpha\beta} = B^{\alpha\beta}_{ab} \left[\nabla_\alpha \sigma_\beta + \nabla_\beta \sigma_\alpha \right] = {}'\nabla_a \sigma_b + {}'\nabla_b \sigma_a \quad ,$$

since σ^μ on the initial surface is a tensor field entirely on that surface.

We may express results (iv-3) and (iv-4) in words as follows: The effect of taking the Lie derivative with respect to the field v^μ is the sum of two terms. The first is just ρ times the effect of taking the Lie derivative with respect to the normal field; the second is the effect of taking the Lie derivative with respect to an arbitrary surface vector field. Putting these results back into (iv-2), and solving for h_{ab}, we get

$$\text{(iv-5)} \qquad h_{ab} = \frac{1}{2\rho}\left(-\underset{v}{\mathcal{L}}\,{}'g_{ab} + {}'\nabla_a \sigma_b + {}'\nabla_b \sigma_a\right) .$$

We see from this equation that ρ cannot be zero, which is clearly the case, since it it were, the v^μ field would not transvect the initial surface, and could not take us off the initial surface.

In the last section we showed that there were four equations which the $'g_{ab}$ and h_{ab} of any hypersurface in a V_4 with vanishing $G_{\mu\nu}$ had to satisfy; the constraint equations (iii-15) and (iii-17). In the present case we can take one of two points of view with respect to these equations. Either we can regard them as restrictions on the $\underset{v}{\mathcal{L}}\,'g_{ab}$ expressed indirectly through (iv-5); or we can regard the $\underset{v}{\mathcal{L}}\,'g_{ab}$ as being completely arbitrary, but the relationship of the v^μ field to the n^μ field, i.e., the functions ρ and σ^μ, as determined by the constraint equations, through substitution of (iv-5) into the constraint equations. We can express the situation in this way: the projections of the Lie derivatives with respect to the v^μ field onto the unit normal direction must satisfy the constraint equations. However, if the Lie derivatives with respect to the v^μ field are given arbitrarily, we may regard our problem as that of tilting and stretching the v^μ field in such a way as to get the projections to obey the constraint equations. The results of this substitution of (iv-5) into the constraint equations are:[15]

$$(iv\text{-}6)\quad {}'R + \frac{1}{(4\rho^2)}\left(-\underset{v}{\mathcal{L}}\,{}'g_{ab} + {}'\nabla_a\sigma_b + {}'\nabla_b\sigma_a\right)\left(-\underset{v}{\mathcal{L}}\,{}'g_{cd} + {}'\nabla_c\sigma_d + {}'\nabla_d\sigma_c\right)\left({}'g^{ac}\,{}'g^{bd} - {}'g^{ab}\,{}'g^{cd}\right) = 0,$$

and

$$(iv\text{-}7)\quad {}'g^{bn}\left\{{}'\nabla_b\left[\frac{1}{2\rho}\left(-\underset{v}{\mathcal{L}}\,{}'g_{mn} + {}'\nabla_m\sigma_n + {}'\nabla_n\sigma_m\right)\right] - {}'\nabla_m\left[\frac{1}{2\rho}\left(-\underset{v}{\mathcal{L}}\,{}'g_{bn} + {}'\nabla_b\sigma_n + {}'\nabla_n\sigma_b\right)\right]\right\} = 0.$$

Equation (iv-6) can be regarded as an algebraic condition on ρ, for which it may be solved, so long as $'R \neq 0$; while (iv-7) may be looked upon as three differential equations for the σ_m.

Geometrically, this reformulation of the constraint equations, has the following significance. The metric of the initial surface is given arbitrarily. Giving the Lie derivative of $'g_{ab}$ with respect to some vector field v^μ is equivalent to giving the infinitesimally different metric on the "first" neighboring surface which results from the original surface by dragging it infinitesimally along the v^μ field. If the metric of this surface is given arbitrarily, the constraint equations in the form (iv-6) and (iv-7) express the necessary relationship between the two surfaces in order for them to fit into a Riemann space in which $G_{\mu\nu}$ vanishes. If, for example, we demand that the surfaces be geodesically parallel, then $\rho = 1$, and (iv-6) remains a geometrical condition on the two metrics, while (iv-7) expresses the fact that we must drag one of the metrics over its surface an infinitesimal amount with the surface field σ_m until the metric "fits" properly, i.e., until (iv-7) are satisfied. If we demand that the surfaces not only be geodesically parallel but that the metrics shall correspond at the points on each surface connected by the normal field, then σ_m is also zero, all four equations become geometrical conditions on the two metrics; and we are back at the viewpoint of the last section. Wheeler has also examined this way of looking at the constraint equations. He has further conjectured that it might be possible to extend it from a relationship between neighboring surfaces; and that given any two positive-definite surface metrics, there will always exist a Riemann space of vanishing $G_{\mu\nu}$ into which both surfaces may be fitted.[16] However, the narrower question of whether (iv-6) and (iv-7) can always be solved for arbitrary choice of $'g_{ab}$ and $\mathcal{L}_v \, 'g_{ab}$ still needs further investigation.

Having obtained the conditions which a set of initial data in an enlarged sense must satisfy, let us calculate the second Lie derivatives of the $'g_{ab}$ with respect to the v^μ field. We find that

$$\text{(iv-8)} \quad \mathcal{L}_v^2 \, 'g_{ab} = \mathcal{L}_v \left(\mathcal{L}_{\rho n} \, 'g_{ab} + \mathcal{L}_\sigma \, 'g_{ab} \right)$$
$$= \mathcal{L}_{\rho n}^2 \, 'g_{ab} + \mathcal{L}_{\rho n} \mathcal{L}_\sigma \, 'g_{ab} + \mathcal{L}_\sigma \mathcal{L}_{\rho n} \, 'g_{ab} + \mathcal{L}_\sigma^2 \, 'g_{ab},$$

through the use of the rule for Lie derivatives with respect to the sum of two vector fields. We must evaluate the four terms in equation (iv-8).

We start with $\mathcal{L}_{\rho n}^2 \, 'g_{ab}$:

$$\underset{\rho n}{\mathcal{L}}^{2}\,{}'g_{ab} = \underset{\rho n}{\mathcal{L}} B_{ab}^{\alpha\beta} \underset{\rho n}{\mathcal{L}} g_{\alpha\beta} = \underset{\rho n}{\mathcal{L}} B_{ab}^{\alpha\beta}[\rho(\nabla_\alpha n_\beta + \nabla_\beta n_\alpha) + (n_\beta \partial_\alpha \rho + n_\alpha \partial_\beta \rho)]$$

$$= B_{ab}^{\alpha\beta}[\rho \underset{\rho n}{\mathcal{L}}(\nabla_\alpha n_\beta + \nabla_\beta n_\alpha) + (\nabla_\alpha n_\beta + \nabla_\beta n_\alpha)\underset{\rho n}{\mathcal{L}}\rho] +$$

$$+ B_{ab}^{\alpha\beta}[n_\beta \partial_\alpha \underset{\rho n}{\mathcal{L}}\rho + n_\alpha \partial_\beta \underset{\rho n}{\mathcal{L}}\rho + \partial_\alpha \rho \underset{\rho n}{\mathcal{L}} n_\beta + \partial_\beta \rho \underset{\rho n}{\mathcal{L}} n_\alpha$$

$$= \rho B_{ab}^{\alpha\beta}[\nabla_K(\nabla_\alpha n_\beta + \nabla_\beta n_\alpha)(\rho n^K) + (\nabla_K n_\beta + \nabla_\beta n_K)(\nabla_\alpha(\rho n^K)) +$$

$$+ (\nabla_\alpha n_K + \nabla_K n_\alpha)(\nabla_\beta(\rho n^K)) - 2h_{ab}\underset{\rho n}{\mathcal{L}}\rho \, .$$

Thus

(iv-9) $$\underset{\rho n}{\mathcal{L}}^{2}\,{}'g_{ab} = -2\rho^2 \underset{n}{\mathcal{L}} h_{ab} - 2h_{ab}\underset{n}{\mathcal{L}}\rho + 2\partial_a\rho\,\partial_b\rho \, .$$

Next we evaluate $\underset{\sigma}{\mathcal{L}}^{2}\,{}'g_{ab}$:

(iv-10) $$\underset{\sigma}{\mathcal{L}}^{2}\,{}'g_{ab} = \underset{\sigma}{\mathcal{L}}({}'\nabla_a \sigma_b + {}'\nabla_b \sigma_a)$$

$$= [{}'\nabla_k({}'\nabla_a \sigma_b + {}'\nabla_b \sigma_a)\sigma^k + (\nabla_k \sigma_b + {}'\nabla_b \sigma_k)\,{}'\nabla_a \sigma^k +$$

$$+ ({}'\nabla_a \sigma_k + {}'\nabla_k \sigma_a)\,{}'\nabla_b \sigma^k \, .$$

$\underset{\sigma}{\mathcal{L}}\,\underset{\rho n}{\mathcal{L}}\,({}'g_{ab})$ yields:

(iv-11) $$\underset{\sigma}{\mathcal{L}}\,\underset{\rho n}{\mathcal{L}}\,{}'g_{ab} = \underset{\sigma}{\mathcal{L}}(-2\rho h_{ab}) = -2h_{ab}\underset{\sigma}{\mathcal{L}}\rho - 2\rho\underset{\sigma}{\mathcal{L}} h_{ab}$$

Finally, we evaluate $\underset{\rho n}{\mathcal{L}}\,\underset{\sigma}{\mathcal{L}}\,{}'g_{ab}$:

$$\underset{\rho n}{\mathcal{L}}\,\underset{\sigma}{\mathcal{L}}\,{}'g_{ab} = B_{ab}^{\alpha\beta}\underset{\rho n}{\mathcal{L}}(\nabla_\alpha \sigma_\beta + \nabla_\beta \sigma_\alpha) = B_{ab}^{\alpha\beta}[\nabla_K(\nabla_\alpha \sigma_\beta + \nabla_\beta \sigma_\alpha)(\rho n^K) +$$

$$+ (\nabla_K \sigma_\beta + \nabla_\beta \sigma_K)\nabla_\alpha(\rho n^K) + (\nabla_\alpha \sigma_K + \nabla_K \sigma_\alpha)\nabla_\beta(\rho n^K)]$$

$$= \rho B_{ab}^{\alpha\beta}[\nabla_K(\nabla_\alpha \sigma_\beta + \nabla_\beta \sigma_\alpha)n^K + (\nabla_K \sigma_\beta + \nabla_\beta \sigma_K]n^K +$$

$$+ (\nabla_\alpha \sigma_K + \nabla_K \sigma_\alpha)n^K] + B_{ab}^{\alpha\beta}[n^K(\nabla_K \sigma_\beta + \nabla_\beta \sigma_K)\partial_\alpha \rho +$$

$$+ n^K(\nabla_\alpha \sigma_K + \nabla_K \sigma_\alpha)\partial_\beta \rho]$$

$$= \rho B_{ab}^{\alpha\beta}\underset{n}{\mathcal{L}}(\nabla_\alpha \sigma_\beta + \nabla_\beta \sigma_\alpha) + \partial_a \rho B_b^{\beta} n^K(\nabla_K \sigma_\beta + \nabla_\beta \sigma_K) +$$

$$+ \partial_b \rho B_a^{\alpha} n^K(\nabla_\alpha \sigma_K + \nabla_K \sigma_\alpha) \, .$$

Now,

$$B^{\beta}_{b} n^{\kappa} (\nabla_{\beta} \sigma_{\kappa} + \nabla_{\kappa} \sigma_{\beta}) = B^{\beta}_{b} [\nabla_{\beta}(n^{\kappa}\sigma_{\kappa}) - \sigma_{\kappa}(\nabla_{\beta} n^{\kappa})] + B^{\beta}_{b} n^{\kappa} \nabla_{\kappa}\sigma_{\beta}$$

$$= -B^{\beta k}_{bk}\, \sigma^{k} \nabla_{\beta} n_{\kappa} + B^{\beta}_{b} n^{\kappa} \nabla_{\kappa}\sigma_{\beta} \,,$$

and

$$\mathcal{L}_{n} \sigma_{b} = B^{\beta}_{b} \mathcal{L}_{n} \sigma_{\beta} = B^{\beta}_{b} [(\nabla_{\kappa}\sigma_{\beta}) n^{\kappa} + \sigma_{\kappa} \nabla_{\beta} n^{\kappa}]$$

$$= B^{\beta}_{b} n^{\kappa} \nabla_{\kappa} \sigma_{\beta} + B^{\beta\kappa}_{bk} \sigma^{k} \nabla_{\beta} n_{\kappa} \,.$$

Thus

$$B^{\beta}_{b} n^{\kappa} (\nabla_{\beta}\sigma_{\kappa} + \nabla_{\kappa}\sigma_{\beta}) = \mathcal{L}_{n} \sigma_{b} - 2 B^{\beta\kappa}_{bk} \sigma^{k} \nabla_{\beta} n_{\kappa}$$

$$= \mathcal{L}_{n} \sigma_{b} + 2 \sigma^{k} h_{bk} \,.$$

Combining these results:

(iv-12) $$\mathcal{L}_{\rho n} \mathcal{L}_{\sigma} {}'g_{ab} = \rho \mathcal{L}_{n} ({}'\nabla_{a}\sigma_{b} + {}'\nabla_{b}\sigma_{a}) + \partial_{a}\rho (\mathcal{L}_{n}\sigma_{b} + 2\sigma^{k} h_{bk}) + \partial_{b}\rho (\mathcal{L}\, \sigma_{a} + 2\sigma^{k} h_{ak}) \,.$$

Putting (iv-9-12) together, we finally obtain

(iv-13) $$\mathcal{L}^{2}_{v}{}' g_{ab} = \rho^{2} \mathcal{L}^{2}_{n}{}' g_{ab} - 2\rho h_{ab} \mathcal{L}_{n}\rho + 2\partial_{a}\rho\, \partial_{b}\rho + \rho \mathcal{L}_{n}({}'\nabla_{a}\sigma_{b} + {}'\nabla_{b}\sigma_{a}) + \partial_{a}\rho \mathcal{L}_{n}\sigma_{b} + \partial_{b}\rho \mathcal{L}_{n}\sigma_{a} + 2(\partial_{a}\rho\, \sigma^{k} h_{bk} + \partial_{b}\rho\, \sigma^{k} h_{ak}) - 2(h_{ab} \mathcal{L}_{\sigma}\rho + \rho \mathcal{L}_{\sigma} h_{ab}) + \mathcal{L}^{2}_{\sigma}{}' g_{ab} .$$

We see that to find $\mathcal{L}^{2}_{v}{}'g_{ab}$, we need $\mathcal{L}^{2}_{v}{}'g_{ab}$, which we have of course evaluated in the previous section for the case when $G_{\mu\nu}$ van-

vanishes, and in addition the ρ and σ^{m} fields on the initial hypersurface as well as their first Lie derivatives with respect to the n^{μ} field. Lie differentiation of (iv-14) an arbitrary number of times shows that to find the nth Lie derivative of $'g_{ab}$ with respect to the v^{μ} field we need the nth derivative with respect to the n^{μ} field and all Lie derivatives of ρ and σ^{m} fields up the nth with respect to the n^{μ} field. But, clearly, giving the ρ and σ^{m} fields throughout all space is equivalent to giving them on the initial surface, together with all their Lie derivatives with respect to the n^{μ} field; so we have just the degree of arbitrariness in this procedure consistent with really choosing the ρ and σ^{m} fields freely throughout all space.

If we know all the Lie derivatives of the initial surface metric with respect to the geodesic normal field, as well as ρ and σ^{μ} throughout the manifold, we can find all the Lie derivatives of the initial surface metric with respect to V^{μ}. Geometrically, this means we can find the surface metric on an arbitrary family of surfaces (generated by the n^{μ} field); with the surface metric dragged arbitrarily over the surfaces (generated by the σ^{μ} field). The metric of the V_4 is related to this surface metric by:

$$g^{\mu\nu} = B^{\mu\nu}_{mn}\,'g^{mn} + N^{\mu}N^{\nu} \,, \qquad \text{(iv-14)}$$

where N^{μ} is the unit normal of the hypersurface. Only in the case of the geodesic normal field will the dragging field and the normal field coincide everywhere; and it is this circumstance that simplifies the geodesic normal case so much. In every other case, the N^{μ} field and the n^{μ} field will only coincide on the initial surface. The N^{μ} field at other points is found by dragging the surface normal field along the v^μ field, i.e.,

$$N^\mu(\overset{+}{X}_3) = e^{+\frac{t}{v}} N^\mu(\overset{\circ}{X}_3) = e^{+\frac{t}{v}} n^\mu(\overset{\circ}{X}_3) \quad . \qquad \text{(iv-15)}$$

Since, from equation (iv-1) we know that $n^\mu = 1/\rho \, (v^\mu - \sigma^\mu)$, we see that knowledge of the Lie derivatives of n^μ with respect to v^μ is equivalent to knowledge of the Lie derivatives of ρ and σ^μ with respect to v^μ. But, given the Lie derivatives of ρ and σ^μ with respect to n^μ, we can compute their Lie derivatives with respect to v^μ.

Once we know the N^μ field, we can break up the v^μ field into components parallel and perpendicular to N^μ:

$$v^\mu = r N^\mu + s^\mu , \quad N_\mu s^\mu = 0 \quad . \qquad \text{(iv-16)}$$

From the last paragraph, it follows that

$$r(\overset{+}{X}_3) = e^{+\frac{t}{v}} \rho(\overset{\circ}{X}_3) , \quad s^\mu(\overset{+}{X}_3) = e^{+\frac{t}{v}} \sigma^\mu(\overset{\circ}{X}_3) \quad . \qquad \text{(iv-17)}$$

Alternatively, we could have started out by giving <u>r</u> and s^μ on the initial hypersurface, and specified their derivatives of all orders with respect to v^μ . This would enable us to compute the N^μ field directly from (iv-16). This approach would parallel Dirac's method, as we shall indicate later.

As in the last section, we have arrived at a series of results which can be expressed in terms of a bare manifold, an arbitrary initial hypersurface of the X_4, and initial data on this initial hypersurface. Thus, we are able to reverse our steps, and construct a Riemann space which satisfies the empty-space field equations. In this case, four arbitrary functions, ρ and σ^m, enter our description. They describe the orientation of the arbitrary vector field at each point with respect to the geodesic normal vector field passing through the initial hypersurface. In building our Riemann space, this means that we construct the metric on an arbitrary family of hypersurfaces, instead of the unique geodesically parallel family of hypersurfaces, containing the initial hypersurface. The geodesic normal field might be described as acting like a "compass" to give us a fixed direction at each point of the manifold. ρ and σ^m describe how our family of surfaces is oriented at each point with respect to this direction. It is clear, however, that we could solve equations (iv-13) for $\overset{*}{k}{}'g_{ab}$, and then get the metric on the geodesically parallel family of hypersurfaces. The Riemann space is the same in either case; the element of arbitrariness introduced by ρ and σ^m is just that of the family of surfaces on which the metric is constructed. We still have the additional arbitrary elements of choice of coordinates system, initial hypersurface and vector field discussed in Section III, of course.

By adopting a special coordinate system, we can compare our results with those of Dirac.[17] Let us use the three coordinates of the initial hypersurface $'x^a$ as three of the coordinates of the X_4, agreeing to label the corresponding points on all the hypersurfaces resulting from the initial hypersurface by dragging it along the v^μ field with the same values of $'x^a$. For our fourth coordinate, we shall take <u>t</u> the parameter of the family of point transformations generated by v^μ. In this coordinate system:

(iv-18) $$x^\mu = {}'x^a \delta^\mu_a + t\,\delta^\mu_0 \quad ;$$

it is clear that $v^{\mu} \stackrel{*}{=} \delta^{\mu}_{0}$, and $B^{\mu}_{a} \stackrel{*}{=} \delta^{\mu}_{a}$. The following results are then easily proved:

(iv-19) $N^{0} \stackrel{*}{=} (g^{00})^{1/2}\,;\, N^{a} \stackrel{*}{=} (g^{00})^{-1/2} g^{0a}\,;\, N_{\mu} \stackrel{*}{=} \delta^{0}_{\mu} (g^{00})^{-1/2}$

$$'g_{mn} \stackrel{*}{=} g_{mn}\,;\, g^{mn} \stackrel{*}{=} {'g^{mn}} + (g^{00})^{-1} g^{om} g^{on} .$$

From the definition of r and S^{μ}, equation (iv-1), it then follows that in this coordinate system $r \stackrel{*}{=} (g^{00})^{-1/2}$. The contravariant components of $\underline{s}$ in the surface coordinate system are $S^{a} \stackrel{*}{=} -g_{0b}\,{'g^{ab}}$. Thus, $\underline{r}$ and S^{a} correspond to the four arbitrary functions not restricted by the equations of motion in Dirac's formalism, in which these functions are represented by $(g^{00})^{-1/2}$ and $-g_{0b}\,{'g^{ab}}$ (which are equivalent to the $g_{0\nu}$). Their geometrical significance is now clear. Dirac's general expression for the total time (x^{0}) derivative of any dynamical variable not involving the $g_{0\nu}$ corresponds to the equation

(iv-20) $$\mathcal{L}_{v}\eta = \mathcal{L}_{rN}\eta + \mathcal{L}_{s}\eta ,$$

which holds for any function of the $'g_{ab}$ and h_{ab}.[18]

If the v^{μ} field is taken equal to the n^{μ} field, i.e., ρ set equal to one and $\sigma^{\mu} = 0$, then $n^{\mu} = N^{\mu}, \rho = r, S^{\mu} = \sigma^{\mu} = 0$ and the above coordinate system reduces to geodesic normal coordinates. No information is lost thereby, of course, since we can still construct the Riemann space via geodesically parallel hypersurfaces.

V. THE CAUCHY PROBLEM IN THE INTERIOR REGION

So far, we have considered only the case of regions in which the Ricci tensor $R_{\mu\nu}$ vanishes. Physically, they represent those regions where no matter (including in this term, as usual, all non-gravitational fields present) is to be found, only the pure gravitational field. This region is often called the exterior region, on the assumption that the gravitational field here is generated by non-gravitational sources in some other region or regions. Of course, everywhere non-singular, and therefore presumably source-free solutions of the empty-space field equations are known;[19] and even with the imposition of Minkowskian boundary conditions on the gravitational field at infinity it has not been proven that non-stationary solutions of this type do not exist.[20] So the problem of everywhere regular solutions of the empty-space field equations is neither an unimportant nor a fully resolved one. All that can be said, on the basis of work done on the local exterior Cauchy problem, is that sufficiently regular initial data will evolve for some finite time without the development of such singularities.

However, another set of problems of great interest arises in connection with the evolution of systems which do have matter sources. In these regions, due to the presence of the sources, the Ricci tensor does not vanish. More specifically, to each point in such regions there are attached a certain number of dynamical variables, such as densities, pressures, velocity fields (in the case of a fluid), charge and current densities and the electromagnetic field tensor. These dynamical variables, usually tensors themselves, will obey some set of covariant field equations, such as the hydrodynamical equations or Maxwell's equations, usually involving the metric tensor explicitly. In addition, a stress-energy tensor is constructed from these dynamical variables and the metric tensor. Coupling between the metric, as the expression of the gravitational field, and the dynamical variables is then established by the Einstein field equations, which set $G_{\mu\nu}$ equal to a gravitational coupling constant (which we pick to be 1 for simplicity) times the stress energy tensor. There thus arises a set of coupled equations, the dynamical field equations and the gravitational field equations. Solving this set means finding some set of metrical and dynamical variables which together satisfy the equations.

Obviously, different choices of the dynamical fields to be considered lead to rather different problems. Some work has been done on the Cauchy problem for the hydrodynamical, electromagnetic, etc., cases.[21] However, we shall confine ourselves to that part of the general problem arising from the Einstein field equations. That is, we shall assume that a certain set of dynamical variables has given rise to a stress tensor, and shall then investigate the modifications in our previous work that arise from the fact that $G_{\mu\nu}$ now no longer vanishes but must be set equal to $T_{\mu\nu}$. We only consider the case of the geodesic normal field, since all the essential points are thereby brought out. Generalization to the case of the arbitrary field is straightforward.

First of all, let us examine the form taken by the constraint equations on the initial hypersurface. Equations (iii-14) and (iii-16) give us expressions for $G_{\mu\nu}\, n^\mu n^\nu$ and $G_{\mu\nu}\, B^\mu_m n^\nu$ respectively, that are correct independently of any field equations; therefore, in this case, all we need do is to set these expressions equal to the corresponding projections of $T^{\mu\nu}$ in order to get the form of the constraint equations:

(v-1) $$-\frac{1}{2}({}'R + h_{ab}h^{ab} - h^2) = T_{\mu\nu} n^\mu n^\nu ,$$

and

(v-2) $${}'\nabla_b h^b_m - {}'\nabla_m h = T_{\mu\nu} n^\nu B^\mu_m \quad .$$

It should be noted that, on the right-hand side, we may raise or lower the μ, ν and $\underline{m}$ as needed to contract $T_{\mu\nu}$ in the most convenient way, since we know n_μ and B^m_μ. For example, in the case of the so-called incoherent matter stress tensor, $\rho v^\mu v^\nu$, we may choose to regard the contravariant components of the velocity field and the density as our fundamental dynamical variables, which would lead us to contract $\rho v^\mu v^\nu$ with $n_\mu n_\nu$. Should the metric tensor of V_4 occur explicitly on the right in $T_{\mu\nu}$, even with our choice of fundamental dynamical variables, we use the fact that $g_{\mu\nu} = {}'g_{\mu\nu} + n_\mu n_\nu$ to eliminate it from the constraint equations.

These equations may be interpreted physically as follows: Suppose a family of observers situated on test particles to start out from each point of our initial hypersurface, with velocity vector given by n^μ. Then $T_{\mu\nu} n^\mu n^\nu$ is the local density of matter-energy which each such observer would find in his local instantaneous three-space. Equation (v-1) then states that the local density of matter-energy, as defined by these observers, must equal the geometrically given quantity on the left-hand side. $T_{\mu\nu} n^\nu B^\mu_m$, for such observers, is the momentum density, so that (v-2) states that the momentum density is given by the divergence on the left. If the observers were to be connected together, say by a network of ropes, the intrinsic and extrinsic geometry of this rope network would be given by $'g_{ab}$ and h_{ab} respectively. Since test particles move along geodesics, and since the n^μ field's trajectories are geodesics, the observers will continue to move, with n^μ as their velocity vector. The rope network will then move in such a way as to trace out the family of hypersurfaces generated from our initial hypersurface by the n^μ field; their intrinsic and extrinsic geometry will be given by the geometry of the hypersurfaces, and (to anticipate) since the constraint equations must hold on each hypersurface, the varying matter-energy and momentum densities encountered by the observers are correlated with the changing extrinsic and intrinsic geometry of the rope network by (v-1) and (v-2).

Assume that we have a first and second fundamental form and a set of initial values for the dynamical variables which satisfy the constraint equations, and let us examine the evolution of the metric. Equations (iii-4) and (iii-7), which hold in an arbitrary Riemann space, allow us to express the second Lie derivative of the surface metric with respect to n^μ as follows:

(v-3) $\underset{n}{\mathcal{L}}^{2}{}'g_{ab} = 2({}'R_{ab} + 2h_a^{\gamma} h_{b\gamma} - h h_{ab} - B_{ab}^{\mu\nu} R_{\mu\nu})$.

Now, as is well known, the field equations $G_{\mu\nu} = T_{\mu\nu}$ are equivalent to the set $R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$, where T is the trace of the stress tensor. Thus in the presence of matter, we have

(v-4) $\underset{n}{\mathcal{L}}^{2}{}'g_{ab} = 2[{}'R_{ab} + 2h_a^{\gamma} h_{b\gamma} - h h_{ab} - B_{ab}^{\mu\nu}(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)]$.

Now

(v-5) $$B_{mn}^{\mu\nu}(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} T_{\alpha\beta}) = B_{mn}^{\mu\nu} T_{\mu\nu} - \frac{1}{2}{}'g_{mn}({}'g^{\alpha\beta} + n^{\alpha} n^{\beta}) T_{\alpha\beta}$$
$$= B_{mn}^{\mu\nu} T_{\mu\nu} - \frac{1}{2}{}'g_{mn}{}'g^{ab} B_{ab}^{\alpha\beta} T_{\alpha\beta} - \frac{1}{2}{}'g_{mn} n^{\alpha} n^{\beta} T_{\alpha\beta}$$

But (v-1) expresses $n^{\alpha} n^{\beta} T_{\alpha\beta}$ in terms of $'g_{ab}$ and h_{ab}; and when we substitute this into (v-5), and the result into the last term of (v-4), we finally get:

(v-6) $$\underset{n}{\mathcal{L}}^{2}{}'g_{ab} = 2[{}'R_{ab} + 2h_a^{\gamma} h_{b\gamma} - h h_{ab} + \frac{1}{2}{}'g_{ab}({}'R + h_{mn}h^{mn} - h^2) - B_{ab}^{\mu\nu} T_{\mu\nu} - \frac{1}{2}{}'g_{ab}{}'g^{mn} B_{mn}^{\alpha\beta} T_{\alpha\beta}] .$$

Thus, we see that it is the surface-surface components of the stress tensor (which our observers would see as the three-dimensional stress tensor) $B_{mn}^{\mu\nu} T_{\mu\nu}$ which drive the evolution of the metric, as far as the influence of the matter dynamical variables is concerned. Clearly, if we know the successive Lie derivatives of these components, we can continue computing the higher Lie derivatives of the $'g_{ab}$. How far these Lie derivatives of $B_{mn}^{\mu\nu} T_{\mu\nu}$ are freely specifiable, and how far they are already determined by data already given will, of course, depend on the field equations for the dynamical variables (a problem we are not considering), as well as on their relation to the Einstein field equations. Thus we can say no more about this question without referring to specific examples.

However, there is one question that can be still discussed in general, namely, the evolution of the constraint equations. For whatever the nature of the dynamical field equations may be, we know that they demand the vanishing of the divergence of the stress tensor; which is also a consistency condition for the existence of solutions to the Einstein field equations alone. We can show (see Appendix II) that as a consequence of $\nabla_K T^K_\mu = 0$, that

$$\text{(v-7)}\quad {}'g^{nk'}\nabla_k\left[(G_{\mu\lambda}-T_{\mu\lambda})n^\mu B^\lambda_n\right] - {}'g^{nk} B^1_\nu B^K_k\left[(G_{\mu\lambda}-T_{\mu\lambda})n^\mu B^\lambda_1\right]\nabla_k B^\nu_n -$$
$$-{}'g^{nk} h_{kn}\left[(G_{\mu\lambda}-T_{\mu\lambda})n^\mu n^\lambda\right] + h^{nk}\left[B^{\mu\nu}_{nk}(G_{\mu\nu}-T_{\mu\nu})\right] +$$
$$+ \mathcal{L}\left[(G_{\mu\nu}-T_{\mu\nu})n^\mu n^\nu\right] = 0 .$$

Thus, if the constraint equations hold initially, and the surface-surface components of the field equations hold initially (i.e., $B^{\mu\nu}_{mn}G = B^{\mu\nu}_{mn}T_{\mu\nu}$), then the Lie derivative of the constraint equation (v-1) will vanish as a consequence of the vanishing divergence of the stress tensor. A similar result holds for (v-2). Reiteration of this procedure shows us that if the constraint equations hold initially, and $B^{\mu\nu}_{mn}(G_{\mu\nu} - T_{\mu\nu})$ vanishes together with all its Lie derivatives, then the constraint equations will hold everywhere off the initial surface.

Example.

As a very simple example of this procedure, we shall take the case of incoherent matter. As indicated earlier, here the stress tensor is of the form $T^{\mu\nu}=\rho v^{\mu}v^{\nu}$. As relations restricting our dynamical variables we have the requirement that $v_{\mu}v^{\mu}=1$, and the equation of continuity; as well as the inequality $\rho \geqslant 0$. However, the equation of continuity follows from the consistency condition for the Einstein field equations that the divergence of the stress tensor vanish. It is thus not an independent field equation. This leads us to suspect that everything may be determined by the gravitational field equations alone in this case, as indeed we shall see to be the case. Instead of using the four v^{μ} as our dynamical variables, we shall find it convenient to decompose v^{μ} into its surface and normal components:

(v-8) $$v^{\mu} = a n^{\mu} + b^{\mu} \quad , n_{\mu} b^{\mu} = 0 \quad .$$

We shall use the surface components of b^{μ}, $b^{m} = B^{m}_{\mu}b^{\mu}$ (or their covariant components b_m), as well as $\underline{a}$, and ρ as our dynamical variables. From the condition that v^{μ} be of unit length, it follows that

(v-9) $$a^2 + b^m b_m = 1 \; .$$

Consequently, if $'g_{mn}$ and b^m are known, $\underline{a}$ can be found, Similarly, by taking the Lie derivative of (v-9), it is easily seen that if $'g_{ab}$, h_{ab} and $\underset{n}{\mathcal{L}} b^m$ are known, $\underset{n}{\mathcal{L}} \underline{a}$ is known. Therefore, $\underline{a}$ is not a fundamental dynamical variable and may be eliminated from the following equations. $T_{\mu\nu} n^{\mu} n^{\nu}$ is easily seen to equal ρa^2, while $T_{\mu\nu} n^{\mu} B_n^{\nu}$ equals $\rho a b_n$. The constraint equations then become

(v-10) $$-\tfrac{1}{2}({}^{\prime}R + h_{ab}h^{ab} - h^2) = \rho a^2 ,$$

(v-11) $${}^{\prime}\nabla_m h^m_n - {}^{\prime}\nabla_n h = \rho a b_n$$

Suppose that we have found a set of ${}^{\prime}g_{ab}$, h_{ab}, ρ, b_n and $\underline{a}$, which satisfy (v-10) and (v-11), as well as the algebraic criteria demanded by (v-9) and the fact that ρ must be positive. $T_{\mu\nu} B_{mn}^{\mu\nu}$ is easily seen to equal $\rho b_m b_n$, while T equals ρ. Thus,

(v-12) $$\underset{n}{\pounds}^2 g_{ab} = 2({}^{\prime}R_{ab} + 2h_{ar}h^r_b - h h_{ab} - \rho b_a b_b + \tfrac{1}{2}{}^{\prime}g_{ab}\rho) .$$

The Lie derivative of the surface-surface component of $T_{\mu\nu}$ is

(v-13) $$\underset{n}{\pounds} T_{\mu\nu} B^{\mu\nu}_{mn} = b_m b_n \underset{n}{\pounds}\rho + \rho b_n \underset{n}{\pounds} b_m + \rho b_m \underset{n}{\pounds} b_n .$$

But, as we have seen, the Lie derivatives of the constraint equations are zero if the constraint equations hold, as well as the surface-surface components of the Einstein field equations. Thus the Lie derivatives of $T_{\mu\nu} n^{\mu} n^{\nu}$ and $T_{\mu\nu} n^{\mu} B_n^{\nu}$ are known functions of the variables already supposedly known. They are equal to the Lie derivatives of the normal-normal and normal-surface components of $G_{\mu\nu}$, which involve no higher than the known second Lie derivatives of the surface metric. On the other hand, they involve just the four Lie derivatives we need in order to compute the Lie derivative of $B^{\mu\nu}_{mn} T_{\mu\nu}$; namely, $\underset{n}{\pounds}\rho$, and $\underset{n}{\pounds} b_n$ (as well as $\underset{n}{\pounds}\underline{a}$, which, however, can be eliminated as we have seen). Thus, in the case of an incoherent fluid, the Lie derivative of $B^{\mu\nu}_{mn}T_{\mu\nu}$ is determined by the initial data already given.

VI. THE TETRAD APPROACH TO THE CAUCHY PROBLEM

It is well known that the theory of Riemannian geometry and the general theory of relativity can be formulated in terms of tetrads, and the rotation coefficients. The Cauchy problem has also been discussed from this point of view.[22] We shall use Lie derivatives and a specially chosen tetrad to show that the Cauchy problem in the tetrad formalism may be given a simple geometrical interpretation.

A tetrad is a set of four linearly independent vectors defined at each point of the space, tangent to four congruences of curves and usually chosen to be orthonormal. The metric can be expressed in any coordinate system in terms of the components of the tetrad vectors in that system. If we designate the tetrad vectors, which we shall always take to be orthonormal, by $\underset{\nu}{e}^{\mu}$, where the index under the kernel symbol $\underline{e}$ serves to label the four tetrad vectors, then

$$g_{\mu\nu} = \sum_{\alpha} \underset{\alpha}{e}_{\mu} \underset{\alpha}{e}_{\nu} \quad , \tag{vi-1}$$

with a similar expression for $g^{\mu\nu}$ involving contravariant components of the $\underset{\alpha}{e}^{\mu}$. Any tensorial quantity can be projected onto the tetrad vectors by contracting each of its indices with the indices of one of the tetrad vectors. Thus a set of scalars arises, equal in number to the number of components of the tensor. Among physicists it has become customary to refer to these scalars as the physical components of the tensor with respect to that tetrad system. In particular, the covariant derivatives of the tetrad vectors may be projected onto the tetrad, giving rise to the rotation coefficients, which play a somewhat analogous role to that played by the Christoffel symbols in the metric formulation of Riemannian geometry:

(vi-2) $$\gamma_{\mu\nu\kappa} = (\nabla_\alpha \underset{\mu}{e}_\beta)\, \underset{\nu}{e}^\beta\, \underset{\kappa}{e}^\alpha$$

The coefficients (which, it should be emphasized, are scalars) are antisymmetric in the first two indices since $\underset{\mu}{e}_\beta \underset{\nu}{e}^\beta = \delta_{\mu\nu}$. The properties of the rotation coefficients reflect both the properties of the particular congruences of curves to which the tetrad vectors are tangent, and those of the particular Riemann space in which the curves lie. With a special choice of the congruences, certain of the coefficients can be made to vanish or to take on additional symmetry properties. The remaining rotation coefficients will then better reflect the underlying properties of the space; and if we have not imposed any properties on the congruences of curves which cannot be fulfilled by some congruence or congruences in an arbitrary Riemann space, no generality will have been lost. We shall take advantage of this possibility in our treatment of the Cauchy problem.

We shall need the expression for the physical components of the Ricci tensor for our work. This is conveniently calculated from the formula for reversing the order of covariant differentiation of a vector:[23]

(vi-3) $$\nabla_\mu \nabla_\nu \underset{\kappa}{e}^\lambda - \nabla_\nu \nabla_\mu \underset{\kappa}{e}^\lambda = R_{\mu\nu\rho}{}^{\lambda} \underset{\kappa}{e}^\rho \quad .$$

Contracting μ and λ, we get

$$\text{(vi-4)} \quad R_{\nu\rho}\, \underset{\kappa}{e}^{\rho}\, \underset{\lambda}{e}^{\nu} = \underset{\lambda}{e}^{\nu}\left(\nabla_\mu \nabla_\nu \underset{\kappa}{e}^{\mu} - \nabla_\nu \nabla_\mu \underset{\kappa}{e}^{\mu}\right);$$

expressing the various terms in this expression in terms of the rotation coefficients, we finally get an expression for the physical components of the Ricci tensor involving only the rotation coefficients and their ordinary derivatives projected onto the various tetrad vectors: $(\partial_\mu \gamma_{\kappa\iota\lambda})\underset{\alpha}{e}^{\mu}$. These latter are sometimes referred to as intrinsic derivatives;[24] but since the rotation coefficients are scalars, these are the Lie derivatives of the rotation coefficients with respect to the tetrad vectors. We may finally write:

$$\text{(vi-5)} \quad R_{\nu\rho}\, \underset{\kappa}{e}^{\rho}\, \underset{\lambda}{e}^{\nu} = \sum_{\iota} \pounds_{\underset{\iota}{e}} \gamma_{\kappa\iota\lambda} - \sum_{\iota} \pounds_{\underset{\lambda}{e}} \gamma_{\kappa\iota\iota} + \sum_{\iota,\alpha}\left(\gamma_{\kappa\iota\lambda}\gamma_{\iota\alpha\alpha} - \gamma_{\kappa\iota\alpha}\gamma_{\lambda\iota\alpha}\right).$$

(Note the formal similarity to the expression for the Ricci tensor in terms of the Christoffel symbols.)

Now we are ready to look at the Cauchy problem in terms of the rotation coefficients. We shall restrict ourselves again to the geodesic normal case, which allows important simplifications; and shall again first work in a given Riemann space where the empty space field equations hold. We pick an initial spacelike hypersurface in the space, and choose an orthonormal triad tangent to three orthogonal congruences of curves in the three-space. As the fourth vector of our tetrad, we pick the unit normal to our hypersurface, and continue this field by using the geodesic congruence normal to our hypersurface. Up to now we have called the unit tangent to this congruence n^μ, but in this section we shall denote it by $\underset{0}{e}^{\mu}$, since it is one of our tetrad vectors.

We continue our other three triad vectors off the initial hypersurface by parallel-transport along the e_0^μ field. We now have a tetrad field everywhere in our space. Let us examine the rotation coefficients. It is readily shown that the γ_{omo} are zero and that the $\gamma_{oab} = \gamma_{oba}$ as a result of the fact that e_0^μ is a geodesic normal field; and that the $\gamma_{abo} = 0$ as a result of the parallel propagation of the e_m^μ.[25] Thus, of the 24 possible rotation coefficients, only 15 are left, the nine γ_{abc}, and the six γ_{oab} (six because of the symmetry in the last two indices that parallel propagation induces).

The spacelike triad of our tetrad forms a complete triad for the hypersurface when the triad components are taken in the hypersurface coordinate system. We denote these components by e_M^b. We can then compute the hypersurface rotation coefficients $'\gamma_{abc}$; but since the covariant derivative of a vector in the hypersurface, when projected into the hypersurface, equals the hypersurface covariant derivative of the vector, the $'\gamma_{abc}$ will equal the γ_{abc}. Thus, given the hypersurface metric $'g_{ab}$ and an orthonormal triad onthe hypersurface which enable us to compute the hypersurface rotation coefficients, we know all the rotation coefficients γ_{abc}. Alternatively, we might try to give an orthonormal triad on the hypersurface and the $'\gamma_{abc}$ directly, together with a set of integrability conditions on them which assured their derivation from a surface metric; but we shall not pursue this possibility further.

Now we shall examine equation (vi-5) more closely. The physical components of the Ricci tensor break up into the six surface-surface projections when κ and λ are given the values $\underline{k}$ and $\underline{1}$; the three surface-normal projections when $\kappa=0$ and $\lambda=1$; and the one normal-normal component when $\kappa=\lambda=0$. Using the facts that $\gamma_{\iota\kappa\lambda}=-\gamma_{\kappa\iota\lambda}$, $'\gamma_{abc}=\gamma_{abc}$ and $\gamma_{omo}=0$ since g^{μ} is a tangent to a geodesic congruence, we can show that

$$\text{(vi-6)}\quad R_{\nu\rho}\, \underset{k}{e}^{\rho} \underset{1}{e}^{\nu} = -\underset{\underset{o}{e}}{\mathcal{L}} \gamma_{ok1} + {}'R_{mn}\, \underset{k}{e}^{m} \underset{1}{e}^{n} + \sum_m (\gamma_{okm}\gamma_{1mo} + \gamma_{o1m}\gamma_{kmo}) - \sum_m \gamma_{ok1}\gamma_{omm}\ ;$$

while

$$\text{(vi-7)}\quad R_{\nu\rho}\, \underset{o}{e}^{\rho} \underset{1}{e}^{\nu} = \sum_m \Big(\underset{\underset{m}{e}}{\mathcal{L}} \gamma_{om1} - \underset{\underset{1}{e}}{\mathcal{L}} \gamma_{omm}\Big) + \sum_{m,a} (\gamma_{om1}\gamma_{maa} - \gamma_{oam}\gamma_{1ma})\ ;$$

and

$$\text{(vi-8)}\quad R_{\nu\rho}\, \underset{o}{e}^{\rho} \underset{o}{e}^{\nu} = -\sum_m \underset{\underset{o}{e}}{\mathcal{L}} \gamma_{omm} - \sum_{m,n} \gamma_{onm}\gamma_{omn}\ .$$

$'R_{mn}\, \underset{k}{e}^{m} \underset{1}{e}^{n}$ are the physical components of the hypersurface Ricci tensor, which we shall symbolize by $'R_{(kl)}$. In general, indices in parentheses will be used for hypersurface physical components.

It is easily shown that the physical components of the second fundamental form on the surface, $h_{(kl)}$, are just the negative of the γ_{okl}. For

$$\text{(vi-9)}\quad h_{(kl)} = h_{ab}\, \underset{k}{e}^{a} \underset{1}{e}^{b} = -B^{\alpha\beta}_{ab}(\nabla_{\alpha} \underset{o}{e}_{\beta})\, \underset{k}{e}^{a} \underset{1}{e}^{b}$$

$$= -\underset{k}{e}^{\alpha} \underset{1}{e}^{\beta} (\nabla_{\alpha} \underset{o}{e}_{\beta}) = -\gamma_{o1k}\ .$$

Note again that $\gamma_{okl}=\gamma_{olk}$ because of the geodesic normal character of the $\underset{o}{e}^{\mu}$ field. Thus, we can rewrite equation (vi-6) as (vi-6') $$R_{\nu\rho}\, \underset{k}{e}^{\rho}\, \underset{1}{e}^{\nu} = -\underset{\underset{0}{e}}{\mathcal{L}}\, \gamma_{0kl} + {}'R_{(kl)} - h\, h_{(kl)} ,$$

remembering that $\gamma_{km0} = 0$ if we parallel propagate the $\underset{k}{e}^{\mu}$ along the $\underset{0}{e}^{\mu}$ field, and that $\sum_k h_{(kk)} = h$. Thus, the vanishing of the surface-surface physical components of the Ricci tensor enables us to compute the Lie derivatives with respect to $\underset{0}{e}^{\mu}$ of the γ_{0kl} from their values on the initial hypersurface together with the metric of the hypersurface and the hypersurface triad:

(vi-10) $$\underset{\underset{0}{e}}{\mathcal{L}}\, \gamma_{0kl} = {}'R_{(kl)} - h\, h_{(kl)} .$$

Equation (vi-7) represents the set of three constraint equations on the initial data analogous to equation (iii-16). The vanishing of the surface-normal components of the Ricci tensor requires that

(vi-11) $$-\sum_m \underset{\underset{m}{e}}{\mathcal{L}}\, h_{(ml)} + \underset{\underset{1}{e}}{\mathcal{L}}\, h + \sum_{m,a}\left(-h_{(ml)}\gamma_{maa} + h_{(am)}\gamma_{lma}\right) = 0.$$

The fourth constraint equation can be obtained from (vi-6') and (vi-8) by summing the former over $k = l$, and equating the resulting expression for $\sum_k \underset{\underset{0}{e}}{\mathcal{L}}\, \gamma_{0kk}$ with that from the latter equation (this is equivalent to finding the normal-normal component of $G_{\mu\nu}$), giving:

(vi-12) $${}'R - h^2 + \sum_{a,b} h_{(ab)}\, h_{(ab)} = 0 .$$

Note that $\sum_m {}'R_{(mm)} = {}'R$, and that the last term equals $h_{ab}h^{ab}$. Since this is a scalar equation with respect to the hypersurface metric, it is clear that it must take the same form in the tetrad formalism as in our previous work.

Now let us compute the physical components of the Lie derivatives of the $\underset{m}{e}$ vectors with respect to $\underset{0}{e}^{\mu}$:

(vi-13) $$\underset{\lambda}{e}^{\mu} \underset{\underset{0}{e}}{\mathcal{L}} \underset{m}{e}_{\mu} = \underset{\lambda}{e}^{\mu} \underset{0}{e}^{\kappa} \nabla_{\kappa} \underset{m}{e}_{\mu} + \underset{\lambda}{e}^{\mu} \underset{m}{e}_{\kappa} \nabla_{\mu} \underset{0}{e}^{\kappa}$$

$$= \gamma_{m\lambda 0} + \gamma_{0m\lambda} .$$

If $\lambda = 0$, this gives zero, because of the symmetry properties of the rotation coefficients. If $\lambda = 1$, we get

(vi-14) $$\underset{1}{e}^{\mu} \underset{\underset{0}{e}}{\mathcal{L}} \underset{m}{e}_{\mu} = \gamma_{m10} + \gamma_{0m1} = \gamma_{0m1} ,$$

since the γ_{m10} are zero for parallely propagated $\underset{m}{e}^{\mu}$. The surface physical components of the Lie derivatives of the triad vectors with respect to $\underset{0}{e}^{\mu}$ are thus the negative of the physical components of the second fundamental form. Now,

(vi-15) $$\underset{1}{e}^{b} \underset{\underset{0}{e}}{\mathcal{L}} \underset{m}{e}_{b} = \underset{1}{e}^{b} B^{\beta}_{b} \underset{\underset{0}{e}}{\mathcal{L}} \underset{m}{e}_{\beta} = \underset{1}{e}^{\beta} \underset{\underset{0}{e}}{\mathcal{L}} \underset{m}{e}_{b} ,$$

so that we may perform the calculation entirely with hypersurface quantities. Thus, for parallel propagation of the triad axes with respect to the $\underset{0}{e}^{\mu}$ field, the rotation coefficients γ_{0ab} (or the $h_{(ab)}$) tell us how the triad axes change their orientation relative to the original, arbitrarily chosen, triad directions on the initial hypersurface.

Now we can reverse our analysis, as in Section III. Suppose we are given a manifold X_4, together with an initial hypersurface $\overset{0}{X}_3$, and a transvecting vector field $\underset{0}{e}$ which is to be our geodesic normal field. We give a hypersurface metric $'g_{ab}$, and a triad of orthonormal vectors $\underset{m}{e}^{b}$ at each point of $\overset{0}{X}_3$. This enables us to compute the γ_{abc} , which will equal the γ_{abc} . We also give the γ_{0ab} (or the physical components of the second fundamental form), these data being chosen in such a way as to satisfy the constraint equations (vi-11) and (vi-12). Now we compute the Lie derivatives of the γ_{0ab} from equation (vi-10); and the physical components of the Lie derivatives of the triad vectors from (vi-14) and (vi-15). If we drag along the hypersurface coordinate system of the initial hypersurface to the family of hypersurfaces generated by the $\overset{0}{e}{}^{\mu}$ field, we can then compute the triad vectors on these hypersurfaces, and construct the metric for each hypersurface, from the triad vectors and the hypersurface equivalent of (vi-1). Thus the γ_{abc} are known everywhere, and we set $\gamma_{abc} = \gamma_{abc}$. The γ_{0ab} are known, since their Lie derivatives with respect to $\overset{0}{e}{}^{\mu}$ can be computed to all orders by successive differentiation of equation (vi-10). If the other rotation coefficients are set equal to zero, the Riemann space is entirely determined; and in such a way that $R_{\mu\nu}$ vanishes everywhere. The triad vectors will then be seen to be parallely propagated along the $\overset{0}{e}{}^{\mu}$ field with respect to the metric of this Riemann space. Thus, the Cauchy problem has been solved in the tetrad formalism. It should be noted that the use of parallel propagated triad axes served only to simplify the calculations, but was not essential.

VII. LAGRANGIAN AND HAMILTONIAN FORMULATIONS OF THE PROBLEM

Up to now we have given a formulation of the Cauchy problem in the general theory of relativity which might be called "Newtonian," in the sense that we have actually computed the second Lie derivatives of the appropriate field variables — the "accelerations" — directly from their definitions. Like ordinary mechanics or a Lorentz covariant field theory, general relativity may also be given a Lagrangian and a Hamiltonian formulation. Of course, the usual Lagrangian formulation dates back to the earliest days of the theory;[26] but in this form the $g_{\mu\nu}$ are the basic field variables, with the well-known result that the field equations are not of Cauchy-Kowalewski type.[27] But, as we have seen, the choice of the $'g_{\mu\nu}$ (or the $'g_{ab}$ in the hypersurface coordinate system) as field variables, although associated with a particular breakup of the manifold into a family of hypersurfaces, does enable the Cauchy problem to be uniquely formulated and solved symbolically by a covariant iterative process. What we intend to do is to show how a Lagrangian and Hamiltonian formulation of this procedure can be set up. We shall restrict ourselves to the case of a geodesic normal field.

To do this, we start from the usual variational principle for deriving the field equations:

(vii-1) $$\delta \int (-g)^{1/2} R\, d^4x = 0 ,$$

where the variation in the integral is to be induced by arbitrary variations $\delta g_{\mu\nu}$ in the $g_{\mu\nu}$, and the variation of the integral is to vanish over any arbitrary volume. Somewhat more explicitly, we may formulate the procedure as follows. We start with a four-dimensional manifold, which we break up into "volume" elements arbitrarily (we put quotes around the word volume to indicate that all we mean here are infinitesimal regions of the manifold — no volume being defined until a metric is imposed on the manifold). Then we impose a metric on the manifold, and compute the volume with respect to this metric of the "volume" elements, as well as the value of R within each volume element, and perform the integration. As is well known, in any given coordinate system, the volume element dV formed by the coordinate surfaces will be given by $(-g)^{\frac{1}{2}}d^4x$. Then we look for that class of metrics we can impose on the manifold which has the property that the integrand of (vii-1) is an extremum, with respect to small changes of the metric. Naturally, the result of this process is independent of both the breakup of the manifold into "volume" elements we choose, as well as the coordinate system in which we carry out the calculation, since the integrand is an invariant.

In particular, we may break up our manifold by means of a family of hypersurfaces generated from one initial hypersurface by means of a transvecting vector field V^{μ}, in the usual way. Now suppose we give the hypersurface metric $'g_{ab}$, or the equivalent contravariant components $'g^{ab}$ on each hypersurface. If we pick the V^{μ} field as the geodesic normal field to the family of hypersurfaces, as we have seen to be always possible, then the contravariant metric for the full manifold is given by

(vii-2) $$g^{\mu\nu} = {}'g^{\mu\nu} + n^\mu n^\nu = B^{\mu\nu}_{mn}\,{}'g^{mn} + n^\mu n^\nu \,.$$

We notice that, if we do not change the coordinate systems on the manifold or on the hypersurfaces, variations of the six $'g^{mn}$ and of the four n^μ will produce variations in the $g^{\mu\nu}$ given by

(vii-3) $$\delta g^{\mu\nu} = B^{\mu\nu}_{mn}\,\delta\,{}'g^{mn} + n^\mu \delta n^\nu + n^\nu \delta n^\mu \,.$$

Thus we have a set of ten variables $'g^{mn}$ and n^μ whose variations are all independent of each other, and which may therefore be used instead of the $g^{\mu\nu}$ in applying our variational principle (vii-1). The variations of the $'g^{mn}$ produce different surface metrics on our family of hypersurfaces; while the variation of the n^μ change the family of hypersurfaces into which the space is broken up. δn^μ itself may be broken up into normal and surface components with respect to the original n^μ field:

(vii-4) $$\delta n^\mu = \alpha n^\mu + \beta^\mu = \alpha n^\mu + B^\mu_m \beta^m \,, \quad n_\mu \beta^\mu = 0 \,.$$

We can easily see the result of using these variables in the variational principle (vii-1). If we choose the $g^{\mu\nu}$ as our fundamental variables, the result of variation of (vii-1) can be written

(vii-5) $$\int (-g)^{1/2}\, G_{\mu\nu}\, \delta g^{\mu\nu}\, d^4x = 0 \,.$$

If we substitute in equation (vii-3) for $\delta g^{\mu\nu}$, keeping in mind the value (vii-4) for n^μ, we get finally:

$$\text{(vii-6)} \quad \int (-g)^{1/2}\left[B^{\mu\nu}_{mn} G_{\mu\nu}\,\delta' g^{mn} + 2(G_{\mu\nu} n^\mu n^\nu)\,\alpha + 2(G_{\mu\nu} n^\mu B^\nu_n)\,\beta^n\right] d^4x = 0,$$

and since the $\delta' g^{mn}$, α and β^n are ten independent variations, we see that this form of the variational principle results in field equations of the form:

$$\text{(vii-7)} \quad B^{\mu\nu}_{mn} G_{\mu\nu} = 0\ , \quad G_{\mu\nu} n^\mu n^\nu = 0\ , \quad G_{\mu\nu} n^\mu B^\nu_n = 0\ ,$$

as we might have expected. The variations of the $'g^{mn}$, which are associated with varying hypersurface metric give rise to the field equations taking us from hypersurface to hypersurface, while the variations with respect to the family of hypersurfaces give rise to the constraint equations. The latter ensure that a set of data which make the surface-surface components of $G_{\mu\nu}$ vanish for a particular family of hypersurfaces will also make them vanish for any other set.

Actually all of our results so far will hold true for any family of surfaces, if n^μ represents their unit normal field, geodesic or not. It is only now, when we express the variational principles in terms of Lie derivatives that our restriction to a geodesic normal field becomes important. To do this we need to express R in terms of the Lie derivatives of $'g_{ab}$ with respect to n^μ. Equation (iii-13) gives us

$$\text{(vii-8)} \quad R = {}'R + h_{ab}h^{ab} - h^2 + 2R_{\mu\nu} n^\mu n^\nu\ ;$$

contracting equation (iii-4) with $'g^{ab}$ gives

$$\text{(vii-9)} \quad {}'g^{ab}\,\pounds_n^2\,{}'g_{ab} = 2h^{ab}h_{ab} - 2R_{\mu\nu} n^\mu n^\nu\ ;$$

substituting this value of $R_{\mu\nu} n^{\mu} n^{\nu}$ into (vii-8) gives

(vii-10) $$R = -{}'g^{ab} \underset{n}{\mathcal{L}}^{2}\,{}'g_{ab} + {}'R + 3h_{ab}h^{ab} - h^{2} .$$

Thus we may write our variational principle as

(vii-1') $$\delta \int (-g)^{1/2} \left(-{}'g^{ab} \underset{n}{\mathcal{L}}^{2}\,{}'g_{ab} + {}'R + 3h_{ab}h^{ab} - h^{2}\right) d^{4}x = 0 .$$

h_{ab} is to be regarded as shorthand for $-\frac{1}{2} \underset{n}{\mathcal{L}}\,{}'g_{ab}$ in this expression.

If we adopt that particular coordinate system for the whole manifold consisting of the $'x^{a}$ of the initial hypersurface as dragged to each succeeding hypersurface by the n^{μ} field, and t, the parameter labelling the hypersurfaces (a coordinate system that we used once before in Section IV) then $(-g)^{\frac{1}{2}}\,d^{4}x = (-{}'g)^{\frac{1}{2}}\,d^{3\prime}x\,dt$. If we set

$$L = \int \mathcal{L}\, d^{3\prime}x = \int (-{}'g)^{1/2} \left(-{}'g^{ab} \underset{n}{\mathcal{L}}^{2}\,{}'g_{ab} + {}'R + 3h_{ab}h^{ab} - h^{2}\right) d^{3\prime}x ,$$

then our variational integral is of the form $\int L\,dt$; and it turns out that the Lie derivative in this expression can be treated very much like an ordinary total derivative in handling the variation. In particular, a total Lie derivative may be added to the integrand without altering the field equations resulting from variation of the integral. The field equations will be of the usual form $\frac{\partial \mathcal{L}}{\partial\, {}'g_{ab}} - \underset{n}{\mathcal{L}}\left(\frac{\partial \mathcal{L}}{\partial(\underset{n}{\mathcal{L}}\,{}'g_{ab})}\right) + \ldots = 0$, with $\underset{n}{\mathcal{L}}\,{}'g_{ab}$ playing the role of velocities, the number of terms of course depending on the order of the highest Lie derivative appearing in the integrand. This can be seen from the fact that in the particular coordinate system we have adopted above, the Lie derivative with respect to n^{μ} reduces to the ordinary derivative with respect to t (as is easily seen from rules for Lie derivation of tensors given at the end of Section II), and that δ and $\underset{n}{\mathcal{L}}$ commute for variation of the $'g_{ab}$.

In particular, for our variational integral, a total Lie derivative can be subtracted which will remove the term containing the second Lie derivatives. To find this term, we need the Lie derivative of $(-'g)^{1/2}$ which is found by the method of Appendix I to be given by

(vii-11) $$\underset{n}{\mathcal{L}}(-'g)^{1/2} = -(-'g)^{1/2}h .$$

Then it is easily shown that

(vii-12) $$\mathcal{L} = -\underset{n}{\mathcal{L}}\left[(-'g)^{1/2}\,'g^{ab}\underset{n}{\mathcal{L}}\,'g_{ab}\right] + (-'g)^{1/2}('R + h^2 - h_{ab}h^{ab}) .$$

We are thus able to adopt the new Lagrangian density

(vii-13) $$\mathcal{L}' = (-'g)^{1/2}('R + h^2 - h_{ab}h^{ab}) ,$$

The field equations will then result from variation of $\int \mathcal{L}' d^3{}'x\, dt$; but since our result is independent of coordinate system if we replace $(-'g)^{\frac{1}{2}} d^3{}'x\, dt$ by $(-g)^{\frac{1}{2}} d^4x$, the field equations must result from variation of

(vii-14) $$\int (-g)^{1/2}('R + h^2 - h_{ab}h^{ab})\, d^4x .$$

A first order Lagrangian for the general theory of relativity in terms of the first and second fundamental forms of an arbitrary family of hypersurfaces and the normal field to that family has been given by Arnowitt, Deser and Misner.[28] They actually write the Lagrangian in terms of the momenta canonically conjugate to the hypersurface metric; but, essentially, their Lagrangian reduces to equation (vii-14) (in a special coordinate system) for the case of a geodesically parallel family of hypersurfaces. In their formalism the four constraint equations result from variation of the full Lagrangian with respect to the four arbitrary functions (equivalent to Dirac's four) which we discussed in Section IV. We referred to them there as $\underline{r}$ and s^m, and interpreted them as components of the arbitrary vector field taking us from hypersurface to hypersurface with respect to the normal field. However, as we indicated above and shall now show in detail, the constraint equations may also be obtained from that part of the Lagrangian an which describes only the evolution along a geodesically parallel family of surfaces, if we vary the vector field of the manifold which is to be the geodesic normal field. It is easily checked that the $B^{\mu\nu}_{mn} G_{\mu\nu} = 0$ field equations result from variation of equation (vii-14) with respect to the $'g^{ab}$. We shall now show that the remaining field equations, the constraint equations, result from variation of (vii-14) with respect to the n field. If we write out $(h^2 - h_{ab}h^{ab})$ in terms of Lie derivatives explicitly, it becomes

(vii-15) $$h^2 - h_{ab}h^{ab} = \frac{1}{4}\left(\underset{n}{\mathcal{L}}{}'g_{ab}\underset{n}{\mathcal{L}}{}'g_{cd} - \underset{n}{\mathcal{L}}{}'g_{ac}\underset{n}{\mathcal{L}}{}'g_{bd}\right){}'g^{ab}\,{}'g^{cd}.$$

The variation of this expression induced by a variation in the n^μ field is found, of course, by taking the difference between (vii-15) with all Lie derivatives with respect to n^μ replaced by Lie derivatives with respect to $n^\mu + \delta n^\mu$ and (vii-15), keeping only first order terms in the δn^μ. Using the rule that the Lie derivative with respect to the sum of two vector fields is the sum of the Lie derivatives with respect to each vector field, we see that

(vii-15) $$\delta(h^2 - h_{ab}h^{ab}) = \frac{1}{2}\left(\underset{n}{\mathcal{L}}{}'g_{ab}\underset{\delta n}{\mathcal{L}}{}'g_{cd} - \underset{n}{\mathcal{L}}{}'g_{ac}\underset{\delta n}{\mathcal{L}}{}'g_{bd}\right){}'g^{ab}\,{}'g^{cd}.$$

Now $\underset{\delta n}{\mathcal{L}} = \underset{\alpha n}{\mathcal{L}} + \underset{\beta}{\mathcal{L}}$, and, as we saw in Section IV, $\underset{\alpha n}{\mathcal{L}} = \alpha \underset{n}{\mathcal{L}}$, so that we finally arrive at

(vii-16) $$\delta(h^2 - h_{ab}h^{ab}) = \tfrac{1}{2}\Big[\alpha(\underset{n}{\mathcal{L}}{}'g_{ab}\underset{n}{\mathcal{L}}{}'g_{cd} - \underset{n}{\mathcal{L}}{}'g_{ac}\underset{n}{\mathcal{L}}{}'g_{bd})\,{}'g^{ab}\,{}'g^{cd} + (\underset{n}{\mathcal{L}}{}'g_{ab}\underset{\beta}{\mathcal{L}}{}'g_{cd} - \underset{n}{\mathcal{L}}{}'g_{ac}\underset{\beta}{\mathcal{L}}{}'g_{bd})\,{}'g^{ab}\,{}'g^{cd}\Big]$$
$$= 2\alpha(h^2 - h_{ab}h^{ab}) + (h^{ab} - h\,{}'g^{ab})({}'\nabla_a\beta_b + {}'\nabla_b\beta_a),$$

where we have used the fact that $h_{ab} = -\tfrac{1}{2}\underset{n}{\mathcal{L}}{}'g_{ab}$; and that β^{μ} is a hypersurface vector field, so that the Lie derivative of the hypersurface metric with respect to β^{m} is the Killing form of the β_m field. We also need the variation of $(-g)^{\frac{1}{2}}$ with respect to δn^{μ} to carry out the δn^{μ} variation of (vii-14). Since

(vii-17) $$\delta(-g)^{1/2} = -\tfrac{1}{2}(-g)^{1/2}g_{\mu\nu}\,\delta g^{\mu\nu}$$
$$= -\tfrac{1}{2}(-g)^{1/2}({}'g_{\mu\nu} + n_{\mu}n_{\nu})(B^{\mu\nu}_{mn}\,\delta' g^{mn} + n^{\mu}\delta n^{\nu} + n^{\nu}\delta n^{\mu})$$
$$= -\tfrac{1}{2}(-g)^{1/2}\,{}'g_{mn}\,\delta' g^{mn} - n_{\nu}\,\delta n^{\nu}(-g)^{1/2}$$
$$= -\tfrac{1}{2}(-g)^{1/2}\,{}'g_{mn}\,\delta' g^{mn} - (-g)^{1/2}\alpha;$$

the variation induced in $(-g)^{\frac{1}{2}}$ by the δn^{μ} is just $-\alpha(-g)^{\frac{1}{2}}$. $'R$ of course suffers no variation as a result of δn^{μ}. Variation of (vii-14) with respect to δn^{μ} is now seen to give

(vii-18) $$\int(-g)^{1/2}\big[\alpha(-{}'R + h^2 - h_{ab}h^{ab}) + (h^{ab} - h\,{}'g^{ab})({}'\nabla_a\beta_b + {}'\nabla_b\beta_a)\big]d^4x,$$

combining the results of (vii-16) and (vii-17). Now an integration by parts on the last term of (vii-13) yields a total surface divergence, which may be discarded as it has no effect on the field equations, leaving

$$\text{(vii-19)}\quad \int \left[\alpha(-{}'R + k^2 - k_{ab}k^{ab})(-{}'g)^{1/2} - 2\{{}'\nabla_c(-{}'g)^{1/2}(k^{cd} - k\,{}'g^{cd})\}\beta_d \right] d^{3'}x\,dt,$$

when we return to our special coordinate system. Now the coefficients of α and β_m must vanish, since these are the arbitrary variations which together make up the δn^μ variation; and we see that the constraint equations (iii-15) and iii-17) do indeed hold as a result of this variation. Thus equation (vii-13) yields a Lagrangian involving only first order Lie derivatives from which all the field equations may be derived.

The role of the $\underset{n}{\mathcal{L}}\ {}'g_{ab}$ as "velocities" in the variation with respect to the $'g_{ab}$ discussed above suggests the definition of momenta and a Hamiltonian in the usual way. And indeed if we define

$$\text{(vii-20)}\qquad p^{ab} = \frac{\partial \mathcal{L}'}{\partial(\underset{n}{\mathcal{L}}'g_{ab})} = (-{}'g)^{1/2}\left(k^{ab} - {}'g^{ab}k\right),$$

and

$$\text{(vii-21)}\qquad \mathcal{H} = p^{ab}\underset{n}{\mathcal{L}}'g_{ab} - \mathcal{L}' = -(-{}'g)^{1/2}({}'R + k_{ab}k^{ab} - k^2)$$
$$= -(-{}'g)^{1/2}\left[{}'R + (-{}'g)^{1/2}(p^{ab}p_{ab} - p^2)\right],$$

the resulting Hamiltonian equations of motion

$$\text{(vii-22)}\qquad \underset{n}{\mathcal{L}}'g_{ab} = \frac{\partial \mathcal{H}}{\partial p^{ab}}, \qquad \underset{n}{\mathcal{L}}\,p^{ab} = -\frac{\partial \mathcal{H}}{\partial {}'g_{ab}}$$

are, indeed, equivalent to the definition of the momenta and the equations of motion $B^{\mu\nu}_{mn}G_{\mu\nu} = 0$ respectively. Thus we see that $\mathcal{H}$ $(H = \int \mathcal{H} d^{3'}x$

has been called the main part of the Hamiltonian) governs the evolution of the surface metric on a family of geodesically parallel hypersurfaces.[29] If, in addition, the constraint equations hold on one hypersurface, they will hold everywhere, and the field equations resulting from $\mathcal{H}$ will assure the vanishing of $G_{\mu\nu}$. These constraint equations are easily rewritten in terms of the momenta by solving (vii-20) for the h_{ab} and substituting into (iii-15) and (iii-17) giving

(vii-23) $${}^{\prime}\nabla_b p^{ab} = 0 \quad , \mathcal{H} = 0$$

It can be seen from equations (vii-11), and (A1-10) that $\mathcal{H}$, whether it vanishes or not, has a vanishing Lie derivative as a result of the remaining field equations. This is analogous to the existence of the Jacobian integral in ordinary mechanics, the conserved quantity associated with any Lagrangian not explicitly containing the time.

VIII. CONCLUSIONS

The main feature characterizing this work has been the Newtonian approach used in Sections III–VI coupled with the geometrical emphasis gained by the use of the Lie derivative. These have enabled us to develop the relationship between the geometry of the initial hypersurface and the first neighboring geodesically parallel hypersurface; and the relationship of these two and the surface Riemann tensor to the geometry of the second neighboring geodesically parallel hypersurface. The surface components of the Ricci tensor have been shown to determine the evolution of the surface metric in an arbitrary space; and we have shown how the vanishing of these components in the exterior region or their replacement by the surface components of the stress-energy tensor in the interior region serve to determine this evolution.

The constraint equations have been given their usual interpretation in Section III, but in Section IV it has been shown that they may be reinterpreted as posing the problem of embedding two hypersurfaces with infinitesimally differeing arbitrary metrics into a four-dimensional Riemann space with vanishing Ricci tensor. If we choose to make the hypersurfaces geodesically parallel, the constraint equations may be interpreted as imposing one condition on the two metrics.

The Cauchy problem has been interpreted, in the tetrad formalism, as the problem of determining the evolution of the physical components of the second fundamental form with respect to a suitably chosen tetrad.

A Lagrangian and Hamiltonian formalism for handling the evolution of the metric along a family of geodesically parallel surfaces is given in Section VII. It is interesting that, in the Lagrangian formalism, not only do the equations of evolution of the metric follow from variations of the hypersurface metric, but that the constraint equations follow from variation of the vector field which is to be the geodesic normal field.

It is evident that a number of important problems remain unsolved. We shall mention a few of these, which we hope the methods developed herein may be useful in treating.

Perhaps the outstanding problem still unsolved is that of the degrees of freedom of the pure gravitational field. Since the hypersurface metric and the second fundamental form are restricted by four constraint equations, as well as being covariant with respect to hypersurface coordinate transformations, it is evident that, as functions of arbitrary coordinates, they contain more information than needed on a hypersurface for the construction of a Riemann space satisfying the exterior field equations. It is well known that the number of pieces of information needed to specify such a space is four per point of a spatial hypersurface.[30] So far, however, no one has succeeded in isolating four such pieces of information in closed form; nor in giving a geometrical interpretation of what elements of the surface geometry such data might represent.[31] A major difficulty has been in finding a closed-form solution of the constraint equations. Perhaps the geometrical interpretation of the equations would help in understanding what kind of geometrical data on the hypersurface they leave freely specifiable. The reinterpretation of the constraint equations given here seems to offer some hope in this respect.

Should it prove impossible to find closed-form general solutions to the constraint equations, or until such are found, it may be useful to find approximate solutions to the constraint equations in the neighborhood of the known exact solutions to the constraint equations. The method outlined in Appendix I may be used to solve this problem.

Another problem that may be treated by the methods developed here is the Cauchy problem on a null hypersurface. Recent work on this subject has proved it to be one of the most fruitful approaches to the study of gravitational radiation.[32] It is hoped that the use of the approach developed here will lead to a more intuitive understanding of the significance of this work.

We also hope to study the problem of the equations of motion using this method. Suppose that on the initial spacelike hypersurface the stress-energy tensor vanishes everywhere except inside of a number of finite regions. A knowledge of the gravitational field outside these regions, plus certain boundary conditions on the boundaries of the regions inside of which $T_{\mu\nu} \neq 0$, should serve to determine the equations of motion of the sources without detailed knowledge of the interior fields. It may also prove possible to pass to the limit of point singularities in the initial gravitational fields in this way. A class of solutions to the exterior initial value problem exists which seems to represent a number of sources at rest with no gravitational radiation field present, as well as multiple sources initially at rest.[33] It might be possible through examination of the equations of motion of these solutions to gain insight into the important problem of the generation of gravitational radiation by the motion of sources.

APPENDIX I

LIE DERIVATIVES OF FUNCTIONS OF $'g_{ab}$

In this appendix we shall outline a relatively simple method of calculating the Lie derivatives of some function of the $'g_{ab}$, such as the Christoffel symbols or the Riemann tensor, when the Lie derivative of $'g_{ab}$ is known.[34] It is clear that, since these quantities are formed from the $'g_{ab}$ by a number of ordinary differentiations and algebraic operations, that one could evaluate these Lie derivatives straightforwardly, using the rules given at the end of Section II on the Lie derivative. However, the method we shall outline is simpler, and of wider application. We shall also show how the higher order Lie derivatives of $'g_{ab}$, such as $\underset{n}{\mathcal{L}}^3\, 'g_{ab}$ are easily computed in this way; and give an alternate proof of the fact that the Lie derivatives of the constraint equations vanish if the field equations hold on the initial hypersurface, by directly evaluating these Lie derivatives.

Suppose we have any set of functions of the $'g_{ab}$, which we symbolize by $F_\Lambda('g_{ab})$, where Λ stands for any indices that may occur. We should like to find the changes that are induced in the F_Λ by any variation whatsoever $\delta 'g_{ab}$ that we choose, subject to the restriction that it be of the same tensorial character as $'g_{ab}$. Clearly, this is given by

(A1-1) $$\delta F_\Lambda('g_{ab}) = F_\Lambda('g_{ab} + \delta 'g_{ab}) - F_\Lambda('g_{ab}) \ .$$

This variation will be of the same tensorial nature as F_Λ , if F_Λ is a tensor; but it may be a tensor, even if F_Λ is not, as in the case of the Christoffel symbols, where the difference between two sets of Christoffel symbols at a point is a tensor. Now suppose the variation that we have indiced is infinitesimal. Then the change in F_Λ will be infinitesimal, and to first order will be some linear function of the $\delta' g_{ab}$. From now on $\delta' g_{ab}$ will symbolize an infinitesimal variation of $'g_{ab}$ and δF_Λ will symbolize the first order variation in F_Λ induced by this variation.

It can be readily shown that

$$\text{(A1-2)} \qquad \delta' R_{uab}^{\cdots v} = {}'\nabla_u (\delta' \Gamma^v_{ab}) - {}'\nabla_a (\delta' \Gamma^v_{ub}) ,$$

where $\delta' \Gamma^v_{ab}$ is the variation induced in the Christoffel symbols, which in turn is given by

$$\text{(A1-3)} \quad \delta' \Gamma^v_{ab} = \frac{1}{2} {}'g^{vk} \left[- {}'\nabla_k \delta' g_{ab} + {}'\nabla_b \delta' g_{ka} + {}'\nabla_a \delta' g_{bk} \right] .$$

Substitution of (A1-3) into (A1-2) gives:

$$\text{(A1-4)} \quad \delta' R_{uab}^{\cdots v} = \frac{1}{2} {}'g^{vk} \left[- {}'\nabla_u {}'\nabla_k \delta' g_{ab} + {}'\nabla_a {}'\nabla_k \delta' g_{ub} + {}'\nabla_u {}'\nabla_b \delta' g_{ka} - \right.$$
$$\left. - {}'\nabla_a {}'\nabla_b \delta' g_{ku} \right] + \frac{1}{2} {}'g^{vk} \left[- {}'R_{uab}^{\cdots m} \delta' g_{mk} - {}'R_{uak}^{\cdots m} \delta' g_{bm} \right] .$$

Contracting over u and v in (A1-2), and substituting from (A1-3) we get $\delta' R_{ab}$:

$$\text{(A1-5)} \quad \delta' R_{ab} = \frac{1}{2} {}'g^{cd} \left[- {}'\nabla_c {}'\nabla_d \delta' g_{ab} + {}'\nabla_c {}'\nabla_a \delta' g_{bd} + {}'\nabla_c {}'\nabla_b \delta' g_{ad} - {}'\nabla_a {}'\nabla_b \delta' g_{cd} \right] .$$

Now suppose the variation $\delta' g_{ab}$ is that induced by moving to an infinitesimally neighboring hypersurface a distance $n^k dt$ away.

Then $\delta\, 'g_{ab} = \mathcal{L}_n\, 'g_{ab}\, dt = -2h_{ab}\, dt$, and substitution of this expression in (A1-3, (A1-4), and (A1-5) will give us the change in the Christoffel symbols, Riemann tensor and Ricci tensor respectively induced by moving to the neighboring hypersurface. But this is just the Lie derivative of these quantities with respect to n^k times dt. Thus we are able to show that:

$$\text{(A1-6)} \quad \mathcal{L}_n\, 'R_{ab} = 'g^{cd}\, '\nabla_c\, '\nabla_d h_{ab} - '\nabla_a('\nabla_c h^c_b - '\nabla_b h) - '\nabla_c\, '\nabla_b h^c_a + \\ + 'R_{cab}{}^d h^c_d - 'R_{ad} h^d_b \, ,$$

where we have used the rule for the commutation of the order of covariant differentiation in deriving the last result from (A1-5).

We can now take the Lie derivative of equation (iii-9) to find $\mathcal{L}^3_n\, 'g_{ab}$:

$$\text{(A1-7)} \quad \mathcal{L}^3_n\, 'g_{ab} = 2[\,'g^{cd}\, '\nabla_c\, '\nabla_d h_{ab} - '\nabla_c\, '\nabla_b h^c_a + 'R_{cabd} h^{cd} - 2\, 'R_{ad} h^d_b - \\ - 2 h^d_a\, 'R_{db} - 8 h^r_a h_{rm} h^m_b + 4 h h^r_a h_{rb} + h\, 'R_{ab} + \\ + h_{ab} h_{mn} h^{mn} + h^2 h_{ab}\,].$$

This technique may be used for the calculation of all the higher order Lie derivatives of $'g_{ab}$.

In order to compute the Lie derivatives of the constraint equations, we also need the Lie derivative of $'R$ with respect to n^μ :

$$\text{(A1-8)} \quad \mathcal{L}_n\, 'R = \mathcal{L}_n('g^{ab}\, 'R_{ab}) = 'g^{ab} \mathcal{L}_n\, 'R_{ab} + 'R_{ab} \mathcal{L}_n\, 'g_{ab} \\ = '\nabla_c[\,'\nabla_d(h\, 'g^{cd} - h^{cd})] + 'g^{ab}\, '\nabla_a('\nabla_c h^c_b - '\nabla_b h) + \\ + 2\, 'R_{ab} h^{ab} \, .$$

Using (iii-3) and (iii-9), it is easily shown that

$$\text{(A1-9)} \quad \mathop{\mathcal{L}}_{n}(k_{ab}k^{ab}-k^2) = -2k^{cd}\,{}'R_{cd} + 2k({}'R + k_{cd}k^{cd} - k^2).$$

Thus,

$$\text{(A1-10)} \quad \mathop{\mathcal{L}}_{n}({}'R + k_{ab}k^{ab} - k^2) = {}'\nabla_c[\nabla_d(k\,{}'g^{cd} - k^{cd})] + {}'g^{db}\,{}'\nabla_d(\nabla_c k^c_b - \nabla_b k) + 2k({}'R + k_{cd}k^{cd} - k^2).$$

We have used the field equations $B^{\mu\nu}_{mn}R_{\mu\nu}=0$ in establishing (iii-9); so we see that if, in addition the constraint equations (iii-15) and (iii-17) hold on the initial hypersurface, that the Lie derivative of equation (iii-15) will vanish. A similar result can be proven for the remaining constraint equations (iii-17). Repeated Lie differentiation of the constraint equations then shows that if the constraint equations hold initially and the successive Lie derivatives of $B^{\mu\nu}_{mn}R_{\mu\nu}$ also vanish, that the constraint equations hold everywhere.

Although the proof given here is somewhat simpler technically than that given in the text, Section III and Appendix II, it does not bring out the role of the Bianchi identities in the result.

APPENDIX II

THE BIANCHI IDENTITIES

The Bianchi identities provide four differential relations between the field equations. As a consequence, there are four relations between the Lie derivatives of the normal-normal and normal-surface components of $G_{\mu\nu}$. We derive these relationships below.

$$\nabla_\kappa G^\kappa_\mu = g^{\nu\kappa}\nabla_\kappa G_{\mu\nu} \equiv 0$$

$$= ({}'g^{\nu\kappa} + n^\nu n^\kappa)\nabla_\kappa G_{\mu\nu}$$

(A2-1) $$= {}'g^{\nu\kappa}\nabla_\kappa G_{\mu\nu} + n^\kappa n^\nu \nabla_\kappa G_{\mu\nu} .$$

First we deal with $\underset{n}{\mathcal{L}}(G_{\mu\nu}n^\mu n^\nu)$. Multiply equation (A2-1) by n^μ:

$$n^\mu {}'g^{\kappa\nu}\nabla_\kappa G_{\mu\nu} + n^\mu n^\kappa n^\nu \nabla_\kappa G_{\mu\nu} = 0$$

$$= {}'g^{kn} n^\mu B^{\kappa\nu}_{kn}\nabla_\kappa G_{\mu\nu} + n^\kappa \nabla_\kappa (G_{\mu\nu}n^\mu n^\nu),$$

since $n^\kappa \nabla_\kappa n^\mu = 0$. But the last term is the Lie derivative of $G_{\mu\nu}n^\mu n^\nu$:

(A2-2) $${}'g^{kn} n^\mu B^{\kappa\nu}_{kn}\nabla_\kappa G_{\mu\nu} + \underset{n}{\mathcal{L}}(G_{\mu\nu}n^\mu n^\nu) = 0 .$$

Now

$${}'g^{kn} n^{\mu} B^{\kappa\nu}_{kn} \nabla_\kappa G_{\mu\nu} = {}'g^{kn} B^{\kappa}_{k} \nabla_\kappa (G_{\mu\nu} B^{\nu}_{n} n^{\mu}) -$$

$$- {}'g^{kn} B^{\kappa}_{k} G_{\mu\nu} (n^{\mu} \nabla_\kappa B^{\nu}_{n} + B^{\nu}_{n} \nabla_\kappa n^{\mu})$$

$$= {}'g^{kn} {}'\nabla_k (G_{\mu\nu} B^{\nu}_{m} n^{\mu}) - {}'g^{kn} B^{\kappa}_{k} [G_{\mu\lambda} n^{\mu} (B^{\lambda}_{\nu} + C^{\lambda}_{\nu}) \nabla_\kappa B^{\nu}_{n}$$

$$+ G_{\lambda\nu} B^{\nu}_{n} (B^{\lambda}_{\mu} + C^{\lambda}_{\mu}) \nabla_\kappa n^{\mu}]$$

$$= {}'g^{kn} {}'\nabla_k (G_{\mu\nu} B^{\nu}_{m} n^{\mu}) - {}'g^{kn} B^{\kappa}_{k} (\nabla_\kappa B^{\nu}_{m}) [(G_{\mu\lambda} n^{\mu} B^{\lambda}_{l}) B^{l}_{\nu} +$$

(A2-3) $$+ (G_{\mu\lambda} n^{\mu} n^{\lambda}) n_{\nu}] -$$

$$- g^{kn} B^{\kappa}_{k} [(G_{\lambda\nu} B^{\nu\lambda}_{nl}) B^{l}_{\mu} + (G_{\lambda\nu} B^{\nu}_{n} n^{\lambda}) n_{\mu}] \nabla_\kappa n^{\mu};$$

substituting (A2-3) into (A2-4), we get equation (iii-20) of the text.

Next we deal with $\underset{n}{\mathcal{L}} (G_{\mu\nu} B^{\mu}_{m} n^{\nu})$, by multiplying (A2-1) by B^{μ}_{m}:

(A2-4) $$B^{\mu}_{m} {}'g^{\kappa\nu} \nabla_\kappa G_{\mu\nu} + n^{\kappa} n^{\nu} B^{\mu}_{m} \nabla_\kappa G_{\mu\nu} = 0 .$$

Now

$$n^{\kappa} n^{\nu} B^{\mu}_{m} \nabla_\kappa G_{\mu\nu} = n^{\kappa} B^{\mu}_{m} (\nabla_\kappa (G_{\mu\nu} n^{\nu}))$$

$$= B^{\mu}_{m} [n^{\kappa} \nabla_\kappa (G_{\mu\nu} n^{\nu}) + G_{\kappa\nu} n^{\nu} \nabla_\mu n^{\kappa} - G_{\kappa\nu} n^{\nu} \nabla_\mu n^{\kappa}]$$

$$= B^{\mu}_{m} \underset{n}{\mathcal{L}} G_{\mu\nu} n^{\nu} - B^{\mu}_{m} (n^{\nu} G_{\kappa\nu}) (B^{\lambda}_{\kappa} + C^{\lambda}_{\kappa}) \nabla_\mu n^{\kappa}$$

(A2-5) $$= \underset{n}{\mathcal{L}} (B^{\mu}_{m} n^{\nu} G_{\mu\nu}) - B^{\mu}_{m} \nabla_\mu n^{\kappa} [(G_{\lambda\nu} n^{\nu} B^{\lambda}_{l}) B^{l}_{\kappa} +$$

$$+ (G_{\lambda\nu} n^{\lambda} n^{\nu}) n_{\kappa}];$$

and

$$B^{\mu}_{m} {}'g^{\kappa\nu} \nabla_\kappa G_{\mu\nu} = {}'g^{kn} B^{\mu\kappa\nu}_{mkn} \nabla_\kappa G_{\mu\nu}$$

$$= {}^{\prime}g^{kn} B^{K}_{k} \nabla_{K} (G_{\mu\nu} B^{\mu\nu}_{mn}) - {}^{\prime}g^{kn} B^{K}_{k} G_{\mu\nu} (B^{\mu}_{m} \nabla_{K} B^{\nu}_{n} + B^{\nu}_{n} \nabla_{K} B^{\mu}_{m})$$

$$= {}^{\prime}g^{kn\prime} \nabla_{k} (G_{\mu\nu} B^{\mu\nu}_{mn}) - {}^{\prime}g^{kn} B^{K}_{k} [(G_{\mu\lambda} B^{\mu}_{m})(B^{\lambda}_{\nu} + C^{\lambda}_{\nu}) \nabla_{K} B^{\nu}_{n} + + (G_{\lambda\nu} B^{\nu}_{n})(B^{\lambda}_{\mu} + C^{\lambda}_{\mu}) \nabla_{K} B^{\mu}_{m}]$$

$$= {}^{\prime}g^{kn\prime} \nabla_{k} (G_{\mu\nu} B^{\mu\nu}_{mn}) - - {}^{\prime}g^{kn} B^{K}_{k} [n_{\nu} (G_{\mu\lambda} B^{\mu}_{m} n^{\lambda}) + + B^{1}_{\nu} (G_{\mu\lambda} B^{\mu\lambda}_{m1})] \nabla_{K} B^{\nu}_{n} - - {}^{\prime}g^{kn} B^{K}_{k} [n_{\mu} (G_{\lambda\nu} B^{\nu}_{n} n^{\lambda}) + + B^{1}_{\mu} (G_{\lambda\nu} B^{\nu\lambda}_{n1})] \nabla_{K} B^{\mu}_{m} .$$

inserting (A2-6) and (A2-5) into (A2-4), we get equation (iii-21) of the text.

Since the divergence of the stress tensor must also vanish, an exactly analogous argument to that above applied to $\nabla_{K}(G^{K}_{\mu} - T^{K}_{\mu})$ will yield equation (v-7).

FOOTNOTES

(Bibliographical references to books are by author and to articles are by number)

1. See Darmois and 11 for early treatments of the Cauchy problem in general relativity.

2. See Bergmann 5 and 6 for review articles with extensive references to the earlier literature on this point; as well as 2, 4 for related work. See 3 and references therein for the approach of Arnowitt, Deser and Misner; 14 for the work of Wheeler and Misner; and 7 and 7a for the closely related work of Dirac on the Hamiltonian approach. 17 contains a summary of the related problem of the propagation of discontinuities in the gravitational field.

3. See Courant and Hilbert, and Hadamard for detailed discussion of the Cauchy problem for systems of differential equations.

4. See references in footnote 3. For treatments of these problems in general relativity, see 8, 9, 10, 15 and Lichnerowicz.

5. See 5, for example. This paragraph and the next are heavily indebted to the work of Bergmann and collaborators.

6. See 11 for early work; and the references in note 4 for more recent discussions.

7. See Darmois.

8. See 2, 7 and 7a.

9. See 3, as well as references in footnote 8.

10. See Schouten, passim.

11. See Schouten, p. 105.

12. See Schouten, p. 354 for this definition.

13. See Schouten, pp. 237-238.

14. In Appendix I, we give a simple method of computing the Lie derivative of any function of the $'g_{ab}$, as well as an alternate proof, based on this method, of the constraint equations (iii-15) and (iii-17). $\mathcal{L}_n^3 \, 'g_{ab}$ is also explicitly computed as another example of this method.

15. Equ tions equivalent to (iv-6) and (iv-7) are given by Peres and Rosen in a special coordinate system (see 15). From our viewpoint, the possible behavior at spatial infinity of g_{oo} and g_{os} (equivalent to our ρ and σ^m), which troubles Peres and Rosen, is merely a reflection of the fact that once arbitrary values of $'g_{ab}$ are chosen the metric on some neighboring surface to the initial surface has been given, and the relation between the two surfaces necessary to fit them both into a Riemann space with vanishing $G_{\mu\nu}$ may be very complicated. This by no means implies, in itself, that once a full metric has been constructed from the initial data, a coordinate system cannot be found in which all the components of the metric are asymptotically Minkowskian.

16. J. A. Wheeler, remarks at Stevens Relativity Conferences, 1962; Lanczos had already discussed the fact that in the case of two geodesically parallel surfaces the constraint equations imply one relation between the metrics of the two surfaces (see 11).

17. See 7 and 7a.

18. See 7a, equation (6).

19. Bonnor, for example, has found cylindrically symmetric solutions free of singularities.

20. See Lichnerowicz, Book I, Chap. VIII for a proof that no singularity-free stationary asymptotically Minkowskian solution exists.

21. See Lichnerowicz, Book I, Chap. II, and 9 for examples.

22. For the tetrad formalism, see Schouten, Eisenhart, or Weatherburn. For the Cauchy problem, see 9.

23. See Schouten, p.139, remembering that $S_{\alpha\beta}{}^{\gamma}$ is zero for a symmetric connection.

24. See Eisenhart, p. 98.

25. Proofs in Eisenhart and W atherburn.

26. See Einstein and others, pp. 167-173.

27. See Bergmann 5, for example.

28. See 3, and references to earlier papers therein.

29. See 7a. Actually Dirac's H_{main} differs by a surface divergence from our $\mathcal{H}$.

30. See, for example, 1 or 6.

31. See 3, however, for an analytic approach to this question, which may lead to a geometrical solution.

32. See 13 and 14, for example.

33. See 12, p. 594.

34. The method outlined here can be seen to be of much wider scope than the calculation of Lie derivatives. It can be used to find the effect of any small variation of the metric, in any number of dimensions. It was worked out together with Dr. J. Plebanski in connection with the search for approximate solutions of the field equations in the neighborhood of exact non-flat solutions.

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