Classical particles with spin. I. The WKBJ approximation

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This is the first of a series of papers developing the classical theory of a spinning particle. The equations of motion will be derived from a Lagrangian, and solutions for the classical trajectory and spin precession in external fields will be given. In this paper an abstract spin vector is introduced to characterize the spin of a classical particle. Lagrangians for the classical trajectories and for the motion of the abstract spin vector are derived from corresponding quantum-mechanical Lagrangians by the WKBJ approximation method for nonrelativistic and relativistic particles. The equations of motion for the trajectory and the abstract spin vector following from the extremalization of these Lagrangians are given. The equations of motion for the precession in an external electromagnetic field of the spin vector (or tensor) in space–time is derived from the equations of motion for the abstract spin vector. In the relativistic case, they are equivalent to the Bargmann–Michel–Telegdi equations [Phys. Rev. Lett. **2**, 435 (1959)]. The relationship between the ensemble and single-particle points of view is also elucidated.

I. INTRODUCTION

In this series of papers we discuss the theory of classical particles with spin. That subject has a long history, which we will not review here, although we will give some references to the most relevant literature in the course of our work.[1] What distinguishes our approach from most of the existing work in the field is that we do not consider the spin tensor (or vector) as a primary quantity in defining the theory, but rather as derived from some more fundamental representation of the rotation or Lorentz group, depending on whether it is a nonrelativistic or special-relativistic particle that is being treated. This, of course, is the way that spin enters into quantum mechanics, where the wavefunction corresponding to a particle with spin is taken to be a multicomponent entity, with the approriate transformation properties under the relevant groups. The point is that there is nothing fundamentally quantum mechanical about such a concept of a particle, and the same ideas may be applied at the classical level.

So our basic concepts are a trajectory in space—time, to be picked out by some equation of motion, and a spinor, vector, tensor—what have you—attached to each point of that trajectory with appropriate transformation properties under the rotation group (for nonrelativistic theories) or the homogeneous Lorentz group (for special-relativistic theories) which also obeys some equation of motion along the trajectory. We shall refer to this entity as the *abstract spin—vector* since it is a vector in some abstract space on which a representation of the appropriate group acts. Then, the usual spin tensor (or vector) is derived from this basic spin representation by some operation on it which produces an antisymmetric tensor (or vector) in the Galilei—Newtonian or Minkowski space of the trajectory.

In the earlier literature, we find this point of view in Schiller,[2,3] who starts from such a classical theory of the electron; and implicitly in Pauli,[4] Rubinow and Keller,[5] who treat the classical motion by means of a WKB expansion of the Dirac equation. Rafanelli and Schiller[6] show that the classical equations of motion for the electron may be derived from the WKB approximation to either the Dirac equation or the squared Dirac equation.

Our approach is also characterized by the assumption that the trajectory is not influenced by the spin characteristics of the particle. Thus, we eschew all those theories of the spinning particle in which momentum need not be parallel to velocity, with their accompanying classical Zitterbewegungen. Such theories have their interest, and indeed may also be motivated by certain types of approximation to quantum mechanical equations of motion, just as we shall motivate our approach in this paper, by a discussion of the WKBJ or eikonal type of approximation. However, they are not the type of theory that we wish to develop here, in which the trajectory of the particle is not affected by its spin.[7]

We could, at this point, just begin to consider such classical systems, for example, by writing down a Lagrangian giving rise to the desired equations of motion. However, we shall motivate our approach by showing that the equations that we shall consider can be looked upon as the WKBJ or quasiclassical limit of well-known quantum mechanical equations.

The WKBJ approximation[4] consists in making an asymptotic expansion of the wavefunction in powers of $\hbar$,

$$\psi=\exp(iS/\hbar)=\exp(i/\hbar)[S_0+(\hbar/i)S_1+(\hbar/i)^2S_2+\cdots], \tag{1.1}$$

with the assumption that S_0 is a real scalar function of the coordinates and time, while S_i $(i=1, 2, 3, \cdots)$ are abstract spin vectors like ψ itself. This is equivalent

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to assuming the expansion to be of the form

$$\psi = [R_0 + (\hbar/i)R_1 + \cdots] \exp(iS_0/\hbar) \tag{1.2}$$

as Pauli[4] points out, where the R_i $(i = 0, 1, 2, \cdots)$ are abstract spin vectors. This expansion is then usually inserted into the wave equation for ψ. However, we shall insert it into the variational principle for the wave equation, thus getting an expansion of the variational principle to various orders in $\hbar$. We shall refer to the terms of this expansion as zeroth order, first order, etc., meaning order in powers of $\hbar$, and not the order of the highest derivatives in the variational integrand. Variation of the nth order term will then yield the corresponding order in the expansion of the wave equation. Of course, since the zeroth order in the WKBJ expansion of a wavefunction corresponds to a classical ensemble,[8] we must expect to get the ensemble form of our classical equations, involving the action function S, Hamilton—Jacobi equations, etc., rather than getting the trajectories directly. But, of course, since any solution to the Hamilton—Jacobi equation corresponds to an ensemble of mechanical trajectories which can be derived from it, this constitutes no problem.

In this paper, we shall first discuss the spinless particle, nonrelativistic and relativistic, in an external electromagnetic field in order to demonstrate some features of our approach, which works from the action principle directly, in the simplest possible context. Then we shall discuss the Pauli and Dirac equations for nonrelativistic and relativistic particles of spin $\frac{1}{2}$, both interacting with external electromagnetic fields. We could easily extend our results formally to particles of any spin interacting with the electromagnetic field. However, in view of the well known difficulties with the external field problem for higher spin particles,[9] it is doubtful if these results would have more than formal significance. It is interesting, of course, that these difficulties do not manifest themselves at the level of the quasiclassical approximation. Of course, there is no difficulty with extending the results of this paper to free particles of arbitrary spin, but the results then are rather trivial: The abstract spin—vector is just parallel transported along the free particle trajectory. Finally, we shall consider the transition from the ensemble to the single-particle Lagrangian.

In the next paper we shall generalize the particle Lagrangian for the relativistic particle with spin $\frac{1}{2}$ interacting with an external electromagnetic field, developed here, to the most general possible relativistically invariant interaction, and discuss the solution of the resulting equations of motion.

II. NONRELATIVISTIC SPINLESS PARTICLE

We start from the well-known variational principle for the Schrödinger equation,

$$\delta \int \psi^* \left(i\hbar \frac{\partial}{\partial t} - \hat{H}\right) \psi \, d^3x \, dt = 0, \tag{2.1}$$

where $\hat{H}$ is the Hamiltonian for the particle

$$\hat{H} = \frac{1}{2m} \left[\nabla \frac{\hbar}{i} - \frac{e}{c}\mathbf{A}\right]^2 + V, \tag{2.1'}$$

where V includes $e\phi$, the electric potential energy, as well as any other scalar potentials, and A is the magnetic potential. We now insert the WKBJ ansatz[10] $\psi = R_0 \exp(iS/\hbar)$ directly into the variational principle, giving

$$\delta \int R_0^* \left[-\frac{\partial S}{\partial t} - \frac{1}{2m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right)^2 - V\right] R_0 \, d^3x \, dt = 0, \tag{2.2}$$

where we have omitted all terms of first or higher order in $\hbar$.

Variation of (2.2) with respect to R_0^*, yields

$$\left[\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right)^2 + V\right] R_0 = 0, \tag{2.3}$$

and we see that if R_0 does not vanish, the nonrelativistic Hamilton—Jacobi equation must hold for S. Variation of (2.2) with respect to S gives

$$\frac{\partial |R_0|^2}{\partial t} + \nabla \cdot \left[\frac{|R_0|^2}{m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right)\right] = 0. \tag{2.4}$$

The Hamilton equations for the trajectories corresponding to solutions of the Hamilton—Jacobi equation (2.3) show that

$$m\mathbf{v} = \nabla S - \frac{e}{c}\mathbf{A}, \tag{2.5}$$

so that (2.4) is just the equation of continuity for $|R_0|^2$, the density of trajectories in configuration space.

Thus, we have derived the equations of motion for an ensemble of trajectories, from the zeroth-order WKBJ approximation to the Lagrangian for the Schrödinger equation. The density of trajectories $|R_0|^2$ is also determined by the equation of continuity, which is easily converted into an equation for the ordinary derivative of $|R_0|^2$ along a mechanical trajectory determined by S,

$$\frac{d|R_0|^2}{dt} + \frac{|R_0|^2}{m} \nabla \cdot \left[\nabla S - \frac{e}{c}\mathbf{A}\right] = 0. \tag{2.6}$$

But this does not enable us to determine R_0 itself, which contains a phase factor, undetermined so far. As we shall see, this phase factor can be determined from the first-order approximation to the Lagrangian. This is the reflection, at the spinless particle level, of the same feature we shall find for particles with spin: To determine the trajectories, we only need S, which is fixed by the zeroth-order approximation to the Lagrangian as a solution to the Hamilton—Jacobi equation. However, to fix the motion of the abstract spin vector (in this case just the phase of R_0), the next approximation must be calculated, even though the resulting equation of motion for the spin is independent of $\hbar$, and indeed of any other quantities characterizing the next approximation.

We shall not bother to give the derivation of the equation of motion for R_0 from the first-order approximation, since it can be deduced immediately from our discussion for the Pauli equation in Sec. III, by setting the terms with $\sigma = 0$ in Eq. (4.7). We merely note that the result is

$$\frac{dR_0}{dt} = -\frac{1}{2m} R_0 \nabla \cdot \left(\nabla S - \frac{e}{c}\mathbf{A}\right). \tag{2.7}$$

Comparing (2.6) and (2.7), it follows that the phase of R_0 is constant along a mechanical trajectory,

$$\frac{d}{dt}\frac{R_0}{|R_0|}=0. \tag{2.8}$$

Summarizing our results, we see that the zeroth-order WKBJ approximation yields the nonrelativistic Hamilton—Jacobi equation; a class of mechanical trajectories can be derived from a solution to this in the well-known way. It also gives the continuity equation, which enables us to determine the evolution of the magnitude of R_0, along a mechanical trajectory, given a solution to the Hamilton—Jacobi equation. Thus, the magnitude of R_0 is determined for an ensemble of trajectories; it corresponds to the density in configuration space of the particles in the ensemble, but it is not a quantity which has any meaning for an individual trajectory independently of an ensemble. So it is not surprising that its evolution cannot be determined independently of S. On the other hand, the phase of R_0 along a mechanical trajectory is obtained from the first-order WKBJ approximation. Its evolution is meaningful for an individual trajectory, quite apart from any ensemble to which the latter may belong. In our case, this equation is trivial—the phase stays constant. But this feature of the results will generalize to other cases with spin: The magnitude of the abstract spin—vector will be meaningful only for the ensemble point of view, while the evolution of the "unit" abstract spin vector will be determined by an equation of motion along a single trajectory.

III. RELATIVISTIC SPINLESS PARTICLE

We start from the Lagrangian for the Klein—Gordon equation with external electromagnetic field[11]:

$$\int\left[\left(\frac{-h}{i}\nabla-\frac{e}{c}\mathrm{A}\right)\psi^*\cdot\left(\frac{\hbar}{i}\nabla-\frac{e}{c}\mathrm{A}\right)\psi-m^2c^2\psi^*\psi\right]d^4x. \tag{3.1}$$

Inserting the WKBJ ansatz $\psi=\phi\exp(iS/\hbar)$ into (3.1) and again keeping only terms independent of $\hbar$, we get

$$\int\phi^*\left[\left(\nabla S-\frac{e}{c}\mathrm{A}\right)\cdot\left(\nabla S-\frac{e}{c}\mathrm{A}\right)-m^2c^2\right]\phi d^4x. \tag{3.2}$$

Variation with respect to ϕ^* gives

$$\left[\left(\nabla S-\frac{e}{c}\mathrm{A}\right)^2-m^2c^2\right]\phi=0; \tag{3.3}$$

and again, if ϕ does not vanish, the relativistic Hamilton—Jacobi equation must hold (variation of ϕ again leads to the conjugate equation). Variation with respect to S leads to

$$\nabla\cdot\left[(\phi^*\phi)\left(\nabla S-\frac{e}{c}\mathrm{A}\right)\right]=0. \tag{3.4}$$

Since the Hamilton—Jacobi equation (3.3) implies Hamilton's equations of motion for the trajectories, we again see that (2.5) holds, now as a 4-vector equation; and thus (3.4) is a continuity equation for $\phi^*\phi$, the density of trajectories. Thus, the zeroth-order WKBJ approximation again determines the relativistic Hamilton—Jacobi equation, a solution to which yields an ensemble of mechanical trajectories; as well as the equation of continuity, which determines the evolution of the magnitude of ϕ along each trajectory, given a solution S. We omit the details of the proof that the first-order WKBJ approximation determines the evolution of ϕ along a trajectory given S; from which it follows that the phase along each trajectory is constant.

IV. NONRELATIVISTIC PARTICLE OF SPIN $\frac{1}{2}$ (PAULI EQUATION)

We start from a variational principle for the Pauli equation,

$$\delta\quad\psi^*\left(\pm i\hbar\frac{\partial}{\partial t}-\hat{H}\right)\psi\, d^3x\, dt=0, \tag{4.1}$$

where

$$\hat{H}=\frac{1}{2m}\left[\sigma\cdot\left(\frac{\hbar}{i}\nabla-\frac{e}{c}\mathrm{A}\right)\right]\left[\sigma\cdot\left(\frac{\hbar}{i}\nabla-\frac{e}{c}\mathrm{A}\right)\right]+V. \tag{4.2}$$

Here, ψ is a two-component spinor, ψ^* its Hermitian adjoint, σ the Pauli matrices, A the electromagnetic vector potential, and V a scalar potential energy which includes $e\phi$, where ϕ is the scalar electrostatic potential. We again insert the WKBJ ansatz, but this time the coefficient of $\exp(iS/\hbar)$ is a two-component spinor. Since we shall have to consider both the zero- and first-order approximations, we include two terms in our ansatz:

$$\psi=(D_0+{}^*y_iD_1)\exp(iS/\hbar),\qquad \psi^*=\left(D_0^*-\frac{\hbar}{i}D_1^*\right)\exp(-iS/\hbar), \tag{4.3}$$

where D_0 and D_1 are two-component spinor fields. Inserting this into (4.1) we expand the Lagrangian up to first order in $\hbar$,

$$\begin{aligned}L=&\int D_0^*\left[-\frac{\partial S}{\partial t}-\frac{1}{2m}\left(\nabla S-\frac{e}{c}\mathrm{A}\right)^2-V\right]D_0\,d^3x\\&+\int\Bigg\{\frac{\hbar}{i}(D_0^*D_1-D_1^*D_0)\left[-\frac{\partial S}{\partial t}-\frac{1}{2m}\left(\nabla S-\frac{e}{c}\mathrm{A}\right)^2-V\right]\\&-\frac{\hbar}{i}D_0^*\left[\frac{\partial D_0}{\partial t}+\frac{1}{2m}(\nabla^2S)D_0+\frac{1}{m}\left(\nabla S-\frac{e}{c}\mathrm{A}\right)\cdot\nabla D_0\right.\\&\left.-\frac{ie}{2mc}\sigma\cdot\mathrm{B}\,D_0\right]\Bigg\}d^3x.\end{aligned} \tag{4.4}$$

Variation of the first-, or zeroth-order term gives

$$\delta D_0^*\Rightarrow\left[\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\nabla S-\frac{e}{c}\mathrm{A}\right)^2+V\right]D_0=0, \tag{4.5a}$$

$$\delta S\Rightarrow\frac{\partial}{\partial t}(D_0^*D_0)+\nabla\cdot\left[D_0^*D_0\left(\frac{\nabla S-(e/c)\mathrm{A}}{m}\right)\right]=0. \tag{4.5b}$$

Variation with respect to D_0 yields the Hermitian conjugate of (4.5a) and thus nothing new. If $D_0\neq 0$, we see that (4.5a) implies that S obeys the nonrelativistic Hamilton—Jacobi equation; while (4.5b) is again a conservation law for the magnitude of D_0 from which it follows that

$$\frac{d}{dt}(D_0^*D_0)^{-1/2}=\frac{1}{2m}\nabla^2S(D_0^*D_0)^{-1/2}. \tag{4.6}$$

Thus, the evolution of the magnitude of D_0 along each mechanical trajectory is again fixed by S which determines an ensemble of trajectories. To determine the evolution of $d_0=D_0/(D_0^*D_0)^{1/2}$, the "unit" abstract spin vector ($d_0^\dagger d_0 = 1$), we need to look at the first-order variation of (4.4).

Notice that variation of the term in $(D_0^\dagger D_1 - D_1^\dagger D_0)$ with respect to the D's will yield nothing new, since its coefficient vanishes by virtue of the zeroth-order equation (4.5a). Thus, it may be omitted from the Lagrangian if only results involving D_0 are desired.[12] Variation of the remaining term with respect to $D_0^\dagger$ yields

$$\frac{\partial D_0}{\partial t} + \frac{1}{m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right)\cdot\nabla D_0 + \frac{1}{2m}(\nabla^2 S)D_0 - \frac{ie}{2mc}(\sigma\cdot\mathbf{B})D_0 = 0. \tag{4.7}$$

Variation with respect to D_0 yields the Hermitian conjugate of (4.7) [it is easily checked that (4.5b) actually follows as a consequence of these two equations]. Now, the first two terms in (4.7) are seen to equal dD_0/dt, since $m\mathbf{v} = \nabla S - (e/c)\mathbf{A}$ along a mechanical trajectory, as a consequence of the Hamilton—Jacobi equation. Thus, (4.7) is indeed the equation required for determining the evolution of D_0, given a solution of the Hamilton—Jacobi equation. Our previous work suggests that the equation of evolution of d_0 will be independent of S. Indeed it is easily shown that

$$\frac{d}{dt}d_0 = \frac{ie}{2mc}(\sigma\cdot\mathbf{B})\,d_0 \tag{4.8}$$

Thus, the motion of the particle and of its abstract spin—vector in an external electromagnetic field are given. It only remains to see how the spin—vector in Galilei space—time is determined as a consequence of this equation of motion. Since $d_0^\dagger\sigma d_0 \equiv \mathbf{S}$ will transform as a 3-vector in space as a result of the transformation properties of the two-component spinors and the σ matrices, it is natural to take this as the definition of the spin—vector (actually, any multiple of this could be used, since the resulting equation is linear homogeneous in $\mathbf{S}$). Using (4.8) and its Hermitian conjugate for $d_0^\dagger$, we find immediately that

$$\frac{d\mathbf{S}}{dt} = \frac{-e}{mc}(\mathbf{S}\times\mathbf{B}), \tag{4.9}$$

the equation of motion for spin precession in a magnetic field, for a particle with gyromagnetic ratio two, as might be expected from the Pauli equation.

Note that if we break up D_0 into an amplitude R times d_0,

$$D_0 = Rd_0, \quad R = (D_0^\dagger D_0)^{1/2}, \tag{4.10}$$

we get a representation of D_0 similar to the amplitude-phase representation of a complex number. We shall use this breakup in our discussion of particle Lagrangians in Sec. VI.

V. RELATIVISTIC PARTICLE OF SPIN $\frac{1}{2}$ (DIRAC EQUATION)

Now that our approach is (hopefully) clear, we take up the most complicated example we shall consider in this paper, the relativistic spin-$\frac{1}{2}$ particle in an external electromagnetic field, described quantum mechanically by the Dirac equation. We start from the variational principle,

$$\delta\int\left\{\frac{1}{2}\left[\frac{\hbar}{i}\left(\partial_\kappa + \frac{e}{c}A_\kappa\right)\psi^+\gamma^\kappa\psi - \psi^+\gamma^\kappa\left(\frac{\hbar}{i}\partial_\kappa - \frac{e}{c}A_\kappa\right)\psi\right] + mc\psi^+\psi\right\}d^4x = 0, \tag{5.1}$$

where ψ is a four-component spinor field, ψ^+ its "adjoint" field defined by

$$\psi^+ = \psi^*\gamma^4, \tag{5.2}$$

and ψ^* is the Hermitian adjoint of ψ.[13] Inserting the WKBJ ansatz

$$\psi = \left(D_0 + \frac{\hbar}{i}D_1\right)\exp(-iS/\hbar), \quad \psi^+ = \left(D_0^+ - \frac{\hbar}{i}D_1\right)\exp(-iS/\hbar) \tag{5.3}$$

(note that we have used the same notation, D_0 and D_1, for the 4-spinors here that we used for the 2-spinors of the last section), into the variational principle, we get the expansion of the Lagrangian up to first order in $\hbar$,

$$\int D_0^+\left[-\gamma^\kappa\left(\partial_\kappa S - \frac{e}{c}A_\kappa\right) + mc\right]D_0\,d^4x + \int\frac{\hbar}{i}\left[\left(D_1^+\gamma^\kappa D_0 - D_0^+\gamma^\kappa D_1\right)\left(\partial_\kappa S - \frac{e}{c}A_\kappa\right) + mc(D_0^+D_1 - D_1^+D_0) + \tfrac{1}{2}(\partial_\kappa D_0^+\gamma^\kappa D_0 - D_0^+\gamma^\kappa\partial_\kappa D_0)\right]d^4x. \tag{5.4}$$

Variation of the zeroth-order terms in (5.4) gives

$$\delta D_0^+ \Rightarrow \left[\gamma^\kappa\left(\partial_\kappa S - \frac{e}{c}A_\kappa\right) - mc\right]D_0 = 0, \tag{5.5a}$$

$$\delta S \Rightarrow \partial_\kappa(D_0^+\gamma^\kappa D_0) = 0. \tag{5.5b}$$

[Again, variation with respect to D_0 yields the adjoint equation to (5.5a).][14] Equation (5.5a) will not have any solutions, for nonvanishing D_0, unless the determinant of the matrix in brackets vanishes. This condition is easily seen to be equivalent to the relativistic Hamilton—Jacobi equation

$$\eta^{\mu\nu}\left(\partial_\mu S - \frac{e}{c}A_\mu\right)\left(\partial_\nu S - \frac{e}{c}A_\nu\right) - m^2c^2 = 0. \tag{5.6}$$

The matrix is of rank two, as Rubinow and Keller noted,[5] so that there are only two linearly independent solutions to (5.5a), once S satisfies (5.6). We will not have to use the form of these solutions, given by Rubinow and Keller, but will continue to work with an arbitrary solution.

Now we look at the variations of the first order terms in (5.4). Again, variation with respect to D_1 and D_1^+ merely reproduce equations (5.5a) and its adjoint. Thus, the first equations we require result from the variation of the first-order part of (5.4) with respect to D_0 and D_0^+,

$$\gamma^\kappa\partial_\kappa D_0 + \left[\gamma^\kappa\left(\partial_\kappa S - \frac{e}{c}A_\kappa\right) - mc\right]D_1 = 0, \tag{5.7a}$$

$$\partial_\kappa D_0^+\gamma^\kappa + D_1^+\left[\gamma^\kappa\left(\partial_\kappa S - \frac{e}{c}A_\kappa\right) - mc\right] = 0. \tag{5.7b}$$

Note that, because of the first-order derivative form of the Dirac equation, we cannot avoid the appearance of D_1 and $D_1^\dagger$ in our first-order equations, as we could in the previous second-order wave equations. Our task is to derive equations of motion for D_0 which will not include D_1. We can do this by straightforward computation of $d/d\tau(D_0)$, using our previous equations and a little manipulation of γ matrices. By definition,

$$\frac{dD_0}{d\tau}=\partial_\kappa D_0\frac{dx^\kappa}{d\tau} \quad (5.8)$$

along any mechanical trajectory, where τ is the proper time along the trajectory. But the equations for the trajectory following from the relativistic Hamilton—Jacobi equation show that

$$m\frac{dx^\kappa}{d\tau}=\eta^{\kappa\lambda}\left(\partial_\lambda S-\frac{e}{c}A_\lambda\right); \quad (5.9)$$

substituting this into (5.8), and remembering that $\eta^{\kappa\lambda}=\frac{1}{2}(\gamma^\kappa\gamma^\lambda+\gamma^\lambda\gamma^\kappa)$, we get

$$\frac{dD_0}{d\tau}=\frac{(\gamma^\kappa\gamma^\lambda+\gamma^\lambda\gamma^\kappa)}{2m}\partial_\kappa D_0\left(\partial_\lambda S-\frac{e}{c}A_\lambda\right). \quad (5.10)$$

When we expand the parenthesis in (5.10), we get two terms, the second of which is

$$\frac{\gamma^\lambda\gamma^\kappa\partial_\kappa D_0}{2m}\left(\partial_\lambda S-\frac{e}{c}A_\lambda\right). \quad (5.11)$$

By using (5.7a), and (5.6) this can be reduced to

$$-\frac{c}{2}\gamma^\kappa\partial_\kappa D_0. \quad (5.12)$$

The first term in (5.10) is

$$\frac{\gamma^\kappa\gamma^\lambda}{2m}\partial_\kappa D_0\left(\partial_\lambda S-\frac{e}{c}A_\lambda\right). \quad (5.13)$$

By writing this as

$$\frac{1}{2m}\left\{\partial_\kappa\left[\gamma^\kappa\gamma^\lambda\left(\partial_\lambda S-\frac{e}{c}A_\lambda\right)D_0\right]-\frac{\gamma^\kappa\gamma^\lambda}{2m}\partial_\kappa\left(\partial_\lambda S-\frac{e}{c}A_\lambda\right)D_0\right\}, \quad (5.14)$$

and by using (5.5a) (and adopting the Lorentz gauge condition $\partial_\kappa A^\kappa=0$ to avoid some additional steps) the first term reduces to

$$\frac{c}{2}\gamma^\kappa\partial_\kappa D_0-\frac{1}{2m}(\Box S)D_0+\frac{e}{2mc}F_{\kappa\lambda}\sigma^{\kappa\lambda}D_0, \quad (5.15)$$

where $\Box S$ means the D'Alembertian of S, and $\sigma^{\kappa\lambda}\equiv\frac{1}{2}(\gamma^\kappa\gamma^\lambda-\gamma^\lambda\gamma^\kappa)$ means the commutator of the γ's. So finally, the required equation of motion for D_0 is

$$\frac{dD_0}{d\tau}=-\frac{\Box S}{2m}D_0+\frac{e}{2mc}F_{\kappa\lambda}\sigma^{\kappa\lambda}D_0. \quad (5.16)$$

Again, S is required to determine the evolution of D_0 along a mechanical trajectory. However, it is easily shown that

$$\frac{d}{d\tau}(D_0^\dagger D_0)=-\frac{\Box S}{m}(D_0^\dagger D_0); \quad (5.17)$$

so, again defining $d_0=D_0/(D_0^\dagger D_0)^{1/2}$, we find the equation of motion for d_0,

$$\frac{d}{d\tau}d_0=\frac{e}{2mc}F_{\kappa\lambda}\sigma^{\kappa\lambda}d_0. \quad (5.18)$$

Note the close analogy with (4.8), which can be made closer by using the operator $s^{\kappa\lambda}=i\sigma^{\kappa\lambda}$, in Eq. (5.18), which is more directly related to the spin—tensor in Minkowski space,

$$\frac{d}{d\tau}d_0=-\frac{ie}{2mc}F_{\kappa\lambda}s^{\kappa\lambda}d_0. \quad (5.18a)$$

Indeed, we must now relate this equation to the equation of motion of the spin—tensor in Minkowski space. As is well known, $\psi^\dagger s^{\mu\nu}\psi$ transforms like an antisymmetric tensor of second rank, so that it seems natural to define the spin—tensor by

$$S^{\mu\nu}=d_0^\dagger s^{\mu\nu}d_0. \quad (5.19)$$

Thus,

$$\frac{d}{d\tau}S^{\mu\nu}=\left(\frac{d}{d\tau}d_0^\dagger\right)s^{\mu\nu}d_0+d_0^\dagger s^{\mu\nu}\left(\frac{d}{d\tau}d_0\right), \quad (5.20)$$

and substituting (5.18a) and its adjoint equation into (5.20), we get

$$\frac{d}{d\tau}S^{\mu\nu}=\frac{ie}{2mc}d_0^\dagger[s^{\kappa\lambda}s^{\mu\nu}-s^{\mu\nu}s^{\kappa\lambda}]d_0F_{\kappa\lambda}. \quad (5.21)$$

Using the commutation relations between $s^{\mu\nu}$, which are essentially those for the generators of the homogeneous Lorentz transformations

$$s^{\kappa\lambda}s^{\mu\nu}-s^{\mu\nu}s^{\kappa\lambda}=i(s^{\lambda\nu}\eta^{\kappa\mu}-s^{\kappa\nu}\eta^{\lambda\mu}+s^{\mu\lambda}\eta^{\kappa\nu}-s^{\mu\kappa}\eta^{\lambda\nu}), \quad (5.22)$$

we finally get

$$\frac{d}{d}S^{\mu\nu}=-\frac{e}{mc}(S^{\lambda\nu}\eta^{\kappa\mu}-S^{\lambda\mu}\eta^{\kappa\nu})F_{\kappa\lambda}. \quad (5.23)$$

This is the required equation of motion for the spin tensor, which is seen to be the relativistic generalization of (4.9) for the nonrelativistic spin vector. Indeed, we may introduce a relativistic spin vector S_μ by

$$S_\mu=e_{\mu\nu\kappa\lambda}S^{\nu\kappa}v^\lambda, \quad (5.24)$$

where $e_{\mu\nu\kappa\lambda}$ is the Levi-Civita tensor, equal to $(-\eta)^{1/2}\epsilon_{\mu\nu\kappa\lambda}$. Clearly, $S_\mu v^\mu=0$, and the **S** of Sec. IV represents the nonrelativistic version of S^μ. On the other hand, it is easily shown that $S^{\mu\nu}p_\mu^{\text{mech}}=0$, where $p_\mu^{\text{mech}}=\partial_\mu S-(e/c)A_\mu$:

$$\begin{aligned}2i(D_0^\dagger D_0)S^{\mu\nu}p_\mu^{\text{mech}}&=D_0^\dagger(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)p_\mu^{\text{mech}}D_0\\&=D_0^\dagger(\gamma^\mu\gamma^\nu p_\mu^{\text{mech}}-mc\gamma^\nu)D_0=D_0^\dagger(\gamma^\mu p_\mu^{\text{mech}}-mc)\gamma^\nu D_0=0,\end{aligned} \quad (5.25)$$

where we have used (5.5a) and its Hermitian conjugate. This is the well-known Frenkel condition on the spin tensor[1]; it also guarantees, as we shall see in the next section, that p^{mech} and v^μ are parallel. Note that it is not an additional postulate here, but a consequence of the equations of motion. When the Frenkel condition holds, $S^{\mu\nu}$ can be derived from S_μ, so that the two are entirely equivalent,

$$S^{\mu\nu}=e^{\mu\nu\kappa\lambda}S_\kappa v_\lambda, \quad (5.26)$$

where $e^{\mu\nu\kappa\lambda}$ is again the tensor formed from the Levi-Civita tensor density: $e^{\mu\nu\kappa\lambda} = (-\eta)^{-1/2}\epsilon^{\mu\nu\kappa\lambda}$. Equations (5.23), or the corresponding equations for S_μ are equivalent to the Bargmann—Michel—Telegdi equations.[14]

VI. SINGLE-PARTICLE LAGRANGIANS

In the last five sections, we have seen how to develop ensemble Lagrangians, the variation of which lead to partial differential equations of motion for functions describing ensembles of classical particles without and with spin, by WKBJ expansions of the quantum mechanical Lagrangians for relativistic and nonrelativistic particles of spin zero and spin $\frac{1}{2}$. We are now ready to discuss the transition to single-particle Lagrangians, whose variation leads to ordinary differential equations for the mechanical trajectories and abstract spin vector. As we have seen, we cannot hope to find equations of motion for the magnitude of the abstract spin vector along a single trajectory, as this is basically a characteristic of an ensemble density. Thus, we must expect the magnitude of the abstract spin vector to be left undetermined by the equations of motion; however, this indeterminacy can be absorbed by a reparametrization of the equations as we shall see.

If we remember that the integrand of the zeroth order part of our variational principle is essentially the Hamiltonian written in terms of S plus $\partial S/\partial t$ for the nonrelativistic Lagrangians—and a similar expression in the relativistic case—it will not be surprising that we can form a homogeneous particle Hamiltonian by taking this expression, and replacing all derivatives of S by the corresponding particle variables. That is, by letting $\partial S/\partial t \Rightarrow -E$, $\nabla S \Rightarrow \mathbf{p}$ in the nonrelativistic action principles, and letting $\partial_\mu S \Rightarrow p_\mu$ in the relativistic cases, we get a particle Hamiltonian. Subtracting this from $\mathbf{p}\cdot(d\mathbf{r}/d\lambda) - E(dt/d\lambda)$ in the nonrelativistic cases; and from $p_\mu(dx^\mu/d\lambda)$ in the relativistic cases we get a particle Lagrangian. We have here introduced a parameter λ along the path in space—time, to enable us to vary with respect to the time t in the nonrelativistic case, and with respect to all four x^μ in the relativistic case, without worrying about constraints. The variational principle is now homogeneous in λ, and precisely this enables us to get rid of the unwanted freedom in the length of the abstract spin—vector.

We proceed to write down the variation of the Lagrangian for each of our four cases, and briefly discuss the resulting equations of motion.

(a) Nonrelativistic spinless particle:

$$\delta\int\left[\mathbf{p}\cdot\frac{d\mathbf{r}}{d\lambda} - E\frac{dt}{d\lambda} - R_0^*R_0\left(\frac{[\mathbf{p}-(e/c)\mathbf{A}]^2}{2m} + V - E\right)\right]d\lambda = 0. \qquad (6.1)$$

Variation with respect to:

$$\delta E \Rightarrow \frac{dt}{d\lambda} = R_0^*R_0 = |R_0|^2, \qquad (6.2)$$

which relates the parameter λ to the norm of R;

$$\delta\,\mathbf{p} \Rightarrow \frac{d\mathbf{r}}{d\lambda} = |R_0|^2\frac{[(\mathbf{p}-(e/c)\mathbf{A}]}{m}. \qquad (6.3)$$

Using (6.2), this reduces to

$$\frac{d\mathbf{r}}{dt} = \frac{[(\mathbf{p}-(e/c)\mathbf{A}]}{m} \qquad (6.3')$$

(from now on, we shall omit this intermediate step, and write time derivatives directly);

$$\delta\mathbf{r} \Rightarrow \frac{d\mathbf{p}}{dt} = -\nabla V + \frac{e}{mc}\,\mathbf{p}\cdot\nabla\mathbf{A} - \frac{e^2}{2mc^2}\mathbf{A}\cdot\nabla\mathbf{A}, \qquad (6.4)$$

which, using (6.3′) is easily proved equivalent to the Lorentz force law.

$$\delta R_0^* \Rightarrow \left(\frac{[\mathbf{p}-(e/c)\mathbf{A}]^2}{2m} + V - E\right)R_0 = 0, \qquad (6.5)$$

the expression for the total energy as sum of kinetic plus potential energy (with a similar expression from δR_0),

$$\delta t \Rightarrow \frac{dE}{dt} = \frac{\partial V}{\partial t} - \frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)\cdot\frac{\partial\mathbf{A}}{\partial t}, \qquad (6.6)$$

which expresses the rate at which the particle's energy changes in a time-dependent external electric field.

Similarly, the first-order term in the expansion of the Lagrangian for the spinless particle can be converted into a Lagrangian for the phase of R_0. But since the equation of motion for the phase is so trivial (phase $=$const), and the result can be obtained from the Pauli equation results to be given later, we omit the details.

(b) Relativistic spinless particle:

$$\delta\int\left\{p_\mu\frac{dx^\mu}{d\lambda} - \frac{1}{2}\phi^*\phi\left[\eta^{\mu\nu}\left(p_\mu - \frac{e}{c}A_\mu\right)\times\left(p_\nu - \frac{e}{c}A_\nu\right) - m^2c^2\right]\right\}d\lambda = 0, \qquad (6.7)$$

$$\delta p_\mu \Rightarrow \frac{dx^\mu}{d\lambda} = \phi^*\phi\left(p^\mu - \frac{e}{c}A^\mu\right), \qquad (6.8)$$

$$\delta x^\mu \Rightarrow \frac{dp_\mu}{d\lambda} = -\frac{e}{c}\left(p_\nu - \frac{e}{c}A_\nu\right)\frac{\partial A^\nu}{\partial x^\mu}, \qquad (6.9)$$

which is again easily proved equivalent to the Lorentz force law.

$$\delta\phi^* \Rightarrow \left[\eta^{\mu\nu}\left(p_\mu - \frac{e}{c}A_\mu\right)\left(p_\nu - \frac{e}{c}A_\nu\right) - m^2c^2\right]\phi = 0, \qquad (6.10)$$

the relativistic energy—momentum relation for a particle of mass m. It follows from (6.8) and (6.10) that

$$\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = (mc\phi^*\phi)^2, \qquad (6.11)$$

so that $d\tau/d\lambda = m\phi^*\phi$, where τ is the proper time along the world line. Thus, all λ derivatives can be converted to τ derivatives, giving the correct relativistic relationships.

Again, we omit details of the derivation of the trivial equation of motion for the phase of ϕ from the first-order Lagrangian.

(c) Nonrelativistic particles of spin $\frac{1}{2}$:

$$\delta\int\left[\mathbf{p}\cdot\frac{d\mathbf{r}}{d\lambda}-E\frac{dt}{d\lambda}-D_0^*\left(\frac{[\boldsymbol{\sigma}\cdot(\mathbf{p}-(e/c)\mathbf{A})]^2}{2m}+V-E\right)D_0\right]d\lambda=0. \tag{6.12}$$

The analysis for the mechanical trajectories goes much as in the previous cases, except that now variation with respect to E gives

$$\delta E\Rightarrow\frac{dt}{d\lambda}=D_0^*D_0, \tag{6.13}$$

so that it is the norm of the abstract spin vector which is related to λ.

Now we shall derive the Lagrangian for the evolution of the unit abstract spin vector along the trajectory. To do this, we need to consider the first-order terms in the Lagrangian (4.4). As noted in Sec. IV, the term in $(D_0^*D_1-D_1^*D_0)$ may be omitted, since its coefficient vanishes by virtue of the zero-order equations of motion. The second term may be rewritten

$$\int D_0^*\left(\frac{d}{dt}D_0-\frac{ie}{2mc}(\boldsymbol{\sigma}\cdot\mathbf{B})D_0+\frac{1}{2m}(\nabla^2S)D_0\right)d^3x. \tag{6.14}$$

Breaking up D_0 into an amplitude times d_0 ($D_0=Rd_0$), and inserting this into (6.14), we get (remembering that $d_0^*d_0=1$)

$$\int\left[R\left(\frac{d}{dt}R+\frac{1}{2m}(\nabla^2S)R\right)+R^2d_0^*\left(\frac{d}{dt}d_0-\frac{ie}{2mc}(\boldsymbol{\sigma}\cdot\mathbf{B})\,d_0\right)\right]d^3x. \tag{6.15}$$

But the first term vanishes, using (4.6), and since we are interested in a single-particle Lagrangian we may take R^2 as a delta function centered on the position of the particle. So we finally arrive at the variational principle for d_0,[15]

$$\delta\int d_0^*\left(\frac{d}{dt}d_0-\frac{ie}{2mc}(\boldsymbol{\sigma}\cdot\mathbf{B})\,d_0\right)dt=0. \tag{6.16}$$

(d) Relativistic particle of spin $\frac{1}{2}$:

$$\delta\int\left\{p_\mu\,dx^\mu-D_0^*\left[\gamma^\kappa\left(p_\kappa-\frac{e}{c}A_\kappa\right)-mc\right]D_0\right\}d\lambda=0, \tag{6.17}$$

$$\delta p^\mu\Rightarrow\frac{dx^\mu}{d\lambda}=D_0^*\gamma^\mu D_0, \tag{6.18}$$

$$\delta x^\mu\Rightarrow\frac{dp_\mu}{d\lambda}=\frac{e}{c}D_0^*\gamma^\kappa D_0\partial_\mu A_\kappa, \tag{6.19}$$

$$\delta D_0^*\Rightarrow\left[\gamma^\kappa\left(p_\kappa-\frac{e}{c}A_\kappa\right)-mc\right]D_0=0. \tag{6.20}$$

(6.20) can only hold for nonvanishing D_0 if the determinant of the matrix in brackets vanishes, which gives the relativistic energy—momentum relation

$$\eta^{\mu\nu}\left(p_\mu-\frac{e}{c}A_\mu\right)\left(p_\nu-\frac{e}{c}A_\nu\right)-m^2c^2=0. \tag{6.21}$$

Multiplication of (6.20) from the left by $D_0^*\gamma^\mu$ gives [remembering (5.25)]

$$\eta^{\mu\kappa}\left(p_\kappa-\frac{e}{c}A_\kappa\right)D_0^*D_0=mc\frac{dx^\mu}{d\lambda}, \tag{6.22}$$

and we see that by choosing $d\tau/d\lambda=D_0^*D_0$ we can go over to the proper time parameterization of the equations of motion. Letting $v^\mu\equiv dx^\mu/d\tau$, it is easily seen that (6.19) is the Lorentz force law of motion. The equations of motion for the 4-spinor d_0 may be obtained by adjoining the Lagrangian

$$\int d_0^*\left(\frac{d}{d\tau}d_0+\frac{ie}{2mc}F_{\kappa\lambda}s^{\kappa\lambda}\,d_0\right)d\tau, \tag{6.23}$$

which may again be derived from the first-order terms in (5.4).[16] We omit the details.

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[1]Much of the literature may be traced through the references in H.C. Corben, *Classical and Quantum Theory of Spinning Particles* (Holden-Day, San Francisco, 1968), which surveys this work.
[2]R. Schiller, Phys. Rev. **125**, 1116 (1962).
[3]R. Schiller, Phys. Rev. **128**, 1402 (1962).
[4]W. Pauli, Helv. Phys. Acta **5**, 179 (1932).
[5]S.I. Rubinow and J.B. Keller, Phys. Rev. **131**, 2789 (1963).
[6]K. Rafanelli and R. Schiller, Phys. Rev. **135**, B279 (1964).
[7]For a discussion of the conditions of validity of these different types of classical theories, considered as approximations to the quantum mechanical equations of motion of the electron, see Ref. 5, Sec. II.
[8]See, e.g., J. Frenkel, *Wave Mechanics: Advanced General Theory* (Clarendon, Oxford, 1974), Chap. I, especially Secs. 3 and 4; W. Pauli, "Die alligemeinen Prinzipien der Wellenmechanik" in S. Flügge, Ed., *Handbuch der Physik, Band V, Teil 1. Prinzipien der Quantentheorie I* (Springer-Verlag, Berlin, 1958), Sec. 12.
[9]See, for example, the article: Arthur S. Wightman, "Instability Phenomena in the External Field Problem for Two Classes of Relativistic Wave Equations," in E.H. Lieb, B. Simon, and A.S. Wightman, *Studies in Mathematical Physics* (Princeton U.P., Princeton, New Jersey, 1976), pp. 423—60, and the references to his earlier work therein, for an introduction to this problem.
[10]From now on we drop the subscript from S_0, in the WKBJ approximation, since no confusion can arise.
[11]Note that throughout this section vector notation denotes a 4-vector, $\nabla=\partial/\partial x^\mu\equiv\partial_\mu$, the four-dimensional operator, and a dot product is to be taken with the Minkowski metric, $\mathbf{A}\cdot\mathbf{B}=\eta_{\mu\nu}A^\mu B^\nu$.
[12]Its variation with respect to S will yield a term in a conservation equation involving D_1 that would be needed if we were to go to the second order WKBJ approximation.
[13]We have not included an anomalous magnetic moment in the Dirac equation, although it could be included with a small amount of extra work. In the next paper, we shall consider the case of a classical particle of spin $\frac{1}{2}$ in the most general possible interaction with external fields.
[14]These equations, and/or equations equivalent to them and to the first order equations that will follow were derived by Keller and Rubinow in their paper on the WKBJ approximation to the Dirac equation,[5] following the early work by Pauli[4] on this subject. They also derived the Bargmann—Michel—Telegdi equations for the spin vector by a noncovariant approach: V. Bargmann, L. Michel, V. L. Telegdi, Phys. Rev. Lett. **2**, 435 (1959).
[15]Schiller, in Ref. 2, basing himself on the work of H.A. Kramers in his book *Quantum Mechanics* (North-Holland, Amsterdam, 1957), has written down this Lagrangian for the spin equations of motion without deriving it from the WKBJ approximation to the Lagrangian for the Pauli equation.
[16]Schiller, in Ref. 3, has again written down this Lagrangian without deriving it from the WKBJ approximation to the Dirac Lagrangian.
[17]Ph. Choquard, Helv. Phys. Acta, **28**, 89 (1955).