EINSTEIN'S INTUITION AND THE POST-NEWTONIAN APPROXIMATION

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One of the many topics in general relativity, to which Jerzy Plebański has made outstanding contributions, is the problem of the equations of motion in the "slow motion" approximation. In addition to several papers on this topic, he is the co-author, with Leopold Infeld, of the book *Motion and Relativity* (1960). The fate of this book is a subject that caused him considerable embarrassment. When Jerzy first went to the United States in 1959, he left behind in Warsaw a draft manuscript of the book. After he was gone, Infeld - without consulting Jerzy - made some "new and interesting" contributions to it.[a] In particular he added a "proof" of the non-existence of purely gravitational radiation, a view with which Jerzy disagreed. Even though Infeld took "the full responsibility" for the changes in the "Introduction", Jerzy often found himself saddled with Infeld's views.

Even sadder is the fate of one of the most unjustly-neglected papers in the large and ever-growing literature on the post-Newtonian approximation: Plebański and Bażanski 1959. In a recent issue of *Phys. Rev.* **D**, I found an important article on the topic (Poujade and Blanchet 2002). A purported survey of earlier literature "Concerning ... the dynamics of extended fluid systems", cites only the "works of Chandrasekhar and collaborators". To my question why they had not cited the Plebański-Bażanski paper, which appeared much earlier than any of the Chandrasekhar papers, one of the authors replied that he had been completely unaware of it until I mentioned it. So, as a tribute to, and reminder of, Jerzy's work in this field, I decided to speak on some aspects of the Newtonian and post-Newtonian approach.

1 Einstein's Intuition

Shortly after completing work on the final formulation of general relativity, on 21 December 1915 Einstein wrote to his old friend and confidant Michel Besso: *Most gratifying [about the new theory] are the agreement of the perihelion motion [of Mercury] and general covariance; the most remarkable, however, the fact that Newton's theory of the field, even for terms of the 1st order, is incorrect for the field (occurrence of [non-flat]* $g_{11} - - - g_{33}$*). Only the fact that* $g_{11} - - - g_{33}$ *do not occur in the first approximation of the eq[uations] of mot[ion] of a point[-particle] causes the simplicity of Newton's theory.*

A few days earlier, on Dec. 10, he had written: *You will be astonished by the occurrence of* $g_{11} - - - g_{33}$. The subject of Einstein's evident and Besso's presumed astonishment is the occurrence of spatial curvature in the first-order corrections to the Minkowski metric for a spherically symmetric solution to the gravitational field equations. The reason for this astonishment was their previous joint work in 1914 on a calculation of the perihelion precession based on the spherically symmetric solution to the field equations of the non-generally

[a] You probably know the joke about new and interesting: What was interesting was not new, and what was new was not interesting.

covariant Einstein-Grossmann theory. This work of course did not correctly account for the observed precession value; nevertheless it was not done in vain: The techniques developed in the course of their joint work enabled Einstein, after he returned in November 1915 to generally covariant field equations and found the (approximate) Schwarzschild solution, to quickly calculate the precession value predicted by general relativity.

Einstein had good reason for his long-held intuition that, in Newtonian theory, space (as opposed to space-time) should be flat. In his first, scalar theory of the gravitational field, before he had adopted the metric tensor as the correct representation of the gravitational field, and in the subsequent Einstein-Grossmann theory, after he had, static gravitational fields were associated with spatially flat cross-sections. In collaboration with Besso, he had developed an approximation scheme for deriving the first-order corrections to a metric gravitational theory, based on flat Minkowski space-time as its starting point (zeroth approximation). When applied to the Einstein-Grossmann theory, this scheme naturally (since it is true for the exact theory) showed that the spatial cross-sections of a static metric remain flat in the first post-Minkowskian approximation, which they identified with the Newtonian theory.

Accordingly, when he applied the same scheme to the new, general-relativistic field equations in November 1915, he was amazed to find that it gave *non-flat* spatial cross sections at the first post-Minkowskian level. He communicated his astonishment to Besso, in the letters with which I began this section.

To sum up, Einstein's intuition told him that the deviation of the orbits of test bodies from inertial paths (Euclidean straight lines) due to the (static) gravitational field produced by a central body should show up *before* the effects of the central body's gravitation in curving the previously flat spatial metric. Was he wrong? The approximation method he used seems to show that he was. But perhaps there is another approximation method, in which he is right.

2 Approximation Methods

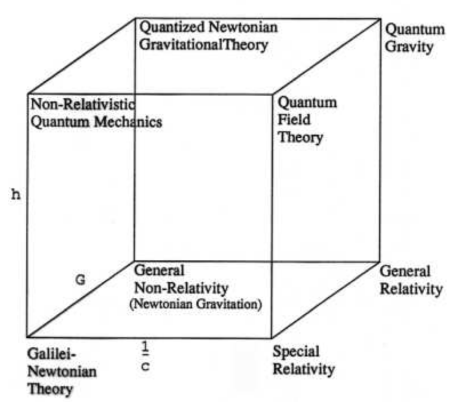

Figure 1. Bronstein cube

To get a better idea of the possibilities, let us start by looking at the Bronstein cube (Figure 1), which shows the relation between a number of space-time theories, starting from Galilei-Newtonian space-time, as we introduce the three dimensional constants c, G and $\hbar$.

For our purpose, we can forget about $\hbar$ and confine our attention to the Bronstein square (Figure 2). We see that there are infinitely many possible paths in the $c-G$ *plane* that lead from Galilei-Newtonian space-time to General-Relativistic space-time, depending on how we "turn on" c and G. We can first turn on c and reach Special-Relativistic space-time, and then turn on G. This approach leads to the "fast approximation" methods, like the one that Einstein used originally in 1914 and again in 1915-1916. In order to find the Newtonian limit by this method, one must make the additional assumption that the gravitational field is weak.

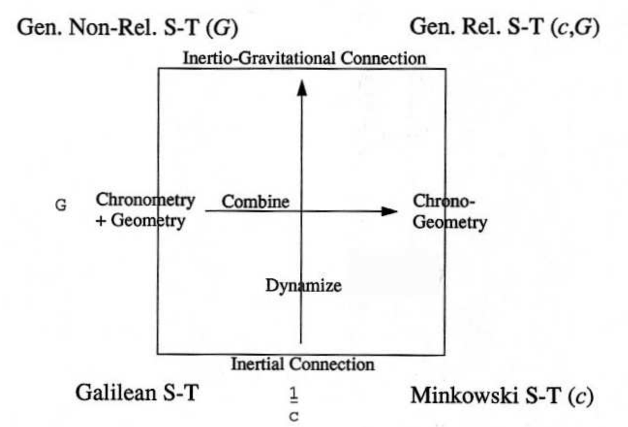

Figure 2. Bronstein square

A careful version of this "weak field, fast approximation method" (see Weinberg 1972, p. 211, for example) is based on noting that, due to the virial theorem applied to a body in a bound Newtonian orbit,

$$(v/c)^2 \approx GM/(c^2 r),$$

where v and r represent average values of the velocity and distance of the body orbiting around a central mass M. So we can expand using a dimensionless parameter ϵ , such that ϵ^2 is of the same order as $(v/c)^2 \approx GM/(c^2 r)$. Although often loosely called a $(1/c)$ expansion, it is better to recognize that we are using a dimensionless expansion parameter based on a path in the $c-G$ parameter plane that links c and G as indicated above.

But clearly, there are other expansions possible based on different paths in this plane. We shall here consider the following path: First introduce G to reach the four-dimensional version of Newtons theory ("general non-relativity" as Ehlers 1973 dubbed it), and proceed towards general relativity by expanding in powers of (v/c) starting from some Newtonian solution. This method allows arbitrarily large values of $GM/(c^2 r)$ using an expansion parameter $\epsilon' \approx (v/c)$ that is now independent of G. What enables us to carry out this expansion, which allows Einstein's intuition to be given a precise mathematical form that was not available to him in 1915, is the concept of an affinely-connected manifold. This concept was only developed after 1915 by Levi-Civita, Weyl and Weizenbock in direct response to the formulation of general relativity. In another paper[b], I have discussed some of the problems that lack of this concept

[b] "*The Story of Newstein, or is Gravity Just Another Pretty Force?*" in Renn and Schimmel 2006.

caused Einstein in the search for a generally-covariant theory of gravitation, and given an account of the four-dimensional version of Newtonian gravitational theory with historical references on its development. So I shall not here go into these matters any further, confining myself to a brief review of the four-dimensional version of Newtonian gravitation theory.

3 Newtonian Space-Time Structures

I remind you that three elements enter into the space-times structure of all the past and current fundamental physical theories summarized in the Bronstein cube:

1. an affine structure, describing the inertio-gravitational field
2. some mathematical structure(s) describing the chronometry and geometry of spacetime:
   (a) In the Newtonian theories this involves a foliation of spacetime into a family of hypersurfaces, each characterized by an absolute time T, describing the chronometry; these hypersurfaces are postulated to be spatially flat, describing their 3-geometry;
   (b) In special and general relativity, this involves a pseudo-metrical structure, describing the fusion of chronometry and geometry into a chrono-geometry.
3. compatibility conditions between the first two structures.

We start with a review of the four-dimensional version of Newtonian gravitation theory, which will be the starting point (zeroth order) of our approximation scheme.

Mathematically we begin with a Galilei Manifold (Ehlers 1973), which consists of three elements:

1. A four-dimensional manifold $\mathcal{M}$ (topologically homeomorphic to $\boldsymbol{R}^4$);
2. A foliation of $\mathcal{M}$, given by a differentiable function $T(x)$, the absolute time. The level surfaces of T define absolute simultaneity; and T provides the chronometry of spacetime: the time interval between any two events is given by $\triangle T$, the difference in absolute time between the two events. Like all the coordinates x, we take T to have dimensions of [length],[c] and introduce a constant c with dimensions of [velocity] to relate it to the ordinary Newtonian time t:

   $$T = ct.$$

   For the present, the value of c is arbitrary and it is not interpreted physically; but, in view of our intended use of a Newtonian solution as a zeroth-order approximation to a solution of the field equations of GR, it does no harm to think of it already as the speed of light– or better, the fundamental velocity of SR.

   The gradient of T, $T_\mu := \partial_\mu T$, $T_\mu = ct_\mu$, gives us a covector that enables us to distinguish between space-like and time-like vectors (vector always means contra-vector):

[c]Schouten's *Tensor Analysis for Physicists* (Schouten 1989) is one of the few mathematics books that discusses "Physical Objects and Their Dimensions" (see Chapter VI, pp. 127-138). He emphasizes that: *Quantities such as scalars, vectors, densities, etc., occurring in physics are not by any means identical with the quantities introduced in Chapter II* [Geometrical Objects in E_n]. *....[Q]uantities in physics have a property that geometric quantities do not. Their components change not only with transformations of coordinates but also with transformations of certain units* (p. 127).]

A vector X^μ is spacelike if $X^\mu T_\mu = 0$, future timelike if $X^\mu T_\mu > 0$, past timelike if $X^\mu T_\mu < 0$.

3. Each leaf of the foliation is a spatial section with the structure of Euclidean 3-space, which (in view of our four-metric sign convention $+ - - -$) we take with negative definite signature.

In contrast to other treatments of Newtonian space-time, instead of now introducing $h^{\mu\nu}$, a contravariant degenerate "metric" tensor of rank three, we introduce a fundamental tetrad field, consisting of one time-like vector field and a triad of space-like vector fields. The time-like vector field will be used to define a frame of reference, and $h^{\mu\nu}$ will be defined with the help of the space-like triad. We shall work with the physical components (Pirani 1957) of any geometrical object with respect to the fundamental tetrad, that is, the set of scalars that result when all co- (contra-) variant tensorial indices are saturated by contraction with the basis vectors (dual basis covectors). At any point of space-time, these components are to be identified with possible measurements of the components of the object made by an observer at that point, moving with the 4-velocity defined by the time-like tetrad vector, and whose laboratory frame and spatial axes are associated with the 3-space spanned by the three spatial triad vectors:

1. We pick a triad of space-like vector fields $e_{(i)}^{\ \mu}$ that span the tangent space at each point of each leaf of the foliation, with a Kronecker delta tangent space metric $\delta_{(i)(j)}$ (with inverse $\delta^{(i)(j)}$), so that

   $$h^{\mu\nu} = e_{(i)}^{\ \mu} e_{(j)}^{\ \nu} \delta^{(i)(j)}$$

   is a symmetric, rotationally invariant space-like tensor: $h^{\mu\nu} T_\mu = 0$.

   The triad defines the geometry of the 3-spatial leaves of the foliation (good measuring rods measure the Euclidean geometry of each leaf).

2. To complete the tetrad, we pick a future time-like vector $V^\mu = e_{(0)}^{\ \mu}$, normalized so that $V^\mu T_\mu = 1 \Leftrightarrow V^\mu t_\mu = 1/c$. We also introduce $v^\mu = e_{(t)}^{\ \mu}$, defined by $e_{(0)}^{\ \mu} = (1/c) e_{(t)}^{\ \mu}$, so that $v^\mu t_\mu = 1$.

For later reference, we note here that the V^μ field provides a rigging (Schouten 1954) of each leaf (hypersurface) of the foliation. With the help of a rigging, we can project 4-dimensional quantities, such as the affine connection, onto these hypersurfaces.

Physically $V^\mu(x)$ represents the state of motion (4-velocity) of an observer at a point x of space-time. Mathematically, a frame of reference corresponds to a fibration and foliation of space-time. The absolute time gives a universal Newtonian foliation. The streamlines of the V^μ field give a fibration of space- time, so choice of a particular V^μ field defines a particular frame of reference, which has been called a kinematic Galileian frame of reference (hereafter kGfor).

Given a tetrad basis $e_{(\alpha)}^{\ \mu}$, there is always an inverse tetrad co-basis $e^{(\alpha)}_{\ \mu}$; and we have normalized $e_{(0)}^{\ \mu}$ in such a way that $e^{(0)}_{\ \mu} = T_\mu = c t_\mu = c e^{(t)}_{\ \mu}$. (Remember in Newtonian space-time there is no non-degenerate 4-dimensional metric, so orthonormality is meaningless except for vectors in the three-spaces, such as the spatial triad vectors.)

If a basis is holonomic, each $e^{(\alpha)}_{\ \mu}$ is a gradient: $e^{(\alpha)}_{\ \mu} = \partial_\mu \phi^{(\alpha)}$, and the four functions $\phi^{(\alpha)}$ would form a coordinate system, and the curl of each $e^{(\alpha)}_{\ \mu}$ would vanish. So the non-vanishing of the curl is a measure of the anholomicity of the basis; hence the collection of these curls is called the anholonomic object Ω:

$$\frac{1}{2}[\partial_{(\mu)}e_{(\nu)}^{\ \mu} - \partial_{\mu(\nu)}e_{(\mu)}^{\ \mu}]e^{(\lambda)}_{\ \alpha} = \Omega^{(\lambda)}{}_{(\mu)(\nu)}.$$

In this definition, we have actually used the Lie bracket of the basis vectors rather than the curl of the dual covectors, but the two definitions are trivially equivalent. The important point is that neither definition involves a metric or connection. So far no dynamical objects have been introduced, and hence we have been defining purely kinematical concepts. We now introduce an affine connection Γ on $\mathcal{M}$, which characterizes the inertio-gravitational field. Rather than its components in a coordinate basis, we work with the tetrad components of the connection:

$$\Gamma^{(\gamma)}_{\ (\alpha)(\beta)} = e^{(\gamma)} \cdot (\nabla e_{(\alpha)}) \cdot e_{(\beta)}.$$

Hereafter, these will be abbreviated "t.c.c.". We have introduced an important notational abbreviation in this equation, which we shall often employ from now on: Tensor indices are omitted; ∇ with no dots before a tensor means exterior covariant differentiation of that tensor; ∇ with a dot means contraction on one index after exterior differentiation of the tensor.

Tetrad components allow us to use an anholonomic basis, the utility of which will soon become evident. Physically, the t.c.c. of Γ represent the components of the inertio-gravitational field relative the frame of reference defined by the tetrad. Note that symmetry of the affine connection does not imply the symmetry of the tetrad connection components. Indeed, for a symmetric connection $\Gamma^{(\gamma)}_{\ [(\alpha)(\beta)]} = \Omega^{(\gamma)}_{\ (\alpha)(\beta)}$, where square brackets represent anti-symmetrization times a factor of 1/2. The time-like covector is already a gradient, so it does not introduce any anholonomicity. As we proceed we shall investigate how much anholonomicity of the space-like covectors we can get rid of by imposing conditions on the space-like triad vector fields, and how much we shall find it advantageous to retain for physical reasons.

4 Compatibility Conditions

Now we can impose the compatibility conditions on the relation between the connection (which, as noted above, represents the inertio-gravitational field) and the tetrad vectors (which represent the chronometry, the geometry, and the kGfor). These conditions are usually imposed in the form:

$$a) \quad \nabla T = 0, \qquad b) \quad \nabla h = 0.$$

(Remember, in our abbreviated notation, T means the covector with covariant index omitted, h the contravariant tensor with two indices omitted, and no dots means exterior covariant differentiation.) But we are interested in their effect on the tetrads, and the relation between h and the triad vectors still allows full rotational freedom for the latter. So let us see to what extent the compatibility conditions limit the t.c.c.:

- It is easily shown that $\nabla T = 0 \Rightarrow \Gamma^{(0)}_{\ (\alpha)(\beta)} = 0$ for all α, β, and hence $\Omega^{(0)}_{(\alpha)(\beta)} = 0$.

  Since their geometry is Euclidean, we should like parallel transport on the flat spatial 3-hypersurfaces T = const. to be independent of space-like path on the surface; so we impose the condition $e_{(a)} \cdot \nabla e_{(b)} = 0$. This implies that $\Gamma^{(c)}_{\ (a)(b)} = 0$, and $\Omega^{(c)}_{\ (a)(0)} = 0$. Hence $\Omega^{(c)}_{\ (a)(b)} = 0$.

So the only remaining non vanishing t.c.c. are $\Gamma^{(c)}_{(0)(0)}$ and $\Gamma^{(c)}_{(0)(a)}$; and the only non-vanishing components of the anholonomic object are $\Gamma^{(c)}_{(0)(a)} = \Omega^{(c)}_{(0)(a)}$. We are of course free to choose a holonomic basis, and make all the Ω vanish. But as we shall now see, dynamical considerations suggest a better choice.

5 What is four-dimensional Newtonian gravitation?

We now have enough concepts available to discuss in more detail the question: "Just what should we require of a four-dimensional version of Newtonian gravitational theory?" Of course, the answer to such a question must be to some degree a matter of definition. We might confine ourselves to a four-dimensional transcription of Newton's original theory, as is usually done. But, although as we shall see it leads us beyond Newton's original theory, the following definition seems to me to do no violence to the concept of a Newtonian-style gravitational theory,:

We shall require Newtonian chronometry and geometry and the compatibility conditions between both and the inertio-gravitational connection to hold. In other words, a Newtonian-style theory is one that is based on a Galileian manifold and a compatible affine connection.

Before proceeding along these lines, allow me a short digression: We might consider dropping the requirement of Euclidicity for the geometry of the space-like hypersurfaces of the foliation of the Galileian manifold. We could admit non-flat Riemannian three-geometries for these hypersurfaces and still regard the resulting gravitation theory as Newtonian in a generalized sense. The well-known argument against this possibility (see Malament 1981) is mathematically correct, but based on much too strong a premise: Once we allow the geometry of the hypersurfaces to be non-flat, we should expect the four-dimensional generalization of Poisson's equation for the gravitational potential to involve the curvature tensor of these hypersurfaces. But this is just what Malament rules out. Indeed, an investigation of this problem (Gonzalez 1970) showed that any static metric with Lorentz signature can be given a quasi-Newtonian interpretation in this generalized sense. In order to make the quasi-Newtonian time (i.e., the parameter of the trajectories of the static Killing vector field) into the affine parameter of the connection, a projective transformation of the connection is needed– but this is another story.

Now I shall return to Newtonian theory, as defined above before the digression, and show that it allows us to go a bit further than traditional Newtonian gravitation theory. As we have seen in the previous section, analysis of the compatibility conditions on the t.c.c. shows that they allow the $\Gamma^{(c)}_{(0)(0)}$ to be non-vanishing; this is well known, since they correspond physically to the "electric-type" Newtonian gravitational field produced by masses at rest, i.e., the ρ term in the T^{00} component of the stress-energy tensor– all that conventional Newtonian theory considers. But the compatibility conditions also allow non-vanishing $\Gamma^{(c)}_{(0)(b)}$, which does not seem to have been noted. Physically, these components correspond to a "magnetic-type" Newtonian gravitational field, produced by moving masses, corresponding to the ρv or T^{0i} components and not present in traditional Newtonian theory. As might be expected, these terms enter at first order in (v/c), i.e., one order higher than the "electric-type" fields.

On the other hand, once non-vanishing $\Gamma^{(c)}_{(a)(b)}$ appear, the spatial hypersurfaces of the foliation no longer remain flat; and once the three-dimensional stress tensor, embodied in the T^{ij} components of the stress-energy tensor, enter the field equations at the next (second) order in (v/c), they will produce terms of this type in the connection. So we may say that it is at this order that Newtonian theory ends (in my sense of the term "Newtonian" at any rate), and post-Newtonian theory proper begins.

6 Passive Newtonian Dynamics

Now we can start to look at some dynamics. The passive reaction of matter to the inertio-gravitational field depends on the connection; in particular, a monopole test particle moves along a time-like geodesic of the connection. Let W be the tangent vector field defined along some time-like curve parametrized by the preferred affine parameter λ and normalized so that $W \cdot T = 1$. If the components of W are referred to the basis vectors $e_{(\alpha)}$:

$$W = W^{(\alpha)} e_{(\alpha)}, \quad \text{with} \quad W^{(0)} = 1,$$

then the geodesic equation takes the form:

$$\frac{DW^{(\nu)}}{d\lambda} + \Gamma^{(\nu)}_{\ (\alpha)(\beta)} W^{(\alpha)} W^{(\beta)} = 0, \qquad D = W \cdot \nabla.$$

We decompose this equation into 1) its time-like and 2) its space-like components.
1) $(\nu) = (0)$: We see that, since $\Gamma^{(0)}_{\ (\alpha)(\beta)} = 0$ for all α, β , then $DW^{(0)}/d\lambda = 0$; this means that, up to a linear rescaling of the origin and unit of time, the affine parameter λ agrees with the chronometric time T; so from now on we shall use $T = ct$ as the affine parameter.
2) $(\nu) = (m)$: Noting that $W^{(m)} = w^{(m)}/c$, the components of the (three-) velocity with respect to the kGfor, the (m) components of the geodesic equation take the form (remember the $\Gamma^{(c)}_{\ (a)(b)}$ and the $\Gamma^{(c)}_{\ (a)(0)}$ have been made to vanish):

$$\frac{1}{c^2}\frac{Dw^{(m)}}{dt} + (\frac{1}{c^2})\Gamma^{(m)}_{\ (t)(t)} + (\frac{1}{c})\Gamma^{(m)}_{\ (t)(n)}\frac{w^{(n)}}{c} = 0,$$

or cancelling out the factor $(1/c^2)$:

$$\frac{Dw^{(m)}}{dt} + \Gamma^{(m)}_{\ (t)(t)} + \Gamma^{(m)}_{\ (t)(n)} w^{(n)} = 0.$$

The first term on the left is the acceleration of the particle wrt the kGfor , i.e., the inertial term. So the next term in the equation should be (minus) the gravitational force term with respect to the kGfor; and, when we get to the active dynamics, we expect it to be generated by the Newtonian mass density.

What does the third and last term signify? Since the previous term is the analogue of an "electric-type" force term in electrodynamics, we guess by analogy that the final term is a "magnetic-type" force term; and expect it to be generated by moving charge density and to be of one order higher in v/c than the electric type term. But before turning to the field equations, let us see exactly what its effect is. Consider the evolution of one of the triad of spacelike vectors as a function of the affine parameter along our geodesic curve: $e_{(b)}(T)$. It is easily shown that $(De_{(b)}/dT) \cdot e^{(0)} = 0$; so $(De_{(b)}/dT)$ is spacelike and hence itself can be expanded in terms of the triad of spacelike vector fields along the curve:

$$\frac{De_{(n)}(T)}{dT} = \omega^{(m)}_{\ (n)} e_{(m)}(T).$$

Remembering that the spatial triad is orthonormal, i.e., $e_{(i)} \cdot e_{(j)} = \delta_{(i)(j)}$, and lowering the first index on $\omega^{(m)}_{\ (n)}$ with $\delta_{(i)(j)}$, it is easily seen that $\omega_{(m)(n)}$ is antisymmetric. Thus, it represents a rate of rotation, and $\omega_{(m)(n)}dT$ represents an infinitesimal rotation of the triad during the time dT.

On the other hand, going back to the definition of the Γs, it is easy to show that

$$\omega^{(m)}_{\ \ (n)} = \Gamma^{(m)}_{\ \ (0)(n)}.$$

So the last term of the geodesic equation represents (using the usual correspondence between antisymmetric three-tensors and three-vectors) a Coriolis type ($\boldsymbol{\omega} \times \boldsymbol{v}$) force.

- Now we can turn to the other compatibility condition, $\nabla h = 0$ (remember, no dot means exterior differentiation!). Since $h = e_{(i)} e_{(j)} \delta^{(i)(j)}$, this condition implies certain restrictions on the triad vectors, which we investigate by taking the tetrad components of the equation.

First, $e_{(0)} \cdot (\nabla h) \cdot e_{(m)} \cdot e_{(n)} = 0$ implies that:

$$\Gamma^{(m)}_{\ \ (0)(n)} + \Gamma^{(n)}_{\ \ (0)(m)} = 0;$$

using the equation $\omega^{(m)}_{(n)} = \Gamma^{(m)}_{\ \ (0)(n)}$ discussed above, this means that the triad can be allowed to rotate rigidly as it moves along the time-like world line, to which $e_{(0)}$ is tangent (note that this world line is arbitrary - nothing requires it to be geodesic). If we consider two different time-like paths with the same starting and ending points, both starting off with the same initial triad (each corresponding to a different frame of reference, of course), the rotations may differ. Thus, parallel transport of the triad vectors is not independent of path if the $\Gamma^{(m)}_{\ \ (0)(n)}$ terms do not vanish. If we were to require that $\Gamma^{(m)}_{\ \ (0)(n)} = 0$, then such rotation would be excluded, and parallel transport of the triad vectors along time-like paths would also be independent of path:

$$De_{(i)}/dT = e_{(0)} \cdot \nabla e_{(i)} = 0.$$

As we shall see in the next section, when there is moving mass, the dynamics of the gravitational field forces us to keep $\Gamma^{(m)}_{\ \ (0)(n)}$ terms.

7 Field Equations

Finally, we turn to the field equations. We shall write them in terms of the affine t.c.c.'s and the compatibility conditions between metric and connection, which assure– at the general-relativistic level– that the connection is metric. We need an expression for the tetrad components of the Ricci tensor, which can be found in Papapetrou-Stachel 1978:

$$R_{(\lambda)(\mu)} = \Gamma^{(\kappa)}_{\ \ (\lambda)(\mu)},_{(\kappa)} - \Gamma^{(\kappa)}_{\ \ (\kappa)(\mu)},_{(\lambda)} + \Gamma^{(\kappa)}_{\ \ (\kappa)(\rho)} \Gamma^{(\rho)}_{\ \ (\lambda)(\mu)} - \Gamma^{(\kappa)}_{\ \ (\lambda)(\rho)} \Gamma^{(\rho)}_{\ \ (\kappa)(\mu)} + 2\Omega^{(\rho)}_{\ \ (\kappa)(\lambda)} \Gamma^{(\kappa)}_{\ \ (\rho)(\mu)}.$$

[Note that here, by definition $,_{(\kappa)} := e^{\rho}_{(\kappa)} \partial_\rho$.]

In the previous section we have seen that, in our four-dimensional formulation of Newtonian theory, we were led to eliminate all t.c.c.'s except $\Gamma^(c)_{(0)(0)}$ and $\Gamma^{(c)}_{(0)(a)}$; and hence the only non-vanishing components of the anholonomic object are $\Omega^{(c)}_{(0)(a)} = \Gamma^{(c)}_{(0)(a)}$. Our strategy is to start by calculating a set of t.c.c.s that constitute an exact Newtonian connection for the Newtonian field equations with some given Newtonian stress-energy tensor as source. Then we shall use this Newtonian solution as the starting point for an iteration process leading to higher-order corrections to the Newtonian t.c.c.'s and stress-energy tensor that bring us closer and closer to a solution to the general-relativistic field equations with a general-relativistic stress-energy tensor as source. We know, of course that this can only lead to an approximate solution that valid in the near zone; and that if we want to consider gravitational radiation processes, this near-zone solution must be coupled to a solution in the far (radiation) zone by the method of matched asymptotic expansions. But we shall not here enter into any of the details of this procedure[d].

Calculating the tetrad components of the Ricci tensor, with the Newtonian Ansatz given above for the t.c.c.'s, we find:

$$R_{(0)(0)} = \Gamma^{(k)}{}_{(0)(0)},_{(k)} \quad , \quad R_{(0)(m)} = \Gamma^{(k)}{}_{(0)(m)},_{(k)} \quad , \quad R_{(m)(0)} = R_{(m)(n)} = 0.$$

[Note that symmetry of the connection does not imply symmetry of the tetrad components of the Ricci tensor.]

In the previous section, by eliminating various components of the connection, we have already satisfied the compatibility conditions between the Newtonian chronometry and geometry and the affine connection. So all that remains is to look at the right-hand side of the Newtonian field equations, that is, at some Newtonian stress-energy tensor.

I shall make the Ansatz that the stress-energy tensor (SET) takes the form of that for elastic matter. For a review of work on general-relativistic treatments of a perfectly elastic body, see Morrill 1991, Chapter 3, pp. 72-73; and for a derivation of the post-Newtonian equations of motion, see Chapters 3 and 4. Often a further simplifying assumption is made by taking the SET of a perfect fluid. As far as the external equations of motion are concerned, it doesn't make much difference, and the fluid Ansatz certainly simplifies the calculations. But the internal equations of motion for an elastic body allow for the possibility of treating such important astrophyical events as neutron star quakes; and since I will not be doing any detailed calculations, I shall stay at this level of generality.

Before actually calculating anything, we can use a dimensional argument to see what to expect. Let us take the dimension of all the components of the SET as those of the mass density ρ. Then the mass current density vector $\rho v/c$ has the same dimensions, as do the components of the three-dimensioneal stress tensor σ^{ij}/c^2. So I shall make the following Ansatz for the Newtonian SET:

$$T^{\mu\nu} = \rho W^\mu W^\nu + \sigma^{\mu\nu} \,, \quad \text{with} \quad \sigma^{\mu\nu} T_\mu = 0.$$

Here, I choose "W" and not "V" to symbolize the Newtonian four-velocity of the fluid, because "V" is the symbol for the time-like fundamental tetrad vector. Of course we might choose to identify the two; but the point is that we need not do so. The components of W^μ are $(1, w^i/c)$, the components of $\sigma^{\mu\nu}$ are p^{ij}/c^2. One might think that, by introducing "c" into these expressions we are going beyond the Newtonian framework; but remembering that $x^0 = T = ct$, we find that the c's cancel out of the Newtonian equations of motion.

Taking the tetrad components, $e^{(\mu)}{}_\nu(\nabla_\mu T^{\mu\nu}) = 0$, of the conservation equation with respect to the fundamental tetrad, for $(\mu) = (0)$ we get:

$$\partial_{(0)}(\rho) + \partial_{(i)}[\rho w^{(i)}/c] = 0,$$

and remembering that $x^0 = T = ct$, we see that this is just the Newtonian equation of continuity.

[d]For recent accounts using traditional, purely metric methods, see Blanchet 2002 and Poujade and Blanchet 2002.

For $(\mu) = (m)$ we get:

$$\partial_{(0)}[\rho w^{(m)}/c] + \partial_{(n)}[\rho(w^{(m)}/c)(w^{(n)}/c) + p^{(m)(n)}/c^2] + \rho^0\Gamma^{(m)}{}_{(0)(0)} + [\rho w^{(n)}/c]^0\Gamma^{(m)}{}_{(0)(n)} = 0.$$

Remembering that the convective derivative $D_{(0)} = W \cdot \partial = \partial_{(0)} + \frac{w^{(i)}}{c}\partial_{(i)}$, we see that the first equation can be written:

$$D_{(t)}(\rho) + \rho\partial_{(i)}w^{(i)} = 0;$$

and the second equation can be rewritten (using the first to eliminate several terms):

$$\rho D_{(t)}[w^{(m)}] + \partial_{(n)}[p^{(m)(n)}] + \rho^0\Gamma^{(m)}{}_{(t)(t)} + [\rho w^{(n)}]^0\Gamma^{(m)}{}_{(t)(n)} = 0.$$

The first term on the left is the continuum analogue of the inertial term $m\,dv/dt$; the second, the divergence of the three-dimensional stress-tensor, is the negative of the net stress density on faces of a volume element; the third is the usual "electric-type" gravitational force density on a static mass density; and the fourth is the new "magnetic-type" gravitational force density on a moving mass density.

Note the difference with the usual treatment, assuming a special-relativistic starting point in interpreting the SET for an elastic body. In that case, $W = \gamma[1, w^{(m)}/c]$, where $\gamma = \sqrt{1-(w/c)^2}$, and one proceeds to expand γ in powers of (w/c): $\gamma = 1 + 1/2(w/c)^2 + ...$ to get the Newtonian limit. But we are starting our expansion from an exact Newtonian solution. This can be done by noting that, in our approach, the difference between the Newtonian and general-relativistic cases can be formulated in the tangent and co-tangent spaces at each point of the manifold: A set of basis vectors can give rise to the two degenerate Newtonian space and time "metrics" if a degenerate metric of rank three is introduced in the tangent space, and a degenerate metric of rank one in the cotangent space. The same set of basis vectors can be used in the general-relativistic case: For general relativity, of course, special relativity holds in the tangent and cotangent spaces, i.e., a non-degenerate (pseudo-) metric with Minkowski signature is introduced in either and induces a corresponding metric in the other– i.e., special-relativistic chrono-geometry holds in the tangent and cotangent spaces.

For the approximation procedure, the Newtonian degenerate "metrics" in the tangent and cotangent spaces can be taken as the starting point for the addition of terms that convert them into the special-relativistic ones.

Returning to the problem of the four-dimensional form of the Newtonian field equations, we need the tetrad components of the SET for the right-hand side of the field equations. It is easily seen that:

$$T^{(0)(0)} = \rho\,, \quad T^{(0)(n)} = \rho w^{(n)}/c\,, \quad T^{(m)(n)} = \rho w^{(m)}w^{(n)}/c^2 + p^{(m)(n)}/c^2.$$

We want to set these terms equal to various tetrad components of the. Ricci tensor, but just how should we do it? Since our aim is to use the Newtonian solution as the starting point for approximating a solution to the Einstein equations of general relativity, our approach is to let the latter equations tell us how to proceed. We take the field equations in the form in which Einstein originally wrote them, with the Ricci tensor on the left-hand side:

$$R_{(\lambda)(\mu)} = (const)G[T_{(\lambda)(\mu)} - \frac{1}{2}\eta_{(\lambda)(\mu)}T],$$

where the value of the constant is to be determined by the Newtonian limit. In the t units, the co-metric in the tangent space $\eta_{(\lambda)(\mu)} = \mathrm{diag}\, c^2[1, -1/c^2, -1/c^2, -1/c^2]$, and the contra-metric $\eta^{(\lambda)(\mu)} = diag[1/c^2, -1, -1, -1]$. Using them, we find:

$$T = c^2T^{(0)(0)} - \sum T^{(i)(i)} = \rho c^2 + \rho w^{(i)}w^{(i)}/c^2 + p^{(i)(i)}/c^2.$$

The components of $T_{(\lambda)(\mu)}$ are:

$$T_{(0)(0)} = c^4 T^{(0)(0)} = c^4 \rho \quad , \qquad T_{(0)(n)} = c^2 T^{(0)(n)} = c\rho w^{(n)}$$
$$T_{(m)(n)} = T^{(m)(n)} = \rho w^{(m)} w^{(n)}/c^2 + p^{(m)(n)}/c^2.$$

Finally, the components of $T_{(\lambda)(\mu)} - \frac{1}{2}\eta_{(\lambda)(\mu)}T$ are:

$$\begin{aligned}
(0)(0) &= (1/2)c^4[\rho + \rho w^{(i)} w^{(i)}/c^4 + p^{(i)(i)}/c^4] \\
(0)(n) &= c\rho w^{(n)} = c^4[\rho w^{(n)}/c^3] \\
(m)(n) &= c^4\{\frac{1}{2c^2}\delta_{(m)(n)}\rho + \\
&\quad [\rho w^{(m)} w^{(n)}/c^6 + p^{(m)(n)}/c^6 + 1/2\delta_{(m)(n)}(\rho w^{(i)} w^{(i)}/c^6 + p^{(i)(i)}/c^6)]\}
\end{aligned}$$

In all these terms, we have taken out the common factor c^4 in view of the fact that all of them must be inserted into the right-hand side of the field equations. It is clear that the leading term is the $(0)(0) - \text{term}(1/2)c^4\rho$; so this is the term that we use to determine the numerical factor in the equation.

Remembering that :

$$R_{(0)(0)} = \Gamma^{(k)}_{\ \ (0)(0)},_{(k)} \quad , \quad R_{(0)(m)} = \Gamma^{(k)}_{\ \ (0)(m)},_{(k)} \quad , \quad R_{(m)(0)} = R_{(m)(n)} = 0 \ ,$$

the resulting field equation for $(0)(0)$ is :

$$R_{(0)0)} = \Gamma^{(k)}_{\ \ (0)(0)},_{(k)} = (\text{const})G[T_{(0)(0)} - \frac{1}{2}\eta_{(0)(0)}T].$$

Or, converting to t,

$$\Gamma^{(k)}_{\ \ (t)(t)},_{(k)} = 1/2(\text{const})Gc^6\rho.$$

Similarly for the $(0)(n)$ component, we get:

$$R_{(t)(m)} = \Gamma^{(k)}_{\ \ (t)(m)},_{(k)} = (\text{const})Gc^6[\rho w^{(n)}/c^4].$$

Now, to get from these equations to the usual Newtonian gravitational scalar potential φ_{gr} and the new Newtonian "magnetic type" gravitational vector potential A_{gr}, we must make some assumptions about the existence of "connection potentials". For elegance, these can be formulated as further conditions on the Riemann tensor (see Trautman 1965, Kűnzle 1972, 1976, Ehlers 1981; and for a useful summary Malament 1986); but we shall simply assume that

$$\Gamma^{(k)}_{\ \ (0)(0)} = \delta^{(k)(j)}\partial_{(j)}\varphi_{\text{gr}} \quad \text{and} \quad \Gamma^{(k)}_{\ \ (0)(m)} = \frac{1}{c}\delta^{(k)(j)}[\partial_{(j)}A_{\text{gr}\,(m)} - \partial_{(m)}A_{\text{gr}(j)}].$$

[One might worry about substituting the tetrad components of the curl for the curl in the last expression. But a short calculation shows that, since $\Omega^{(c)}_{\ \ (a)(b)} = 0$, this is OK.] Then, if we take the constant $= 8\pi G/c^6$, the field equations reduce to:

$$\nabla^2\varphi_{\text{gr}} = 4\pi G\rho \qquad \text{and} \qquad \nabla^2 A_{\text{gr}} = 4\pi G\rho v$$

with the condition $\nabla \cdot A_{\text{gr}} = 0$. Thus, $\Gamma^{(k)}_{\ \ (0)(m)}$ is one order higher in powers of $(1/c)$ than $\Gamma^{(k)}_{\ \ (0)(0)}$. But, as discussed earlier, this still does not affect the flat spatial character of the hypersurfaces of simultaneity.

Our next step is to use a solution to these equations as the starting point for a post-Newtonian approximation. We start from a quite general result: If ${}^0\Gamma^{\kappa}_{\ \ \mu\nu}$ is any given symmetric connection and $\Gamma^{\kappa}{}_{\mu\nu}$ another symmetric connection, then their difference $A^{\kappa}{}_{\mu\nu} = \Gamma^{\kappa}{}_{\mu\nu} - {}^0\Gamma^{\kappa}{}_{\mu\nu}$ is a tensor of third rank, and the relation between the Ricci tensor of the two connections is given by:

$$R_{\mu\lambda} = {}^0R_{\mu\lambda} + {}^0\nabla_{\kappa}A^{\kappa}{}_{\mu\lambda} - {}^0\nabla_{\mu}A^{\kappa}{}_{\kappa\lambda} - A^{\rho}{}_{\kappa\lambda}A^{\kappa}{}_{\mu\rho} + A^{\rho}{}_{\mu\lambda}A^{\kappa}{}_{\kappa\rho}.$$

Taking tetrad components of this equation, we get:

$$\begin{aligned} R_{(\mu)(\lambda)} = {} & {}^0R_{(\mu)(\lambda)} + {}^0\partial_{(\kappa)}A^{(\kappa)}{}_{(\mu)(\lambda)} - {}^0\partial_{(\mu)}A^{(\kappa)}{}_{(\kappa)(\lambda)} \\ & + A^{(\rho)}{}_{(\mu)(\lambda)}\,{}^0\Gamma^{(\sigma)}{}_{(\sigma)(\rho)} - A^{(\rho)}{}_{(\mu)(\sigma)}\,{}^0\Gamma^{(\sigma)}{}_{(\rho)(\lambda)} - A^{(\rho)}{}_{(\sigma)(\lambda)}\,{}^0\Gamma^{(\sigma)}{}_{(\rho)(\mu)} \\ & - A^{(\rho)}{}_{(\rho)(\sigma)}\,{}^0\Gamma^{(\sigma)}{}_{(\mu)(\lambda)} - A^{(\rho)}{}_{(\kappa)(\lambda)}A^{(\kappa)}{}_{(\mu)(\rho)} + A^{(\rho)}{}_{(\mu)(\lambda)}A^{(\sigma)}{}_{(\sigma)(\rho)}. \end{aligned}$$

Applying this result to our problem, we take the Newtonian t.c.c.'s and components of the Ricci tensor as the background connection and Ricci tensor:

$${}^0R_{(0)(0)} = {}^0\partial_{(\kappa)}\,{}^0\Gamma^{(\kappa)}{}_{(0)(0)}\ , \quad {}^0R_{(0)(m)} = {}^0\partial_{(\kappa)}\,{}^0\Gamma^{(\kappa)}{}_{(0)(m)}\ , \quad {}^0R_{(m)(0)} = {}^0R_{(m)(n)} = 0.$$

We start with a Newtonian solution to the equations of motion for the elastic body, but now we include the "post-Newtonian" source-term components of $\sigma^{\mu\nu} = p^{ij}/c^2$; as mentioned above, the components of $A^{(\kappa)}{}_{(\mu)(\lambda)}$ are also assumed to be of one higher order ("$1/c^2$"). When we go to this order in the SET, the new term that appears is $T^{(m)(n)}$. So we shall need to solve the $R_{(m)(n)}$ term in the field equations to this order, and make the Ansatz that only the spatial components $A^{(k)}{}_{(m)(n)}$ will be needed.

With this assumption, and using the fact that only ${}^0\Gamma^{(\kappa)}{}_{(0)(0)}$ and ${}^0\Gamma^{(\kappa)}{}_{(0)(m)}$ are non-vanishing, and that products of the As are of higher order, we find that, to this order:

$$R_{(m)(n)} = {}^0\partial_{(k)}A^{(k)}{}_{(m)(n)} - {}^0\partial_{(m)}A^{(k)}{}_{(k)(n)}.$$

This can be solved by requiring the second term to vanish. But we shall not pursue the details of the post-Newtonian approximation any further here.

8 Concluding Remarks

We might prefer to derive the field equations and the compatibility conditions from a Lagrangian as the most efficient technique. As soon as one recognizes that the affine connection is the immediate representation of the inertio-gravitational field in both four-dimensional Newtonian theory and in general relativity, it is clear that the optimal choice of a Lagrangian should be one of the Palatini type, in which both connection and metric are varied independently. This results in what Infeld and Plebański call equations of motion of the second kind, in which both field and matter variables occur.

One might object that for efficiency of calculation, at least up to orders of approximation at which gravitational radiation starts to play a role, the Plebański-Bażanski technique for deriving equations of motion of the third kind, in which only matter variables enter, is clearly superior; and indeed their way of deriving these equations from a Lagrangian is superior to other methods, such as that of Chandrasekhar and co-workers. But equations of motion of the third kind leave obscure some points of principle. These equations implicitly involve some coordinate system. What is the physical significance of these coordinates? Without at least a three-metric, these coordinates have no physical meaning. What order of metric is to be used in interpreting the equations of motion of a given order? And what is the relation between the order of the affine connection that governs the equations of motion of a given order and the order of the metric? The clearest way to answer such questions is by means of equations of motion of the second kind, in a form in which both connection and metric are kept in the equations. It proves advantageous to work with a Lagrangian introduced by Papapetrou and Stachel (1978) for the tetrad formalism, in which tetrad vectors, tetrad metric and tetrad components of the affine connection can all be varied. (Of course, variations of the tetrad metric and of the tetrad vectors are not independent of each other: the resulting two sets of field equations are equivalent.)

Another question is how to correctly formulate the relation between a Newtonian and a general-relativistic space-time? The most mathematically correct way is to take each as a boundary of a five-dimensional manifold, which is foliated by a family of 4-dimensional manifolds, each endowed with a metric and a compatible connection; and fibrated in such a way that points on different four-dimensional hypersurfaces of the foliation may be identified. Such a formulation gives sufficient "rigidity" to the problem to make the concept of the limits of space-times rigorously meaningful (see Geroch 1969), and this is the way that the relation between Newtonian and general-relativistic space-times should be formulated.

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