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RESEARCH ARTICLE

Measurement of the space–time interval between two events using the retarded and advanced times of each event with respect to a time-like world-line

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Abstract Several recent studies have been devoted to investigating the limitations that standard quantum mechanics and/or quantum gravity might impose on the measurability of space–time observables. These analyses are often confined to the simplified context of 2D flat space–time and rely on a simple procedure for the measurement of space-like distances based on the exchange of light signals. We present a generalization of this measurement procedure applicable to all three types of space–time intervals between two events in space–times of any number of dimensions. We also present a preliminary account of an alternative measurement procedure that can be applied taking into account the gravitational field of the macroscopic measuring apparatus.

Keywords Relativity · Quantum mechanics · Quantum gravity · Uncertainty principles

1 Introduction

The limitations on measurability encountered in ordinary (non-gravitational) applications of quantum mechanics concern conjoint measurement of pairs of non-commuting observables. In particular, any given observable can be measured with arbitrary accuracy at the cost of renouncing any attempt at measurement of a conjugate observable. However, a number of arguments have been offered in support of the possibility

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that gravitational observables, including those depending on the chrono-geometric and inertio-gravitational structures, may be subject to more severe measurability limitations. In particular, it has been suggested that there may be absolute limitations (i.e. not avoidable even by sacrificing all knowledge of their conjugates) on the measurability of certain of these observables. These results have been obtained within the framework of certain approaches to quantum gravity, most notably in string theory [1–3], and they are also supported by certain heuristic analyses [4–11], in which the gravitational degrees of freedom playing a role in the relevant measurement procedures are treated classically, while the non-gravitational degrees of freedom are treated using traditional quantum field theory methods in flat or non-flat background space–times. In addition, there are good heuristic arguments (see, e.g., [12]) essentially going back to Bronstein [13], that the very concept of a smooth space–time should break down at the Planck four-volume. An attempt to localize a massive object to within a three-volume approaching the Planck three-volume will give rise, according to the Heisenberg indeterminacy relations, to energy fluctuations that will produce a black hole lasting a Planck time, making any further space–time localization meaningless (see, e.g., [14, p. 117]). These arguments suggest that space–time smoothness breaks down at the Planck scale; and the same conclusion is reached by exploring heuristically the possible consequences of quantization of the gravitation field (see, e.g., [15, p. 40]).

The main interest of these results lies in the hope that they might provide key insights concerning the structure of a future theory encompassing both present-day quantum field theory and general relativity ("quantum gravity"). Of course, faith in the reliability of these hints depends not only on the rigour of the individual analyses, but also on the generality (or representative nature) of the measurement procedures analyzed in establishing the suggested limitations. We share the point of view, emphasized by Heisenberg [16] and Bohr and Rosenfeld [17], that the limits of *definability* of a quantity within any formalism should coincide with the limits of *measurability* of that quantity for all conceivable (ideal) measurement procedures. For well-established theories, this criterion can be tested. For example, in spite of a serious challenge [18], source-free quantum electrodynamics was shown to pass this test [17]. In the case of quantum gravity, our situation is rather the opposite. In the absence of a fully accepted, rigorous theory, exploration of the limits of measurability of various quantities can serve as a tool to provide clues[1] in the search for such a theory: If we are fairly certain of the results of our measurability analysis, the proposed theory must be fully consistent with these results.

This paper is concerned with the generalization of one of the most frequently-cited procedures in such discussions of quantum limitations on quantum measurement of chrono-geometrical quantities, first discussed in detail by Wigner and Salecker [20, 21]: the measurement of space-like intervals in 2D space–time based on the exchange of light signals.

Analyses based on this measurement procedure have been used extensively (see, e.g., [8–10, 22–24]) as the source of intuitive ideas about how to approach the quantum-gravity problem. In evaluating the reliability of the arguments supporting these

[1] Bryce DeWitt first formulated this program of trying "to develop the [quantum gravity] formalism itself with the aid of ideas of the theory of measurement" [19].

intuitions, several concerns arise. We shall mention three that are discussed in this article. (1) Wigner and Salecker's focus on space-like distances in two dimensions might prove to be too narrow a basis for such general conclusions. (2) Wigner–Salecker's treatment of the delicate issue of the relation between the microscopic and macroscopic aspects of quantum phenomena. (3) Even though their ultimate concern is with gravitation theory, most of the studies based on the Wigner–Salecker procedure assume a flat background space–time, without investigating whether this assumption is compatible with the distribution of masses of the (macroscopic) measuring devices.

The primary objective of this paper relates to the first concern: to provide a generalization of the Wigner–Salecker procedure to the case of space–time intervals of any type defined by two events; and, perhaps more significantly, in space–times of any number of space–time dimensions.[2] Through this generalization we are setting the stage for a re-examination of "quantum-gravity measurability analyses" of the type reported for example in [8–10,22,23] that would consistently rely throughout on results that are meaningful in the 4D context. Some considerations partially based on 4D results have already appeared in the Wigner–Salecker literature (such as the role played by properties of 4D black holes in the analyses reported in [8–10]); but we believe that more reliable results could be obtained by insisting that all aspects of the arguments be meaningful[3] for a 4D space–time.

The second objective of our paper relates to the second and third concerns listed above. Besides generalizing the Wigner–Salecker procedure, we also propose an alternative measurement procedure that is addresses concerns about the way in which Wigner–Salecker handled the relation between the microscopic aspects of quantum phenomena and the macroscopic devices used in recording measurements of them. This alternative measurement procedure still involves processes taking place in a flat region of space–time; but, rather than simply assuming the flat space–time background, we devise an arrangement of the macroscopic measuring apparatus such that the flat space–time region is an acceptable solution of Einstein's equations for the gravitational field it produces. We assume that, as long as the measurement procedure is structured (particularly insofar concerns the masses of macroscopic measuring apparata) in such a way as to be compatible with the adoption of a flat space–time, the results emerging from the measurability analysis should provide reliable heuristic guidance for the quantum-gravity realm. Finally, we suggest that a next step in the analysis would be to take account of the gravitational effect of the processes taking place in the flat region of space–time. Rather than treating these processes as passively responding to the gravitational field created by the macroscopic measuring apparatus, they should be treated as sources of small modifications of that gravitational field; in particular they would produce small curvature effects that could be treated in the linearized approximation to the Einstein equations.

[2] For Wigner's concerns with the limitation to two dimensions, see [20, pp. 261–262].

[3] Some quantum-gravity measurability analyses that adopted a genuine 4D perspective have been recently proposed in [25,26]; however, these studies did not investigate directly the measurability of intervals, but rather they concerned a possible quantum-gravity limit on the total number of elementary events that can occur in a given volume of spacetime.

This viewpoint finds some support in the much-discussed "double role" of gravitational fields: in general-relativistic theories, the gravitational field is used not just to describe "gravitational interactions" but also characterizes the structure of space–time itself. If we acquire a good understanding of some space–time structures (such as measurability limitations in the flat space–time and linearized curvature cases) this can guide us toward the formulation of some constraints that should be imposed more generally on the properties of gravitational fields.

The paper is organized as follows. Section 2 provides a brief summary of the Wigner and Salecker measurement analysis for space-like intervals in 2D Minkowski space–time. Section 3 is still confined to the 2D case, but generalizes the result to space–time intervals of any type. We could have considered directly the case of arbitrary intervals in a space–time of an arbitrary number of dimensions, but we prefer to adopt a more pedagogical approach, proceeding inductively from the case of 2D space–time, to 3D and then to $(4+n)$D space–times (where $n = 0, 1, 2, \ldots$). Section 4 concerns 3D Minkowski space–time with emphasis on some new elements required for measurability analyses in more than two dimensions. The straightforward generalization to $(4+n)$D space–times is discussed briefly in Sect. 5. The main new result emerging from our analysis is that, when implemented in space–times of dimension greater than 2, the Wigner–Salecker procedure requries not only clock readings of (some advanced and retarded) times, but also an angle measurement. It follows that the great depth at which the clock readings have been investigated from a quantum-gravity measurability perspective still provides only a partial understanding of measurability limitations. In Sect. 6, we argue that the Wigner–Salecker procedure can be improved by letting the macroscopic recording apparatus consist of a spherically symmetric shell of matter entirely surrounding the region under study. This alternative setup may in particular be relevant for addressing at least 3 key points. The first point concerns some limitations of the Wigner–Salecker analysis that result from the nature of the clock they employ (microscopic versus macroscopic), and the way in which the microscopic aspects of the measurement process are eventually recorded by a macroscopic device. The second point concerns the adoption of a flat background space–time, which is unjustified in the standard Wigner–Salecker setup, but can be obtained from a genuine general-relativistic perspective in our proposed alternative measurement procedure. And the third point concerns the possibility (if the shell of matter that surrounds the region under study is populated sufficiently densely with clocks) to replace the angle measurement, which we found necessary for the standard Wigner–Salecker procedure with additional clock readings. Section 7 summarizes our findings and indicates some goals of our future work.

2 Summary of the WS analysis (2D, space-like interval)

Rather than following Wigner–Salecker's original measurement procedure for 2D space-like intervals in detail, we present the relevant portion of their analysis in a language and notation that provide a better starting point for our generalization, emphasizing those aspects of their work that have been cited and used most in recent studies [8–10,22].

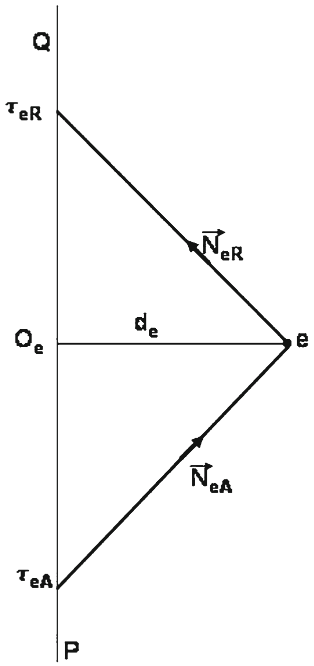


Fig. 1 Proper-time measurements on the inertial time-like world-line PQ used to measure the length d_e of the space-like interval between the event e and the event O_e on the world line

Wigner and Salecker [20,21] determine the space-like interval between two events in terms of certain proper time intervals[4] along a given time-like world line passing through one of the events: namely, the time intervals for massless probes (photons) to traverse the distance back and forth between the world line and the two events.[5] The crucial ingredient is the relation between these proper time intervals and the space-like interval, so we start by expressing a space-like interval between two events in 2D Minkowski space–time in terms of proper times on an inertial time-like world-line PQ (see Fig. 1). The simplest case is when one of the two events lies on PQ and the interval between the two events is orthogonal to PQ. As we shall see in Sect. 3, the interval between any two events can be derived easily from this case.

Let $\tau_{e,A}$ be the (proper) time of the point on PQ such that a light signal sent from it reaches the event e, and $\tau_{e,R}$ the (proper) time a signal sent from e will reach the world-line. (We shall sometimes refer to these as the "advanced" and "retarded" times of the event e with respect to PQ.) Letting

$$\Delta\tau_e \equiv \tau_{e,R} - \tau_{e,A} \tag{1}$$

[4] Note that, like Wigner and Salecker, we take for granted that the time interval between two time-like separated events can be directly measured by the proper time readings of a clock moving on the inertial (time-like) path between the two events. At the quantum level this assumption might call for further analysis.

[5] Wigner and Salecker also provide [20,21] a critique of the use of rigid rods in the measurement of distances. For a defense of legitimacy of their use see [27].

and calling $\vec{\Delta\tau_e}$ the corresponding (2D) space–time vector (in general, $\vec{V}$ will denote a space–time vector of magnitude V), we see that the interval (distance) $\vec{d_e}$ between the event O_e at the midpoint of $\vec{\Delta\tau_e}$ (corresponding to proper time $\tau_e = \tau_{e,A}/2 + \tau_{e,R}/2$) and the event e can be found by "squaring" the vectorial equation:[6]

$$\vec{d_e} + \vec{N_{e,R}} = \vec{\Delta\tau_e}/2 \tag{2}$$

where $\vec{N_{e,R}}$ is the retarded null vector connecting e and the event on the PQ world-line that corresponds to the proper time $\tau_{e,R}$.

Using the indefinite Minkowski metric to "square (2)", and observing that by construction $\vec{d_e}$ is space-like while $\vec{\Delta\tau_e}$ is time-like and that the "square" of a null vector $= 0$, we get

$$d_e = \Delta\tau_e/2 = (\tau_{e,R} - \tau_{e,A})/2 \tag{3}$$

Thus, one can determine d_e, the space-like distance between the event e off the world-line PQ and the event O_e on PQ, by measuring the advanced and retarded times $\tau_{e,R}$ and $\tau_{e,A}$.

A comment is important for our generalization: In 3D Minkowski space, we only need to consider the half-plane defined by PQ and e since one can generate 3D Minkowski space by rotating this half-plane about the line PQ. Put another way, the spatial direction orthogonal to the world-line can be interpreted as a radial direction. In their 2D diagrams, Salecker and Wigner correctly picture the entire 2D Minkowski plane, but, in calculating the distance between two events, they tacitly assume that both lie in the same half-plane, which need not always be the case. This observation is important because they assert [20,21] that, confining themselves to two dimensions, they are able[7] to avoid the problem of ascertaining the direction from which the light signals come when calculating the interval between two events. This is not quite correct, as can be seen by considering how one might distinguish between two events that have the *same* advanced and retarded times with respect to the world-line but lie in *opposite* half-planes. This is the remnant in two dimensions of the fact that, in the higher-dimensional cases, we cannot avoid the problem of ascertaining the angle between the directions of two light signals (rotating a half-plane through the angle π generates the other half-plane).

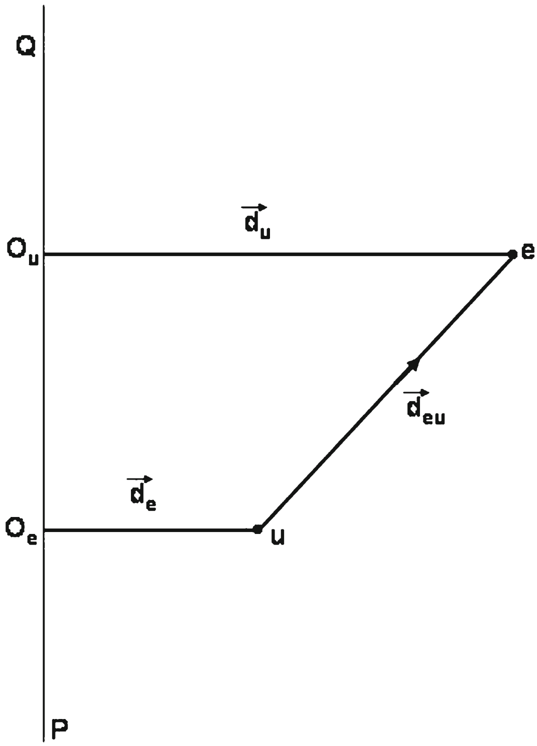
3 Arbitrary interval in 2D space–time

As we have seen, the Wigner–Salecker analysis is satisfactory, and even enlightening. However, it does have definite limitations, notably the restrictions to space-like intervals and 2D space–times. In this section, we remain in 2D Minkowski space–time but generalize the Wigner–Salecker analysis to arbitary space–time intervals defined by two events that may both lie off the time-like world-line PQ.

[6] We use units with $c = 1$.

[7] This also tacitly assumes the availability of a totally-reflecting mirror (i.e., with zero transmission coefficient).

Fig. 2 The space–time interval $\vec{d_{eu}}$ defined by the events e and u in terms of proper time measurements on the world-line PQ. $\vec{d_e}$ and $\vec{d_u}$ are the spatial intervals between the world-line PQ and e and u, respectively



Our measurement procedure is based on the calculation of the space–time interval defined by any two events off PQ in terms of proper time measurements on PQ Let e and u be the two events, d_{eu} the space–time interval between them, and $\vec{d_{eu}}$ the corresponding (2D) space–time vector. In the same way that O_e was defined with respect to e in Sect. 2, one can define an event O_u on the world-line with respect to u, and a vector $\vec{d_u}$ connecting O_u and u. It is then easy to see from Fig. 2 that

$$\vec{d}_{eu} = \vec{D}_{eu} + \vec{d}_u - \vec{d}_e, \tag{4}$$

where $\vec{D_{eu}}$ is the vector connecting the events O_e and O_u on the world-line PQ. Note that by construction $\vec{D_{eu}}$ is time-like and $\vec{d_u}$ and $\vec{d_e}$ are space-like, while $\vec{d_{eu}}$ can be any type of vector (e.g., time-like if $\vec{d_u} = \vec{d_e}$ or space-like if $\vec{D_{eu}} = 0$).

Using simple generalizations of the notation and arguments used in Sect. 2, one easily finds that $d_e = (\tau_{e,R} - \tau_{e,A})/2$, $d_u = (\tau_{u,R} - \tau_{u,A})/2$, and $D_{eu} = (\tau_{e,R} + \tau_{e,A} - \tau_{u,R} - \tau_{u,A})/2$. It is then straightforward to show that (adopting Minkowski signature $+, -, -, -$)

$$\begin{aligned}\vec{d_{eu}}\cdot\vec{d_{eu}} &= D_{eu}^2 - |d_u - d_e|^2 \\ &= \frac{1}{4}[(\tau_{e,R} + \tau_{e,A} - \tau_{u,R} - \tau_{u,A})^2 - (\tau_{u,R} - \tau_{u,A} - \tau_{e,R} + \tau_{e,A})^2] \\ &= \tau_{e,R}\tau_{e,A} - \tau_{e,A}\tau_{u,R} - \tau_{e,R}\tau_{u,A} + \tau_{u,R}\tau_{u,A},\end{aligned} \tag{5}$$

which indeed expresses $\vec{d_{eu}} \cdot \vec{d_{eu}}$ in terms of the (four) advanced and retarded times of the (two) events e and u.

As mentioned in Sect. 1, we postpone to a later paper a detailed analysis of the implications of this result for measurements at the quantum level. It is however reassuring to note that our generalization to arbitrary intervals does not require any qualitatively new type of measurement. All required information can still be obtained by recording some (advanced and retarded) readings of a clock placed close to the reference world-line. It should therefore be straightforward to generalize to arbitrary intervals the arguments previously advanced [8–10,22] in analyses of the implications at the quantum level of the measurement of space-like intervals originally considered by Wigner–Salecker.

4 Arbitrary interval in 3D space–time

As noted in Sect. 2, we can generate 3D Minkowski space–time from the 2D space–time of the previous two Sections by rotating the half-plane about the time-like world-line PQ through the angle 2π. For any given event e, one can still define the retarded and advanced null vectors $\vec{N_{e,R}}$ and $\vec{N_{e,A}}$ that, after the 2π rotation, lie respectively on the retarded and advanced null cones having their origin on PQ and including the point e. These retarded and advanced null cones intersect in a circle that lies in the space-like plane bisecting the interval vector $\vec{\Delta\tau_e}$; the center of the circle is at the bisecting point O_e of the world-line and e of course lies on this circle.

Before discussing the most general space–time interval defined by any two events e and u off PQ, let us first consider the special case, in which e and u are such that the points O_e and O_u coincide (see Fig. 3), i.e. both e and u lie in the same space-like plane intersecting the world line PQ in a single point $O_e = O_u$. In this case, the distance

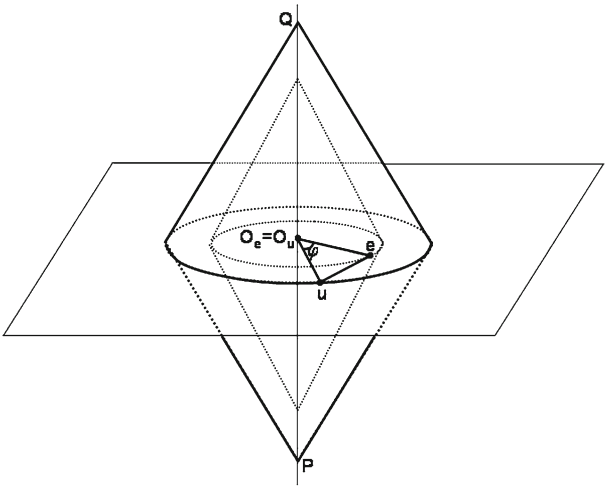


Fig. 3 The special case in which e and u are such that the points O_e and O_u coincide

$\vec{d_{eu}}$ is space-like and may be found from the space-like triangle with sides $\vec{d_{eu}}$, $\vec{d_e}$ and $\vec{d_u}$:

$$\vec{d_{eu}} = \vec{d_u} - \vec{d_e}. \tag{6}$$

"Squaring" this (which amounts to rederiving the law of cosines), we get:

$$\vec{d_{eu}} \cdot \vec{d_{eu}} = -|\vec{d_u} - \vec{d_e}|^2 = -|d_e|^2 - |d_u|^2 + 2|d_e||d_u| \cos(\phi), \tag{7}$$

where d_e and d_u are given respectively by $d_e = (\tau_{e,R} - \tau_{e,A})/2$ and $d_u = (\tau_{u,R} - \tau_{u,A})/2$, and ϕ is the angle (in the plane on which both lie) between $\vec{d_e}$ and $\vec{d_u}$; i.e., the angle between two radii of the circle, discussed above, defined by the intersection of retarded and advanced null cones.

It is not hard now to generalize formula (7) to the case of arbitrary e and u: In general O_e and O_u need not coincide (i.e., e and u do not generally lie in the same space-like plane). In this case, $\vec{d_{eu}}$ can be a vector of any type, but of course $\vec{d_e}$ and $\vec{d_u}$ are (by construction) still space-like. As in Sect. 3, it is also convenient here to introduce the time-like vector $\vec{D_{eu}}$ connecting the points O_e and O_u on the world-line PQ, allowing us to generalize Eqs. (4) and (6) for $\vec{d_{eu}}$:

$$\vec{d_{eu}} = \vec{D_{eu}} + \vec{d_u} - \vec{d_e}. \tag{8}$$

Squaring this, we obtain

$$\vec{d_{eu}} \cdot \vec{d_{eu}} = D_{eu}^2 - |\vec{d_u} - \vec{d_e}|^2 = D_{eu}^2 - |d_e|^2 - |d_u|^2 + 2|d_e||d_u| \cos(\phi), \tag{9}$$

where ϕ is still the angle[8] between $\vec{d_e}$ and $\vec{d_u}$.

Using (9), we can express $\vec{d_{eu}} \cdot \vec{d_{eu}}$ in terms of advanced and retarded times on the world-line PQ:[9]

$$\begin{aligned} \vec{d_{eu}} \cdot \vec{d_{eu}} &= \frac{1}{4}[(\tau_{e,R} + \tau_{e,A} - \tau_{u,R} - \tau_{u,A})^2 - (\tau_{u,R} - \tau_{u,A})^2 - (\tau_{e,R} - \tau_{e,A})^2 \\ &\quad - 2|\tau_{u,R} - \tau_{u,A}||\tau_{e,R} - \tau_{e,A}| \cos(\phi)] \\ &= \tau_{e,R}\tau_{e,A} + \tau_{u,R}\tau_{u,A} - \cos^2(\phi)[\tau_{e,A}\tau_{u,R} + \tau_{e,R}\tau_{u,A}] \\ &\quad - \sin^2(\phi)[\tau_{e,A}\tau_{u,A} + \tau_{e,R}\tau_{u,R}]. \end{aligned} \tag{10}$$

Anticipating the quantum measurability analysis, we observe that the most significant new element that has emerged in generalizing the Wigner–Salecker procedure to 3D space–times is the requirement that the angle ϕ also be measured. All other measurements still involve only (advanced and retarded) readings of a clock placed close to the reference world-line. Interestingly (if perhaps not surprisingly), the way

8 More precisely, taking into account that here $\vec{d_u}$ and $\vec{d_e}$ do not lie in the same plane, ϕ is the angle between $\vec{d_u}$ and the translation (which is actually a translation by $\vec{D_{eu}}$) of $\vec{d_e}$ to the plane, in which $\vec{d_u}$ lies.

9 We thank Andor Frenkel for pointing out to us the possibility of simplifying Eq. (10) to its final form.

to minimize the sensitivity of the interval measurement to a possible uncertainty in the angle ϕ is by means of an arrangement making $\phi \sim 0$ (or $\phi \sim \pi$), i.e. an arrangement in which the experimenter is on a world line such that $\vec{d_e}$ and $\vec{d_u}$ are parallel (or anti-parallel).

5 Generalization to 4D space–time

Having obtained the formulas for the 3D case, the generalization to 4D Minkowski space–time is now rather trivial. Instead of rotation through a circle, to get 4D Minkowski space–time, we need merely rotate the 2D Minkowski half-plane through a two-sphere of spherical angle 4π around the line PQ to get 4D Minkowski space–time. Instead of a circle, the advanced and retarded light cones with vertices on PQ and including the event e now intersect in a two-sphere lying in a space-like hyperplane (i.e., a three-space) orthogonal to PQ and centered on the midpoint O_e of the interval $\Delta\vec{\tau}_e$. The geometrical constructions then proceed just as in the 3D case, but now, instead of circles, e and u lie on spheres centered on the world-line. One can again introduce the space-like vectors $\vec{d_e}$ and $\vec{d_u}$ and the time-like vector $\vec{D_{eu}}$ in complete analogy with the 2D and 3D cases. In case both events e and u lie in the same space-like hypersurface orthogonal to PQ, the two spheres are concentric; and in order to use Eq. (7) in 4D, we merely need to interpret ϕ as the angle between two radii of the sphere.

Similarly, Eqs. (9) and (10) are still valid, and, just as it is in the 3D case, the distance $\sqrt{\vec{d_{eu}} \cdot \vec{d_{eu}}}$ is determined by the (four) advanced and retarded times of the (two) events with respect to the world-line and by the angle ϕ.

Should the need arise for generalization to higher dimensions, it is obvious how to proceed.

6 Beyond Wigner–Salecker

The analysis carried out in the previous Sections eliminates several of the limitations of the original Wigner–Salecker analysis: rather than being restricted to the context of space-like intervals in 2D space–time, the measurement procedure can now be applied to arbitrary space–time intervals in a space–time of Minkowski signature and arbitrary dimension, using an arbitrary[10] inertial time-like world-line.

In this Section, we shall comment on two other limitations of the original Wigner–Salecker analysis, already mentioned in the Introduction: the transfer of information about the microscopic system under observation to the macroscopic devices used to ultimately record the measurement result; and the postulation of a background Minkowski metric. We shall argue that a somewhat different measurement procedure results in an improvement with respect to each of these issues; namely the deployment

[10] The original Wigner–Salecker analysis assumed that one of the two events defining the interval was on the reference world-line.

of a spherically symmetric shell of matter, which entirely surrounds the region under study,[11] as the macroscopic recording apparatus.

In the original Wigner–Salecker setup, as shown in the previous sections, in order to measure intervals in space–times of dimension greater than 2, clock readings must be combined with some angle measurements. We show that, with this new setup, interval measurements in space–times of any number of dimensions can be done exclusively in terms of clock readings.

6.1 Why introduce the microscopic clock?

So far we have only alluded to some limitations relevant to a key aspect of the Wigner–Salecker analysis: the transfer of information about the microscopic system under observation to the macroscopic devices used to ultimately record the measurement result. This information is needed to complete the process[12] for the outcome of which we want to compute the probability. If the process may be treated classically, this probability for each outcome can be computed directly from the ensemble in phase space defined by the process. If the process must be treated quantum mechanically, a probability amplitude for each outcome of the process must be computed.

The existence of such a limitation of their analysis was recognized by Wigner and Salecker. Early in their discussion of the microscopic clock, they remark [21, p. 571]:

> *As is well known, and as was pointed out most clearly by von Neumann, the measurement is not completed until its result is recorded by some macroscopic object. If the macroscopic object were part of the clock, no microscopic clock could exist. The way out of this difficulty is to transmit the signal of the clock to a macroscopic recorder (which can be the 'final observer') which is far away from the clock, considered from the point of view of the average motion of the latter. The transmitting signal will be considered part of the [microscopic] clock, not, however, the recording apparatus.*

This is essentially the reason why they confined their analysis to a world of one spatial dimension [21, pp. 571–572]:

> *If the transmitting signal is to be microscopic, that is, if it is to consist of only a few quanta (actually, our signals will be light quanta), it will reach the recording equipment with certainty only if it does not spread out in every direction. In order to guarantee this, we confine ourselves to a world which has, in addition to the time-like dimension, only one space-like dimension.*

They do not further discuss the nature of the distant macroscopic recording device.

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[11] This actually implements Wigner's suggestion [20, p. 263]: *"In our experiments we surround the microscopic objects with a very macroscopic framework and observe coincidences between the particles emanating from the microscopic system, and parts of the framework."*

[12] We use Feynman's term "process" to describe what Bohr calls a "phenomenon": The preparation by some external apparatus of a quantum system, which then undergoes some interaction(s), the result of which is then "registered" by another external apparatus. The aim of any quantum-mechanical formalism is the computation of a probability amplitude for any such process (see [28]). For a discussion of the relation between classical and quantum ensembles, see [29].

Another, related limitation is not mentioned by Wigner and Salecker. They take as unproblematic the notion of inertial paths, and indeed of parallel inertial paths, in space–time (see [21, Fig. 3, p. 573], for example); as well as the notion of an inertial frame of reference (see [21, Fig. 1, p. 572], for example). The relation between these two limitations is that the specification of the macroscopic recording device can also serve to fix the inertial frame of reference (or at least specify its relation to the macroscopic system that serves this purpose), and hence the inertial paths.

The relation between the microscopic clock, including the light quanta that transmit the signal, and the macroscopic recording device serving to fix the frame of reference, is quite analogous to that described by Bohr when he considers two possible alternative one-slit diaphragm experiments. In one case, the diaphragm is rigidly attached to the apparatus defining the inertial reference frame; and in the other, it is attached to that frame by springs (see [30, pp. 697–698] and [31, pp. 218–221, esp. Figs. 4, p. 219 and Fig. 5, p. 220]). Until the relation of the diaphragm to the macroscopic apparatus defining the inertial frame of reference is fixed, one cannot speak of a definite phenomenon or process involving the diaphragm. As Bohr points out, a rigidly-fixed diaphragm can only be used for position measurements; while a diaphragm capable of motion can also be used for momentum measurement (for a fuller discussion, see [32,33]).

Applying Bohr's point of view to the case of the (microscopic) clock, until the macroscopic recorder has been introduced and its relation to the apparatus used to define an inertial frame of reference has been specified, one cannot meaningfully discuss its use to register the outcome of a process, nor the motion of the clock with respect to to this inertial frame. In the context of special relativity, no more need be added, since the points of space and the instants of time can be individuated relative to the inertial frame (assuming the Poincaré–Einstein convention to define the global time of the inertial frame) quite independently of the quantum physical processes under investigation. This is not the case in general relativity, the theory in which we are ultimately interested. But before turning to that case, let us return to the problem that led Wigner and Salecker to limit themselves to one spatial dimension: the angular spreading of the light signal. We can get around that problem by introducing a macroscopic recording apparatus that entirely surrounds the region under study—for example, a massive spherical shell of matter, within which lies the clock, as well as all the events to be investigated with this clock. Then, a record of the places on the interior wall of the shell[13] and the times at which the signals from the clock are received there, may be treated as final observer results in the Wigner–von Neumann terminology (see above) or just the results of the measurement terminating the process or phenomenon under investigation in the Bohr–Feynman terminology. As we shall show, these data can be used to calculate the clock times needed to define the space–time interval between two events.

[13] From now on, when referring to signals hitting the shell, we shall usually omit explicit mention of the interior wall of the shell.

6.2 Why neglect gravity?

Let us now turn to the concerns based on the fact that Wigner and Salecker [20,21] simply assume a reference background Minkowski space–time metric, completely neglecting gravity. As already suggested in the previous subsection, this assumption is particularly worrisome in light of the fact that they do not discuss the nature of the macroscopic devices used to ultimately record the measurement result: Once gravitation is taken into account, the masses of these macroscopic devices may well be non-negligible; and our proposal of introducing a massive spherical shell offers an obvious way to begin to address this problem. As is well known, inside such a shell[14] the Minkowski metric is a solution to the field equations of general relativity. So it appears that, to a certain extent, we can have our cake and eat it: We can carry over certain results of a special-relativistic analysis into a valid general-relativistic context.

But that is the case only to a certain extent. Due to the universality of gravity, the presence within the shell of a clock, light signals, and whatever other physical processes may be used to define the events, the interval between which we wish to measure, will destroy the spherical symmetry. We must therefore assume that the masses (or more correctly, the physical components of the stress-energy tensors) of all such entities are so small compared to the mass of the shell that, to a good first approximation, we may neglect their effect on the gravitational field inside the shell.[15] To the next approximation, one could use the linearized Einstein equations to take account of modifications of the gravitational field inside the shell.[16]

Perhaps an even more important modification in the general-relativistic case is the loss of a definition of spatial points and instants of time in the region within the shell that is independent of the physical processes going on within the shell. We shall give an example of this problem: assume the shell itself to be static in the sense of general relativity, i.e., there exists a time-like, hypersurface-orthogonal Killing vector field of the metric tensor associated with the shell; and that this Killing vector is also a symmetry of the shell's stress-energy tensor. The metric inside the shell is flat (i.e., its Riemann tensor vanishes), and therefore certainly static, but because of the diffeomorphism invariance of all solutions to the field equations of general relativity, there is no *unique* way to continue this Killing vector inside the shell, i.e. to uniquely associate a physical time variable with the points of the interior. This is just one example of the impossibility of physically identifying the the points of the interior without some further physical specification of these points. In short as a solution to the general-relativistic field equations, the hole argument applies just

[14] i.e. inside a spherical "hole" (i.e., a matter- and non-gravitational field-free region of space–time) within a spherically-symmetric distribution of matter, both hole and shell having the same center of symmetry.

[15] One feature of the general-relativistic version of the material shell model must be noted here. In order that such a static shell be possible, we must ensure that the actual radius of the shell is greater than the Schwarzschild radius associated with the mass of the shell (see the Introduction). That is, we must assure that $2GM/c^2R$ be less than unity. This means, of course, that if for any reason we are forced to increase the mass of the shell (for example, in order to treat objects inside the shell of mass of the order m as test bodies to a better and better approximation, we shall be forced to make the ratio m/M smaller and smaller), we must take care to increase the radius R if it starts to approach the Schwarzschild limit.

[16] For a careful quantum measurability analysis of the linearized gravitational field, see [34] (also see [19]).

as well to the Minkowski metric as to any other empty-space solution (see, e.g., [35,36]). But now the presence inside the shell of non-gravitational physical processes, previously regarded as presenting a difficulty because of their gravitational effects, now proves to be an advantage: Such processes enable us to define physically the spatial points and temporal instants associated with elements of these processes taking place inside the shell by relating them to events on the interior surface of the shell.

6.3 Back to a pure collection of clock readings

This possibility of relating events on the shell to physical events inside the shell provides a valuable alternative to a key result of our previous analysis: the need for an angle measurement in generalizing the Wigner–Salecker procedure to space–times of dimensions greater than 2. There may well be some advantages, especially in the quantum domain, to an alternative measurement procedure that requires only clock readings. The original (2D) Wigner–Salecker analysis was based on clock readings, and ever since the Wigner–Salecker-inspired quantum-gravity literature has focused on plausible quantum-gravity limitations on the accuracy of clock readings. Although a general consensus cannot yet be claimed [8–10,22,24] a certain maturity in the analysis of this difficult task has been reached, with the issues still in dispute clearly defined. In light of our results, the expertise developed in the analysis of quantum limitations on clock readings would provide only part of the tools needed in the case of a Wigner–Salecker-type measurement process for space–times of higher dimensions. While we certainly hope that our analysis will impel a similar discussion concerning quantum-gravity limitations on the accuracy of angle measurements, it is clearly valuable to devise measurement procedures in space–times with dimensions greater than 2 that require only clock readings. Our "spherical-shell setup" appears to be one natural candidate as we shall show by looking at some strategies that can be used to relate physical events inside the shell to events on the shell.

Although here we are mainly concerned with space–times of dimension greater than 2, let us look first at the situation in 2D space–time.[17] Here the sphere reduces to two parallel material strips, representing the histories of the two sides of this "one-sphere." In the inertial frame fixed by these strips, let the distance between the inner surfaces of the two sides of the "one-sphere" be 2R. Let an event e in the interior be marked by the emission of two light rays, one travelling towards each of the two sides. Assuming that clocks are uniformly distributed along each of the two strips, and that the times read by all of the clocks on the two strips have been synchronized (even in general relativity, a unique global time on the shell exists in the case of static metrics),

[17] In 2D space–time, one must simply specify the spatial distance between the two points, the histories of which constitute the interior edge of the two strips. In 3D space–time, (1) the interior radius of the circular shell is defined locally, but only by its extrinsic curvature; and (2) the vanishing of the Ricci tensor already guarantees that the interior is flat no matter what the shape of the shell. It is only in four-or higher-dimensional space–times that: (1) the radius of the interior wall of the spherical shell is defined locally by its intrinsic curvature, and (2) the Einstein equations both on and inside the spherical shell are needed to guarantee flatness of the interior region.

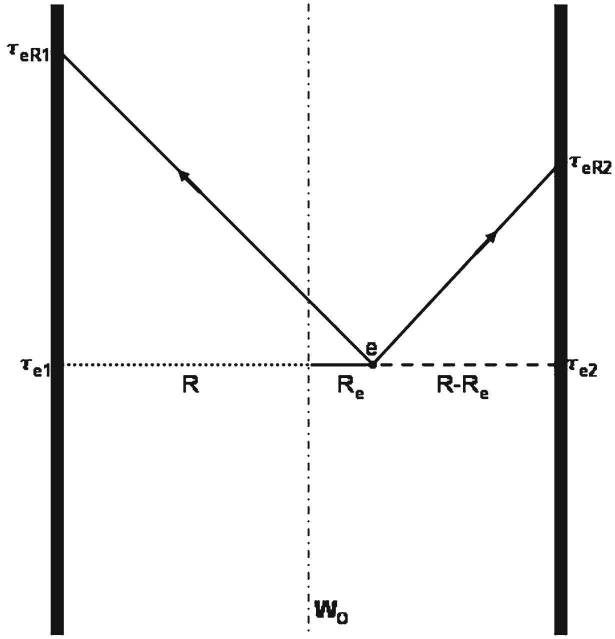

Fig. 4 In 2D space–time, a "spherical shell" reduces to two strips. We denote by R the "radius" (the half of the distance between the strips) and by R_e the position of the event e with respect to the origin of the position axis, which coincides with the "center" of the spherical shell. In the figure W_O is the world-line of this origin of the position axis

we easily see that the time and position of the event, τ_e and R_e respectively, are given by (see Fig. 4)

$$\tau_e = \frac{\tau_{eR1} + \tau_{eR2}}{2} - R, \quad R_e = \frac{\tau_{eR1} - \tau_{eR2}}{2}. \tag{11}$$

This procedure can be generalized to higher dimensions as follows. If we increase the number of dimensions by rotating one of the strips of matter (the other strip is not needed; cf. our discussion in Sect. 2 of the transition from two to three dimensions), then the strip of matter becomes a space–time "cylinder of matter", i.e. the history in time of a circular disk (or more generally a spherical annulus for three spatial dimensions, etc). Again, we shall assume that clocks are distributed uniformly over the surface of this cylinder at known positions and synchronized, enabling the recording of the arrival time at the position of each clock of any light signal from events inside the cylinder.

The light ray emitted from e now becomes a retarded light cone (of the appropriate number of dimensions). That is, we must now postulate that the event e includes the emission of a (retarded) light cone, which intersects the material cylinder in some curve. Let the times of arrival at the material cylinder of the light rays on this cone be noted by the clocks.

We shall now show that our previous 2D analysis can be applied to this situation. First of all, note that, if the event e does not lie on the central axis of the cylinder (the case in which it does is trivial: all the signals of the light cone will arrive at the same time), then the central axis and e define a time-like two-plane. If we are able to pick out the signals of the light cone that lie in this two-plane, then we have reduced the problem to the 2D one. But, as a moment's thought shows, of all the times recording the arrival of the light-cone signals, the two that lie in this plane will be the earliest and the latest recorded, respectively. Taking the 3D case, for example, and looking at the circular spatial cross-section of the cylinder at any time, one sees that points on the circle not at the ends of the diameter through the position of e are further away than the closer point of the diameter.

Thus one need merely determine the latest and the earliest times of arrival on the cylinder of signals from the event. Calling them τ_{eR1} and τ_{eR2}, respectively, we can again apply Eq. (11) to find the time and position of e. Of course, Eq. (11) only gives us the magnitude of R_e, but the positions of the clocks recording the earliest and the latest times already fix the diameter along which e lies.

There may be other and better ways of fixing the position and time of the event e, but at least we have demonstrated that one such procedure exists. It is also interesting to note that, since it uses the entire light cone, this method does not seem to require the measurement of an angle.

Having seen how to define the spatial point and temporal instant associated with an event inside the shell by relating them to events on the shell, we no longer need the Wigner–Salecker microscopic interior clock. As indicated above, to make a microscopic clock reading meaningful, we need to relate it to events on the shell, and we have shown how to do this directly for the events under study.

This might also have interesting implications for the problem of quantum limitations in the gravitational context, which is transformed into the problem of what limitations are imposed by the quantum of action on the ability to define the position and time of an event inside the shell in terms of measurements of positions and times by macroscopic clocks on the shell. And, as anticipated, no angle measurement appears to be needed in our proposed spherical-shell setup.

7 Summary and outlook

The analysis in Sects. 3–5 generalizes the Wigner–Salecker procedure in such a way that it is now applicable to the measurement of arbitrary intervals in space–times of arbitrary dimensions. We have postponed the delicate analysis of quantum limits to the measurability of such intervals, but some of the results presented here are already relevant to the ongoing debate on "Planck-scale uncertainty principles". Several studies [8–10,22] have proposed such uncertainty principles on the basis of the 2D Wigner–Salecker procedure for the measurement of space-like intervals. The key aspect of those studies is the role played by (advanced and retarded) proper time measurements. Our generalization to the measurement of arbitrary intervals renders straightforward a corresponding generalization of the analyses reported in [8–10,22] to the case of non-space-like intervals.

Extending this procedure to more than two space–time dimensions, the measurement of advanced and retarded times is still a key ingredient; but now an angle measurement is also needed. This result suggests a clear priority for generalization of Salecker–Wigner-based quantum-gravity measurability analyses to the case of higher space–time dimensions. The search for possible quantum (-gravity) limitations on the accuracy of this angular measurement would clearly be a key issue in such a generalization.[18]

In Sect. 6, we propose a modification of the Wigner–Salecker procedure using a "spherical shell setup", and emphasize its utility for the discussion of a fully general-relativistic setup that nevertheless involves measurement processes taking place in a flat region of space–time. While we have only carried out a classical analysis of this setup, we hope that a quantum-mechanical analysis will provide a more consistent and theoretically comprehensive derivation of measurability limits relevant to quantum-gravity research than is currently available. This will of course require a detailed and, in large part, novel quantum measurability analysis: Our proposed measurement procedure is "economical" in the sense that it only requires time measurements (no angle measurements even in space–times of dimensions greater than 2), so that it will probably be possible to follow in part the strategies of analysis that have emerged from the literature devoted to such measurements performed by a Wigner–Salecker *microscopic* clock. But we expect some new elements to arise from the requirement that the shell be *macroscopic* and serve to define a static frame of reference; and that the system of clocks on the shell be *macroscopic*, have a fixed position and be capable of recording the time of arrival of signals.

Another important point to be considered for future work is the problem of developing a measurement procedure for the physical components of the space–time curvature, and showing how the existence of the quantum of action leads to a "Planck-scale indeterminacy principle" for the curvature components. This issue will arise in our problem in the next, linear approximation when the gravitational effects of the contents of the shell are taken into account, and its analysis should rely in part on the techniques developed in [19,34]. As argued in [20], this should be possible with a suitable modification of the Wigner–Salecker procedure, even though in its original form, and in our generalization, the procedure assumes the absence of curvature in the space–time region of interest for the measurement procedure.

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References

1. Veneziano, G.: Europhys. Lett. **2**, 199 (1986)
2. Gross, D.J., Mende, P.F.: Nucl. Phys. **B303**, 407 (1988)

[18] Our classical angular measurement analysis can provide the basis for a corresponding quantum-mechanical analysis, but since the studies reported in [8–10,22] do not advocate exactly the same perspective on quantum-measurability analysis of the Wigner–Salecker procedure already in the 2D case, presumably different authors might treat differently the angular analysis at the quantum level.

3. Amati, D., Ciafaloni, M., Veneziano, G.: Phys. Lett. **B216**, 41 (1989)
4. Mead, C.A.: Phys. Rev. **135**, B849 (1964)
5. Padmanabhan, T.: Class. Quantum Grav. **4**, L107 (1987)
6. Doplicher, S., Fredenhagen, K., Roberts, J.E.: Phys. Lett. **B331**, 39 (1994)
7. Ahluwalia, D.V.: Phys. Lett. **B339**, 301 (1994)
8. Ng, Y.J., Van Dam, H.: Mod. Phys. Lett. **A9**, 335 (1994)
9. Amelino-Camelia, G.: Mod. Phys. Lett. **A9**, 3415 (1994) gr-qc/9603014
10. Amelino-Camelia, G.: Mod. Phys. Lett. **A11**, 1411 (1996) gr-qc/9603013
11. Garay, L.J.: Int. J. Mod. Phys. **A10**, 145 (1995)
12. Dowker, F.: New Sci. **180**, 36 (2003)
13. Bronstein, M.P.: Quantentheorie schwacher Gravitationsfelder. Physikalische Zeitschrift der Sowjetunion **9**, 140 (1936)
14. Thiemann, T.: Introduction to modern canonical quantum general relativity. gr-qc/0110034
15. Thiemann, T.: Lectures on loop quantum gravity. gr-qc/0210094
16. Heisenberg, W.: The Physical Principles of the Quantum Theory. University of Chicago Press, Chicago (1930). (cited from the Dover Publications reprint)
17. Bohr, N., Rosenfeld, L.: Zur frage der Messbarkeit der elektromagnetischen Feldgrössen. Kgl. Danske Videnskab S. Nat. Fys. Medd. **12**, 1 (1933) [English translation in selected papers of Leon Rosenfeld, pp. 357–400 (Reidel 1979)]
18. Landau, L.D., Peierls, R.: Erweiterung des Unbestimmtheitsprinzips fur die relativistische Quantentheorie. Zeitschrift Fur Physik **69**, 56 (1931)
19. DeWitt, B.S.: The quantization of geometry. In: Witten, L. (ed.) Gravitation: an Introduction to Current Research, pp. 266–381. Wiley, London (1962)
20. Wigner, E.P.: Rev. Mod. Phys. **29**, 255 (1957)
21. Salecker, H., Wigner, E.P.: Phys. Rev. **109**, 571 (1958)
22. Adler, R.J., Nemenman, I.M., Overduin, J.M., Santiago, D.I.: Phys. Lett. **B477**, 424 (2000)
23. Amelino-Camelia, G.: Nature **398**, 216 (1999) gr-qc/9808029
24. Calmet, X.: Eur. Phys. J. **C54**, 501 (2008) hep-th/0701073
25. Lloyd, S., Ng, Y.J.: Sci. Am. **291**(5), 52 (2004)
26. Christiansen, W.A., Ng, Y.J., van Dam, H.: Phys. Rev. Lett. **96**, 051301 (2006) gr-qc/0508121
27. Stachel, J.: Special relativity from measuring rods. In: Physics Philosophy and Psychoanalysis Essays in Honor of Adolf Gruenbaum, pp. 255–272 (Reidel 1979)
28. Stachel, J.: Feynman paths and quantum entanglement: is there any more to the story? In: Cohen, R.S., Horne, M., Stachel, J. (eds.) Potentiality Entanglement and Passion-at-a-distance/Quantum Mechanical Studies for Abner Shimony, vol. 2, pp. 244–256. Kluwer, Dordrecht (1997)
29. Stachel, J.: Quantum logic. In: Sarkar, S., Pfeiffer, J. (eds.) The Philosophy of Science: an Encyclopedia, vol. 2, pp. 633–644. Routledge, London (2005)
30. Bohr, N.: Can quantum-mechanical description of physical reality be considered complete. Phys. Rev. **48**, 696 (1935)
31. Bohr, N.: Discussion with einstein on epistemological problems in atomic physics. In: Schilpp, P.A. (ed.) Albert Einstein: Philosopher-Scientist, pp. 199–241, Cambridge University Press, London (1949)
32. Bai, T.: Philosophy and physics: action-at-a-distance and locality, doctoral dissertation. Boston University, Boston (2003)
33. Bai, T., Stachel, J.: Bohr's diaphragms (preprint)
34. Bergmann, P.G., Smith, G.J.: Gen. Rel. Grav. **4**, 1131 (1982)
35. Stachel, J.: Prolegomena to any future quantum gravity. In: Oriti, D. (ed.) Approaches to Quantum Gravity- Toward a New Understanding of Space and Time. Cambridge University Press, London (in press) gr-qc/0609108
36. Stachel, J.: A brief history of space–time. In: Ciufolini, I., Dominici, D., Lusanna, L. (eds.) A Relativistic Space–time Odyssey, pp. 15–34 (World Scientific 2003)