Conserving Color Charge

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Noether's Theorem

Symmetry $\longleftrightarrow$ Conserved Quantity

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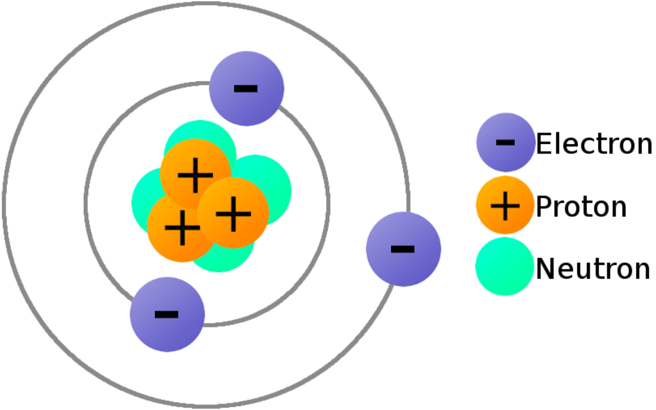
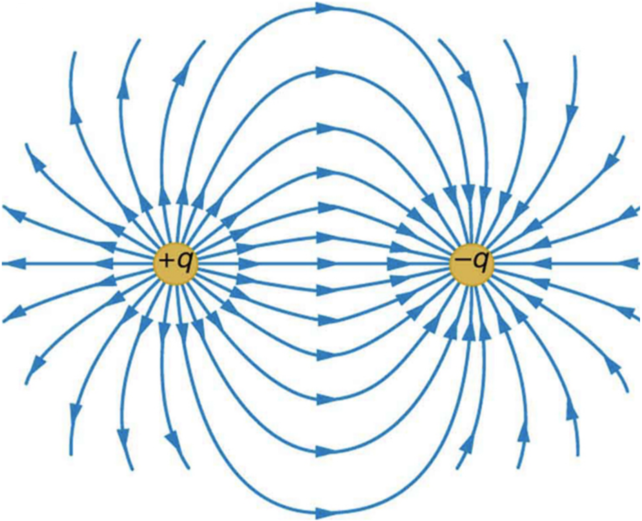
Symmetry $\longleftrightarrow$ Conserved Quantity

Gauge Symmetry	**Charge**
$U(1)$	Electric charge
$SU(3)$	Color charge

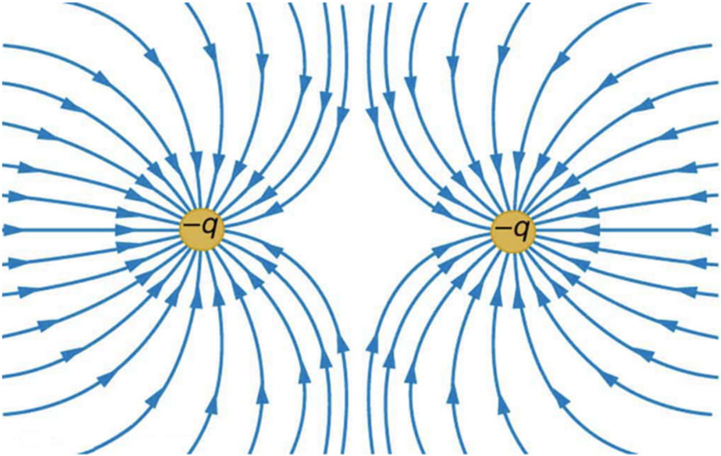
Electric Charge



Electric Charge



Electric Charge



Noether's Theorem

Electric Charge

Charge-current density $j^\mu = j^0 + \vec{j}$.

Total charge $Q = \int j^0 d^3x$.

Noether's Theorem
Electric Charge

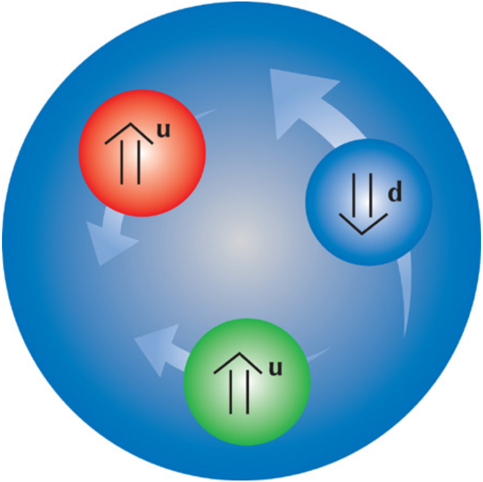
Charge-current density $j^\mu = j^0 + \vec{j}$.

Total charge $Q = \int j^0 d^3x$.

Noether Theorem: $\partial_\mu j^\mu = 0$.

$$\frac{d}{dt}Q = 0.$$

Color Charge



Noether's Theorem

Color Charge

Current for color charge?

Noether's Theorem

Color Charge

Current for color charge?

conservation of red

conservation of blue

conservation of green

Currents

- Electric Charge $\partial_\mu j^\mu = 0$
- Color Charge $\partial_\mu j^{a\mu} = 0$

Currents

- Electric Charge $\partial_\mu j^\mu = 0$
- Color Charge $\partial_\mu j^{a\mu} = 0$

Question: How does $J^{a\mu}$ relate to red, blue, and green?

Claims

- Noether's theorem does *not* show conservation of *red*, *blue*, and *green*.

Claims

- Noether's theorem does *not* show conservation of *red*, *blue*, and *green*.
- Noether's theorem *does* show conservation of a relational quantity of combinations of color and anti-color.

Outline

1. Scalar Electrodynamics

2. Scalar Chromodynamics

3. Group Representations

4. What is Charge?

Outline

Scalar Electrodynamics

charged scalar field

$$L_E = D_\mu \phi (D^\mu \phi)^* - m \phi \phi^*$$

Scalar Electrodynamics

charged scalar field

$$L_E = D_\mu \phi (D^\mu \phi)^* - m\phi\phi^*$$

- ϕ is a matter field
- $D_\mu = \partial_\mu + iqA_\mu$

Scalar Electrodynamics

Noether's Theorem

$$L_E = D_\mu \phi (D^\mu \phi)^* - m\phi\phi^*$$

$$j^\mu = iq(\phi^* D^\mu \phi - \phi (D^\mu \phi)^*)$$

Scalar Electrodynamics

Noether's Theorem

$$j^\mu = iq(\phi^* D^\mu \phi - \phi (D^\mu \phi)^*)$$

$$Q = \int d^3 x j^0.$$

Scalar Electrodynamics

Noether's Theorem

$$j^\mu = iq(\phi^* D^\mu \phi - \phi(D^\mu \phi)^*)$$

$$Q = \int d^3x j^0.$$

Q gives total amount of electric charge.

Scalar Electrodynamics

Interactions

$$L_E = D_\mu \phi (D^\mu \phi)^* - m\phi\phi^*$$

Scalar Electrodynamics

Interactions

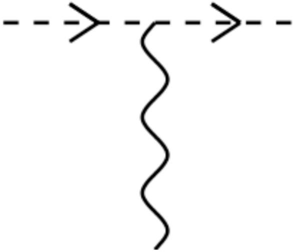
$$L_E = D_\mu \phi (D^\mu \phi)^* - m\phi\phi^*$$

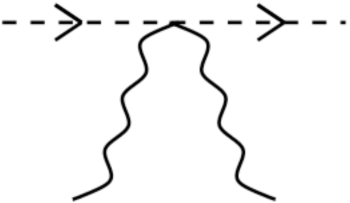
$$D_\mu \phi (D^\mu \phi)^* = -iqA^\mu(\phi^* \partial_\mu \phi + \phi \partial^\mu \phi^*) - q^2 A_\mu \phi A^\mu \phi^* + \partial_\mu \phi \partial^\mu \phi^* .$$

Scalar Electrodynamics

Interactions

$$iqA^{\mu}(\phi^* \partial_{\mu}\phi + \phi\partial^{\mu}\phi^*) \qquad - q^2 A_{\mu}\phi A^{\mu}\phi^*$$





Outline

Color Charge

charged scalar field

$$L_C = D_\mu \phi_i (D^\mu \phi_i)^* - m\phi_i \phi_i^*$$

- ϕ_i matter field
  - i is for *red*, *blue*, or *green*
  - first fundamental representation of $SU(3)$

Color Charge

charged scalar field

$$L_C = D_\mu \phi_i (D^\mu \phi_i)^* - m\phi_i \phi_i^*$$

- ϕ_i matter field
  - i is for *red*, *blue*, or *green*
  - first fundamental representation of $SU(3)$
- $D_\mu = \partial_\mu + igA_\mu^a$
  - a is for Lie algebra
  - adjoint representation of $SU(3)$

Color Charge

charged scalar field

$$L_C = D_\mu \phi_i (D^\mu \phi_i)^* - m \phi_i \phi_i^*$$

$$\begin{aligned} J^{\mu a} &= ig(\phi_i^* D^\mu \phi_i - \phi_i (D^\mu \phi_i)^*) \\ &= ig(\phi_i^* (\partial_\mu + igA_\mu^a)\phi_i - \phi_i(\partial_\mu - igA_\mu^a)\phi_i^*) \end{aligned}$$

Group Representations

- Group representations hardwired into the expressions for the current
- Current transforms according to the adjoint representation

Outline

Groups

Representations

- Abstractly: $(G, \cdot)$.
- Lie groups have lie algebras.

Groups

Representations

- Abstractly: $(G, \cdot)$.
- Lie groups have lie algebras.
- Concretely: representations.
- A representation $\rho : G \to GL(V)$.
- The dimension of ρ is the dimension of V.

Representations of $SU(N)$

For $SU(N)$, there are $N - 1$ many *fundamental* representations.

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For $SU(N)$, there are $N - 1$ many *fundamental* representations.

For $SU(3)$ two fundamental representations: color and anti-color

Representations of SU(3)

The Fundamental Reps

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

These define a basis for $\mathbb{C}^3$.

Representations of SU(3)

A basis for the Lie algebra:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \boldsymbol{\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}$$

Cartan Subalgebra

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Distinguish representations of $SU(3)$ by the subalgebra's eigenvalues, called **weights**.

Representations of SU(3)

The Fundamental Reps

$$\lambda^3(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda^8(b) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Representations of SU(3)

The Fundamental Reps

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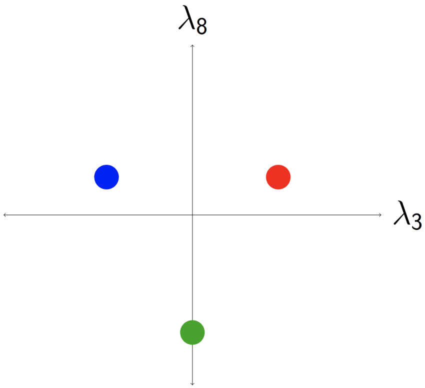
$$\lambda^8(b) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We say that the ***weight*** of b is $(-1, \frac{1}{\sqrt{3}})$.

Representations of SU(3)

The Fundamental Reps



Representations of SU(3)

The Fundamental Reps

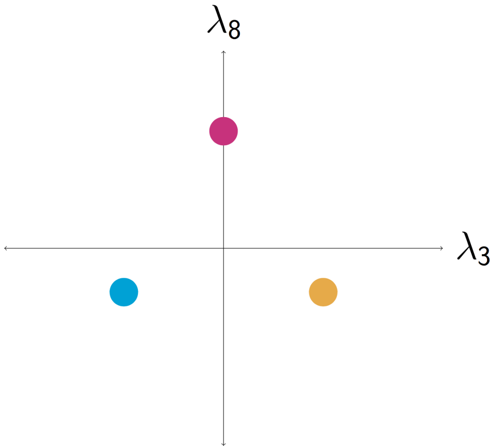
For anti-colors we set:

$$\bar{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \bar{b} = -\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The representation on this space is $\bar{\rho}(g) = -\rho(g)^{tr}$

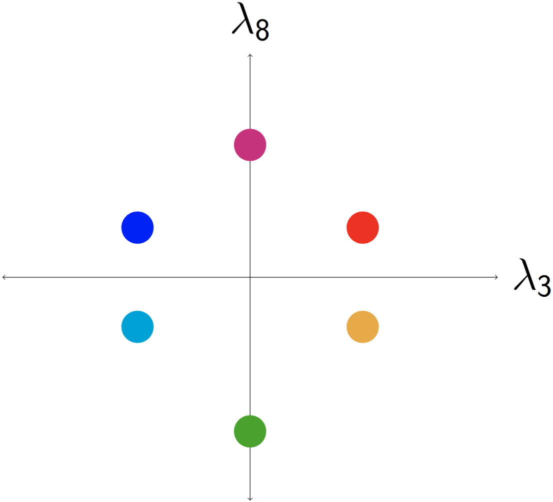
Representations of SU(3)

The Fundamental Reps



Representations of SU(3)

The Fundamental Reps



We distinguish between inequivalent representations using this ***weight space*.**

Representations of SU(3)

$$L_C = D_\mu \phi_i (D^\mu \phi_i)^* - m\phi_i \phi_i^*$$

- i = r, b, g **fundamental representation**
- a = 1, 2, . . . 8 **adjoint representation**

Representations of SU(3)

- A representation $\rho : G \longrightarrow GL(V)$
- We call V the "carrier space."

The adjoint representation: V is the ***Lie algebra*.**

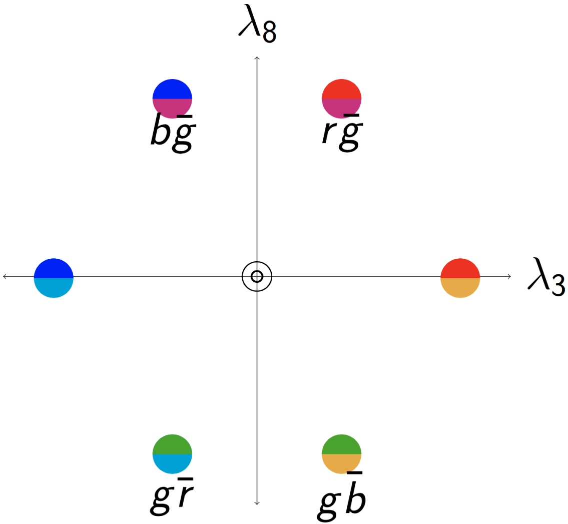
The action ρ_{adj} is conjugation: $\rho_{adj}(g)(v) = gvg^{-1}$.

Adjoint Representation

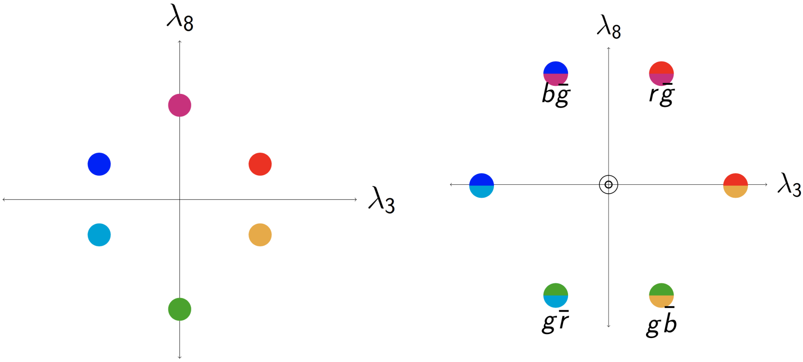
SU(3)

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}, \cdots \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}.$$

The Adjoint Representation



Representations of SU(3)



The Adjoint Representation

- The weights correspond to basis vectors of V
- The carrier space V is the Lie algebra
- So: any element of Lie algebra can be written in this basis

Interpretation

Lie algebra valued quantities are combinations of color and anti-color.

Color Charge

charged scalar field

$$L_C = D_\mu \phi_i (D^\mu \phi_i)^* - m\phi_i \phi_i^*$$

$$\begin{aligned} J^{\mu a} &= ig(\phi_i^* D^\mu \phi_i - \phi_i (D^\mu \phi_i)^*) \\ &= ig(\phi_i^* (\partial_\mu + igA_\mu^a)\phi_i - \phi_i(\partial_\mu - igA_\mu^a)\phi_i^*) \end{aligned}$$

Noether's Theorem

Color Charge

Given $SU(3)$ gauge symmetry, Noether's thoerem give us

$$\partial_\mu J^{\mu a} = 0.$$

The conserved quantity is a ***combination*** of color and anti-color.

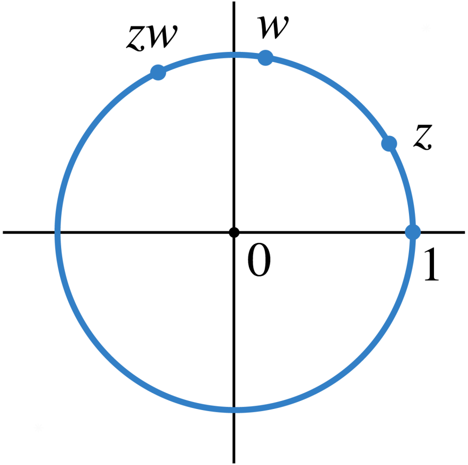
Outline

Color Charge and Electric Charge

$$L_E = D_\mu \phi (D^\mu \phi)^* - m\phi\phi^*$$

$$L_C = D_\mu \phi_i (D^\mu \phi_i)^* - m\phi_i\phi_i^*$$

$U(1)$



Representations of $U(1)$

$$\rho_n(e^{i\theta}) = e^{in\theta}.$$

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- Each ρ_n has $\mathbb{C}$ as its carrier space.
- The adjoint representation is ρ_0.

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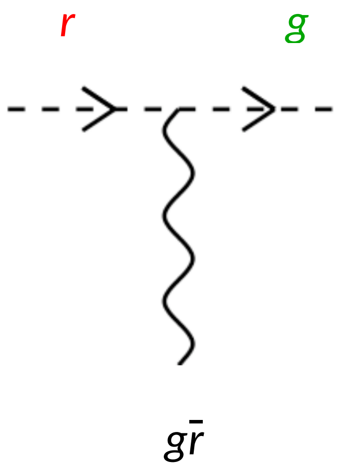
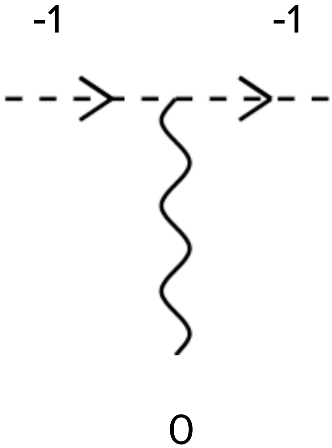
$$\rho_n(e^{i\theta}) = e^{in\theta}.$$

- Each ρ_n has $\mathbb{C}$ as its carrier space.
- The adjoint representation is ρ_0.
- $\rho_0 = \rho_{-1+1}$.
- The Lie algebra is $\mathbb{R}$.

Charge exchange



Electric charge



Conclusions

- Red, blue, and green are not individually conserved.
- The conserved quantity is a combination of charge and anti-charge.
- Conservation of charge is relational.

Thank you!

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$SU(3)$ Adjoint

Three pairs of raising and lowering operators:

$$E_{\pm 1,0} = (1/2)(\lambda_1 \pm i\lambda_2) \qquad \text{red states} \rightleftarrows \text{blue states.}$$

$$E_{\pm 1/2,\pm\sqrt{3}/2} = (1/2)(\lambda_4 \pm i\lambda_5) \qquad \text{red states} \rightleftarrows \text{green states.}$$

$$E_{\mp 1/2,\pm\sqrt{3}/2} = (1/2)(\lambda_6 \pm i\lambda_7) \qquad \text{blue states} \rightleftarrows \text{green states.}$$

$SU(3)$ Adjoint

$$\begin{array}{c}
r\bar{b} \\
\swarrow \qquad \searrow \\
r\bar{g} \qquad\qquad g\bar{r} \\
\swarrow \;\; \downarrow \qquad\qquad \downarrow \;\; \searrow \\
0 \quad g\bar{g} - r\bar{r} \qquad b\bar{b} + g\bar{g} \quad 0 \\
\swarrow \;\; \downarrow \qquad\qquad \downarrow \;\; \searrow \\
b\bar{g} \quad -2g\bar{r} \qquad 2b\bar{g} \quad -g\bar{r} \\
\swarrow \;\; \downarrow \qquad \downarrow \;\; \searrow \quad \swarrow \;\; \downarrow \qquad \downarrow \;\; \searrow \\
0 \quad -b\bar{r} \quad -b\bar{r} \quad 0 \quad -b\bar{r} \quad -b\bar{r} \quad 0
\end{array}$$

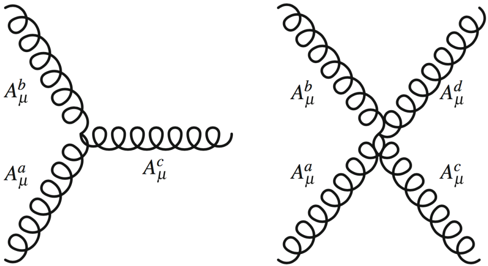
Yang-Mills Theory

$$L_{YM} = -\frac{1}{4}F^{a\mu\nu}F_{a\mu\nu}$$

$$J^{a\mu} = \frac{1}{g^2}D_\mu F^{a\mu\nu}$$

Charged Gauge Field

$$L_C = D_\mu \phi_\mathbf{i} (D^\mu \phi_i)^* - m\phi_i \phi_i^* - \frac{1}{4} F^{a\mu\nu} F_{a\mu\nu}$$



Covariant Derivative