A Theory of Strategic Voting in Runoff Elections*

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Abstract

This paper analyzes the properties of runoff electoral systems when voters are strategic. A model of three-candidate runoff elections is presented, and two new features are included: voters participating in the two rounds may be different, and many types of runoff systems are considered (e.g. any threshold for first-round victory). Three main results emerge. First, runoff elections produce equilibria in which only two candidates receive a positive fraction of the votes. Second, the sincere voting equilibrium does not always exist. Finally, runoff systems with a threshold below 50% produce an Ortega effect that may lead to the systematic victory of the Condorcet loser.

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1 Introduction

In a runoff election, the candidate with the greatest number of votes wins outright in the first round if she obtains more than a predefined fraction of the votes (called the *threshold for first-round victory*). If no candidate wins in the first round, then a second round is held between the two candidates with the most first-round votes. The winner of that round wins the election.

The runoff electoral system is the single most used electoral system for presidential elections: 61 out of 91 countries that elect a president directly have a runoff provision (Blais et al. 1997), France being a notorious example. Moreover, its popularity has continued to rise over the past decades: about 70% of the presidential elections held in the 90s' were runoff elections, compared to only 30% in the 60s (Golder 2005). The widespread use of the runoff system is also striking in the U.S.: runoff primaries are a trademark in southern states, and most large American cities have a runoff provision (Bullock III and Johnson 1992, Engstrom and Engstrom 2008).[1]

The perceived rationale for the worldwide use of runoff systems is twofold: first, runoff elections are expected to be more conducive to preference and information revelation than plurality elections, and second, they are intented to prevent the victory of minority candidates.[2] Nonetheless, despite their relative ubiquity, our understanding of their properties and of voters' behavior is limited and mostly informal. The few formal models of runoff elections left important features aside (Cox 1997 and Martinelli 2002). In this paper, I propose a new model of three-candidate runoff elections which challenges the conventional wisdom in several important ways.

My model includes two new features. First, the voters participating in the two rounds are not necessarily the same (for empirical evidence, see Wright 1989, Bullock III and Johnson 1992, Morton and Rietz 2006). I show that this leads voters to believe that all candidates participating in the second round have a positive probability of winning.

[1] Actually, the U.S. presidential electoral system is a runoff system: if no candidate receives a majority of the electoral votes, the House of Representatives chooses the President among the top three contenders.

[2] The latter rationale can be found in the literature in a slightly different form. In particular, it is often argued that runoff electoral systems should be used because they guarantee that the elected president has a mandate of a large part of the population. This is intented to legitimate her position once elected.

This contrast with previous models which assumed no risk of upset victory in the second round. Second, the model allows for the analysis of many different types of runoff systems: any threshold for first round victory between 0% and 100% as well as more sophisticated rules (e.g. moving thresholds and victory margin requirements). In practice, thresholds below 50% are typical. For instance, the threshold is 40% in Costa Rica, North Carolina State and New York City. Some countries use sophisticated thresholds: in Argentina, for example, a candidate wins outright if she gains 45% of the votes or if she gains 40% of the votes as well as 10% more than the runner-up.[3]

Three main results emerge. First, runoff elections produce multiple *Duverger's Law equilibria* – that is, equilibria in which only two candidates receive votes in the first round. In those, some voters abandon their most preferred candidate and vote to ensure the outright victory of a strong candidate. They do so to avoid the risk of an upset victory of a less-preferred candidate in the second round. In at least one Duverger's Law equilibrium, the Condorcet winner[4] does not receive any votes.

The second main result is that the sincere voting equilibrium – that is, the equilibrium in which all voters vote for their most preferred candidate in the first round – does not always exist. The reason is the same as the one explaining the existence of Duverger's Law equilibria. Together these two first results prove that the Duverger's Law equilibria may be the only (pure strategy) equilibria in runoff elections.

The third main result is that runoff elections with a threshold below 50% may produce an equilibrium in which all voters cast a ballot on their most preferred candidate, thus allowing the outright victory of the Condorcet loser in the first round.[5] The Condorcet loser could instead be beaten if a sufficiently large coalition of voters deviate and coordinate their votes on one of the trailing candidates. This excessive vote dispersion happens because, conditional on being pivotal, voters over-estimate the likelihood that a second round will be held. They thus vote for their preferred candidate to qualify her for the second round. I call this the *Ortega effect*, after Daniel Ortega, the winner of the

[3] Threshold above 50% are more seldom: the only case I am aware of is the 1996 presidential election in Sierra Leone for which the threshold was defined at 55%.

[4] The Condorcet winner is a candidate that would win a one to one contest against any other candidate.

[5] The Condorcet loser is a candidate that would lose a one to one contest against any other candidate.

2006 Presidential election in Nicaragua. In contrast, the Ortega effect does not exist in plurality elections.

Together, these three results form a twofold contradiction of the conventional wisdom regarding runoff systems. First, it is commonly believed that runoff elections are more conducive to preferences and information revelation than plurality (Duverger 1954, Riker 1982, Cox 1997, Piketty 2000, Martinelli 2002). As stated by Duverger in hiswell-known Law and Hypothesis, respectively: "*the simple-majority single-ballot system* [the plurality electoral system] *favors the two-party systems*" whereas the "*simple majority with a second ballot* [the runoff electoral system] *favors multipartyism*". The rationale is that voters' incentives to abandon their most preferred candidate to instead rally behind a serious candidate are more powerful in plurality than in runoff elections. Indeed, there are two serious candidates in plurality elections, i.e. those who have a serious chance to tie for victory, whereas it is believed that there are three candidates who are serious competitors for the second round. The existence of Duverger's Law equilibria in runoff elections contradicts this belief and, therefore, Duverger's Hypothesis.

Second, the choice of a threshold level has traditionally been based on a perceived trade-off between costs of organization and the risk of a minority candidate victory.[6] This trade-off arises mechanically when voters are not strategic: higher thresholds reduce the risk of a minority candidate winning but increase the expected costs of organization since the likelihood of an outright victory in the first round is lower. For lower values of the threshold, the converse holds. On the basis of this perceived trade-off, it has been argued that lower-than-50% thresholds are desirable: they represent a compromise between plurality and runoff with a threshold at 50%. Indeed, they allow for better revelation of preferences but prevent "useless" second rounds in which the candidate who ranks first in the first round eventually wins the election in the second round (Shugart and Taagepera 1994, O'Neil 2007).[7] This perceived trade-off does not exist when strategic voters are taken into account. Indeed, the Ortega effect that I identify for runoff elections with a

[6] Many governments have indeed adopted runoff provisions in response to such a victory (see e.g. Bullock and Johnson 1992 and O'Neil 2007).

[7] This happens regularly. For instance, Bullock and Johnson (1992) report empirical evidence on U.S. data according to which the election winner corresponds to the first-round winner approximately 70% of the times.

threshold below 50% demonstrates that intermediate values of the threshold may actually increase the risk of a minority candidate victory (with respect to plurality elections).

My model of three-candidate runoff elections allows voters to have any possible preference ordering over candidates (i.e. there are up to twelve types of voters). However, for the sake of expositional clarity, I present the main results using a stylized version of the model in which there are three types of voters and a divided majority facing a unified minority. All majority voters prefer either candidate A or candidate B to a third candidate, C, but they are divided as to whether A or B is preferable. The minority is instead unified behind candidate C.[8] I then extend the main results to the general model with more types of voters.

My findings are empirically relevant. First, my result that runoff elections produce multiple equilibria featuring different number of candidates receiving a positive fraction of the votes sheds new light on the mixed empirical evidence vis-à-vis Duverger's Law and Hypothesis. There is evidence supporting Duverger's argument (Wright and Riker 1989, Golder 2006, Clark and Golder 2006, Goncalves et al. 2008, Fujiwara 2009), but the numerous counterexamples have remained puzzling (e.g. Shugart and Taagepera 1994).[9] Moreover, there is systematic evidence that does not support Duverger's argument: Cox (1997) finds no statistically significant effect of the runoff system on the "effective number of presidential candidates" in 16 democracies in the 80s. Similarly, Engstrom and Engstrom (2008) find that the mean effective number of candidates for gubernatorial and senatorial U.S. primary elections (between 1980 and 2002) with runoff provision is notably below three and not significantly different from the mean effective number of candidates with plurality rule. The existence of Duverger's Law equilibria in runoff elections helps

[8] This particular class of situations – that is, the problem of the "divided majority" – merits study in and of itself, as it captures the essence of coordination problems in multicandidate elections. This issue is often considered in the literature on electoral systems. It is, for instance, at the heart of Borda (1781)'s demonstration that plurality may fail to aggregate preferences. See also Myerson and Weber (1993), Piketty (2000), Myerson (2002), Martinelli (2002), Dewan and Myatt (2007), and Myatt (2007).

[9] Shugart and Taagepera (1994) find many instances in which the effective number of candidates in presidential runoff elections is less than three (Chile 1989, Portugal 1976-1986 and Costa Rica 1953-1986).

to make sense of these otherwise puzzling outcomes.[10,11]

Second, the puzzling result of the 2006 presidential election in Nicaragua (see Lean 2006) may be reinterpreted in light of the Ortega effect. In this election, right-wing voters formed a majority of the electorate but were divided between two candidates: Eduardo Montealegre (Alianza Liberal Nicaraguense) and José Rizo (Partido Liberal Constitucionalista). There was only a minority of left-wing voters, but they were staunchly supporting the sole serious contender: Daniel Ortega (Frente Sandinista de Liberacion Nacional). In other words, Ortega was the Condorcet loser of this election. Nicaragua's presidential electoral system is a runoff where a candidate wins outright if she obtains more than 40% of the votes or more than 35% of the vote and a victory margin over the nearest competitor of 5%. Before the election, polls indicated that, due to a division among right-wing voters, Ortega would win outright. Despite this information, right-wing voters persisted in dividing their votes and Ortega won the presidential race with 38% (Montealegre and Rizo obtained 28.3% and 27.1%, respectively). According to traditional models, this result was due to a non-rational reaction of right-wing voters who should not have divided their votes. Yet, the Ortega effect outlined in this paper instead demonstrates how it was individually rational for right-wing voters to behave in this way.

The rest of the paper is organized as follows: Section 2 lays out the setup. Section 3 details how voters decide for whom to vote. Section 4 analyzes equilibrium behavior in runoff elections. Section 5 discusses the assumption about how the two groups of voters differ. Section 6 extends the analysis to runoff elections with victory margin requirements. Section 7 concludes. Proofs are relegated to the appendix.

[10] Spatial models of electoral competition under the runoff rule can also help to make sense of some of these outcomes through the strategic behavior of candidates (Osborne and Slivinski 1996, Callander 2005).

[11] Notice that the existence of sincere/partisan voters does not help to make sense of this empirical puzzle. Indeed, if (all) voters vote for their most preferred candidate, then there should not be any difference between plurality and runoff elections with respect to the number of candidates receiving a positive fraction of votes.

2 Setup

This section describes a new model of runoff elections with Poisson distribution of voters.[12] There are three candidates, $\{A, B, C\}$, and twelve types of voters,

$$t \in T = \{t_{AB}, t'_{AB}, t''_{AB}, t_{AC}, t'_{AC}, t_{BA}, t''_{BA}, t_{BC}, t'_{BC}, t_{CA}, t''_{CA}, t_{CB}\}.$$

I denote the utility of a type-t voter by the function $U(W|t)$, where W is the candidate winning the election. Thus, voters do not directly derive a benefit from the ballot they cast: they are instrumental. The twelve types of voters allow for the representation of every possible preference ordering over the three candidates. I explicitly define the preference ordering of three types of voters:

$$\begin{aligned} U(A|t_{AB}) &> U(B|t_{AB}) > U(C|t_{AB}), \\ U(A|t'_{AB}) &= U(B|t'_{AB}) > U(C|t'_{AB}), \text{ and} \\ U(A|t''_{AB}) &> U(B|t''_{AB}) = U(C|t''_{AB}). \end{aligned} \tag{1}$$

There is no confusion about the preferences of the other types.

Runoff elections are held in one or two rounds. In the first round ($\rho = 1$), each voter either casts a ballot in favor of one of the candidates or abstains. The action set of the voters is denoted by $\Psi^1 = \{A, B, C, \varnothing\}$. If the candidate who ranks first obtains more than a pre-defined fraction, ζ, of the votes (called the *threshold for first-round victory*), she wins outright and there is no second round.[13] A second round is held if no candidate passes the threshold for first-round victory. In the second round ($\rho = 2$), each voter either casts a ballot in favor of one of the participating candidates or abstains. In this round, however, not all candidates participate: only the two candidates who received the most votes in the first round (called the top-two candidates) are included. The action set of voters is denoted by $\Psi^2 = \{P, Q, \varnothing\}$, where P and Q refer to the candidates who ranked first and second in the first round, respectively. The candidate who obtains the most votes

[12] The results do not depend upon the assumption of a Poisson distribution of voters. In particular, results hold if the size of the population is known and fixed. Proof available upon request.

[13] In Section 6, I show that the results hold when first-round victory requires a victory margin over the second-ranked candidate.

in this round wins the election. To lighten notation, I assume without loss of generality that ties are resolved by alphabetical order: A wins over both B and C, B wins over C.[14]

In a three-candidate setup, a runoff electoral system with a threshold below $\frac{1}{3}$ is equivalent to the plurality electoral system (a.k.a. *first-past-the-post).* Since at least one candidate receives $\frac{1}{3}$ or more of the votes, a second round is never held: the first round always determinates a winner. Therefore, I only consider runoff electoral systems with a threshold $\zeta \geq \frac{1}{3}$.[15]

I conduct the analysis under the assumption that the size of the electorate, k, is distributed according to a Poisson distribution of mean n: $k \sim P(n)$ (see Appendix A1 for a summary of important properties of Poisson games). Each voter is assigned a type t by i.i.d. draws. The probability that a randomly drawn voter is assigned type t is $r(t)$, with $\sum_{t \in T} r(t) = 1$. These probabilities are common knowledge.

In runoff election, voters participating in the first round may differ from those participating in the second round (Wright 1989, Bullock III and Johnson 1992, Morton and Rietz 2006). First-round voters may not return to the ballot in the second round for a variety of reasons (e.g. business or private appointment, sickness or accident). Some individuals may participate only in the second round. Therefore, even after observing the first round results, the distribution of preferences in the electorate remains uncertain. This feature of runoff elections must be included in the model. Indeed, voter behavior is dramatically affected by the precision of information regarding the distribution of preferences in the electorate, as conveyed by the first-round outcome.

Ideally, the model should include three groups of voters: those participating (i) only in the first round, (ii) only in the second round, and (iii) in both rounds. However, excluding voters of group (iii) greatly simplifies the analysis. Therefore, in my model none of the first-round voters participate in the second round, and conversely. I assume that there is a (complete) new draw of voters between the two rounds. The expected size of the electorate, n^{ρ}, and the probabilities of the different types, $r^{\rho}(t)$, remain the same in both rounds (that is, there is no Bayesian updating). In section 5, I discuss this assumption and

[14] Results hold if I assume that ties are resolved by the toss of a fair coin.

[15] For details about voters behavior in multicandidate plurality elections, see Myerson and Weber (1993) and Fey (1997).

show that it is not necessary for the main results to hold. The necessary condition is the existence of some uncertainty regarding the distribution of preferences in the electorate after the first round; not the particular structure of this uncertainty.

Strategies for first round voters as well as for second round voters are defined. In the first round, a type t's strategy is a mapping $\sigma^1 : T \rightarrow \Delta \{A, B, C, \varnothing\}$ that specifies a probability distribution over the set of actions in round $\rho = 1$. In the second round, a type t's strategy is a mapping $\sigma^2 : T \times \{\{A, B\}, \{A, C\}, \{B, C\}\} \rightarrow \Delta \{A, B, C, \varnothing\}$ such that $supp\{\sigma^2(t, \{P, Q\})\} = \{P, Q, \varnothing\}$ specifies a probability distribution over the set of actions in round $\rho = 2$ (which depends on which candidates are participating in the second round). For the sake of readability, I henceforth omit $\{P, Q\}$ from the notation $\sigma^2(t, \{P, Q\})$. Given the strategy σ^ρ, a fraction:

$$\tau^\rho_\psi(\sigma^\rho) = \sum\nolimits_t r\left(t\right)\ \sigma^\rho\left(\psi|t\right) \tag{2}$$

of the electorate is expected to play action ψ in round ρ. I call $\tau^\rho_\psi(\sigma^\rho)$ the *expected share of voters* who choose action ψ in round ρ given the strategy σ^ρ.

The number of players who choose action ψ in round ρ is denoted by x^ρ_ψ, where $\psi \in \Psi^\rho$. This number is random (voters do not observe it before going to the polls) and its distribution depends on the strategy, through $\tau^\rho_\psi(\sigma^\rho)$. The expected number of votes in favor of ψ in round ρ is therefore:

$$\mathsf{E}\left[x^\rho_\psi|\sigma^\rho\right] = \tau^\rho_\psi(\sigma^\rho) \cdot n.$$

For the sake of readability, I henceforth omit σ^ρ from the notation.

Given the intrinsic properties of population uncertainty, the equilibrium mapping $\sigma\left(t\right)$ *must* be identical for all voters of a same type t (see Myerson 1998b, p377, for more detail). Therefore, for this voting game, I analyze the limiting properties of sequences of symmetric (weak) Perfect Bayesian Nash equilibria when the expected population size n becomes infinitely large.[16]

[16]This does not mean that the identified properties of runoff elections only hold for infinitely large electorate. It means that there is always a n sufficiently large (but potentially small) such that runoff electoral systems feature the identified properties.

3 Pivot Probabilities and Payoffs in Runoff Elections

Since voters are instrumental, their behavior depends on the probability that a ballot affects the final outcome of the elections, i.e. its probability of being *pivotal*. In runoff elections, a ballot may be pivotal in both rounds. This section identifies all the pivotal events and derives their probabilities. Using the pivot probabilities, I compute voters' expected payoffs of the different actions in the two rounds. Both subsections start with the analysis of the second round.

3.1 Pivot Probabilities

3.1.1 Second Round

In a second round opposing P to Q, a ballot can change the outcome of the elections in two ways: from a victory of P to a victory of Q, and conversely. A ballot is pivotal between P and Q in the second round if an additional ballot in favor of P allows her to win instead of Q. This event, denoted piv^2_{PQ}, happens when P trails behind Q by exactly one vote: an additional ballot in favor of P leads to a tie between P and Q and thus to the victory of the former (since ties are broken alphabetically). Similarly, a ballot is pivotal between Q and P in the second round if an additional ballot in favor of Q allows her to win instead of P. This event, denoted piv^2_{QP}, happens when P and Q obtain exactly the same number of votes: an additional ballot in favor of Q breaks the tie with P and ensures the victory of Q. Using Property 1 (and Theorem 2, Myerson 2000[17]), the following Lemma is proven:

Lemma 1 *The magnitudes of the second-round pivot probabilities PQ and QP are:*

$$mag(piv^2_{PQ}) = mag(piv^2_{QP}) = -\left(\sqrt{\tau^2_P} - \sqrt{\tau^2_Q}\right)^2.$$

Lemma 1 reformulates a known result for two-candidate elections (as is the second round): the larger the difference in the expected vote shares of the two candidates, the smaller the magnitude of the pivot probability. The intuition is straightforward: for a

[17] Theorem 2 in Myerson 2000 shows that two events that differ only by a small number of votes, as do piv^2_{PQ} and piv^2_{QP}, have the same magnitude.

ballot to be pivotal, candidates have to receive (almost) the same number of votes. This is more likely to happen if the second round is expected to be close.

3.1.2 First Round

The first round influences the final result either directly (if one candidate wins outright in the first round) or indirectly (through the identity of the candidates participating to the second round).

A ballot can affect the outcome of the election directly in two ways. First, a ballot is *threshold pivotal* i/ij, denoted $piv^1_{i/ij}$, if candidate i lacks one vote (or less) to pass the threshold for first-round victory and the other candidates are all below that threshold. Thus, without an additional vote in favor of i, a second round opposing i to j is held. The complementary event is the threshold pivotability ij/i, denoted $piv^1_{ij/i}$, that refers to an event in which any ballot against candidate i, i.e. in favor of either j or k, prevents an outright victory of i in the first round and ensures that a second round opposing i to j is held. Second, a ballot is *above-threshold pivotal* i/j, denoted $piv^1_{i/j}$, if candidates i and j have (almost) the same number of votes and are both above the threshold. An additional vote in favor of candidate i allows her to win outright in the first round but, without any other ballot in favor of i, candidate j wins outright. Since two candidates cannot simultaneously obtain more than 50% of the votes, the above threshold pivotability is possible if and only if the threshold is below 50%.

A ballot may also affect the final outcome if it changes the identity of the two candidates participating in the second round. This happens when a ballot changes the identity of the candidate who ranks third in the first round. A ballot is *second-rank pivotal* ki/kj when candidate k ranks first and candidates i and j tie for the second rank. An additional vote in favor of candidate i allows her, instead of j, to participate in the second round with k.

Table 1 below summarizes the different first-round pivotal events that influence the first-round voting behavior.[18]

Table 1: first-round pivotal events.

[18] I consider three-way ties as a specific case of two-way ties.

Event	Notation	Condition*
Threshold pivotal i/ij	$piv^1_{i/ij}$	$x^1_i + 1 > \zeta\left(x^1_i + x^1_j + x^1_k\right) \geq x^1_i \geq x^1_j \geq x^1_k$
Threshold pivotal ij/i	$piv^1_{ij/i}$	$\zeta\left(x^1_i + x^1_j + x^1_k + 1\right) \geq x^1_i > \zeta\left(x^1_i + x^1_j + x^1_k\right)$ $x^1_i > x^1_j \geq x^1_k$
Above-threshold pivotal i/j	$piv^1_{i/j}$	$x^1_i = x^1_j - 1 \geq \zeta\left(x^1_i + x^1_j + x^1_k\right) \geq x^1_k$
Second-rank pivotal ki/kj	$piv^1_{ki/kj}$	$x^1_i = x^1_j - 1$ $\zeta\left(x^1_i + x^1_j + x^1_k\right) \geq x^1_k > x^1_j$

* The conditions for the pivotabilities take into account the alphabetical order i, j, k.

Using Property 1 (and Theorem 2 in Myerson 2000), I can prove the following Lemma:

Lemma 2 *The magnitudes of the first-round pivot probabilities are:*

(a) Threshold pivot probability* i/ij *and* ij/i*:

$$mag(piv^1_{i/ij}) = mag(piv^1_{ij/i}) = \begin{cases} \left(\frac{\tau^1_j+\tau^1_k}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_i}{\zeta}\right)^{\zeta} - 1 \ \textit{if}\ \frac{\zeta}{1-\zeta} \geq \frac{\tau^1_j}{\tau^1_j+\tau^1_k} \geq \frac{1}{2} \\ \left(\frac{\sqrt{\tau^1_i\tau^1_j}}{\zeta}\right)^{2\zeta}\left(\frac{\tau^1_k}{1-2\zeta}\right)^{1-2\zeta} - 1 \ \textit{if}\ \frac{\tau^1_j}{\tau^1_j+\tau^1_k} > \frac{\zeta}{1-\zeta} \geq \frac{1}{2} \\ \left(\frac{2\sqrt{\tau^1_j\tau^1_k}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_i}{\zeta}\right)^{\zeta} - 1 \ \textit{if}\ \frac{\zeta}{1-\zeta} \geq \frac{1}{2} > \frac{\tau^1_j}{\tau^1_j+\tau^1_k} \end{cases} \tag{3}$$

***(b) Above-threshold pivot probability* i/j *and* j/i** *(for $\zeta < 1/2$):*

$$mag(piv^1_{i/j}) = mag(piv^1_{j/i}) = \begin{cases} -\left(\sqrt{\tau^1_i} - \sqrt{\tau^1_j}\right)^2 \ \textit{if}\ \sqrt{\tau^1_i\tau^1_j} \geq \tau^1_k\frac{\zeta}{1-2\zeta} \\ \left(\frac{\sqrt{\tau^1_i\tau^1_j}}{\zeta}\right)^{2\zeta}\left(\frac{\tau^1_k}{1-2\zeta}\right)^{1-2\zeta} - 1 \ \textit{if}\ \tau^1_k\frac{\zeta}{1-2\zeta} > \sqrt{\tau^1_i\tau^1_j} \end{cases} \tag{4}$$

(c) Second-rank pivot probability* ki/kj *and* kj/ki*:

$$mag(piv^1_{ki/kj}) = mag(piv^1_{kj/ki}) = \begin{cases} -\left(\sqrt{\tau^1_i} - \sqrt{\tau^1_j}\right)^2 \ \textit{if}\ 2\frac{\zeta}{1-\zeta}\sqrt{\tau^1_i\tau^1_j} > \tau^1_k > \sqrt{\tau^1_i\tau^1_j} \\ \left(\frac{2\sqrt{\tau^1_i\tau^1_j}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_k}{\zeta}\right)^{\zeta} - 1 \ \textit{if}\ \tau^1_k \geq 2\frac{\zeta}{1-\zeta}\sqrt{\tau^1_i\tau^1_j} > \sqrt{\tau^1_i\tau^1_j} \\ 3\left(\tau^1_i\tau^1_j\tau^1_k\right)^{\frac{1}{3}} - 1 \ \textit{if}\ 2\frac{\zeta}{1-\zeta}\sqrt{\tau^1_i\tau^1_j} > \sqrt{\tau^1_i\tau^1_j} \geq \tau^1_k \end{cases} \tag{5}$$

Lemma 2 shows that the magnitude of a pivotal event *piv* is larger when the expected outcome of the first round, τ^1, is close to the conditions necessary for event *piv* to occur.

For instance, the pivotal event $piv^1_{i/ij}$ is more likely to occur when $\zeta = \tau^1_i \geq \tau^1_j \geq \tau^1_k$ than when $\zeta > \tau^1_k \geq \tau^1_j \geq \tau^1_i$. Indeed, the occurrence of the pivotal event requires a "larger deviation with respect to the expected outcome".

3.2 Payoffs

The value of each ballot, and thus voters' behavior, depends on its probability of being pivotal. In the first round, it also depends on voters' expectations about the outcome of the second round.

3.2.1 Second Round

Let $G^2(\psi|t)$ denote the *expected gain* of playing action $\psi \in \Psi^2$ in the second round. This gain depends on the voter's preference, summarized by $U(\cdot|t)$, and on the strategy functions of second-round voters, σ^2. The strategies of other voters determine the expected number of votes received by each candidate in the second round, and thereby the pivot probabilities in that round. For a type t, the expected gain of voting for candidate P in the first round is:

$$G^2(P|t) = \Pr\left(piv^2_{PQ}\right)[U(P|t) - U(Q|t)]. \tag{6}$$

This reads as follows: a ballot in favor of candidate P can be pivotal in favor of P against candidate Q. If this happens, then P is elected instead of Q and voter t's payoff is $U(P|t) - U(Q|t)$.

3.2.2 First Round

Let $G^1(\psi|t)$ denote the *expected gain* of playing action $\psi \in \Psi^1$ in the first round. This gain depends on the voter's preference, summarized by $U(\cdot|t)$, and on the strategy functions of all voters: σ^1 and σ^2. First-round strategies determine the expected number of votes received by each candidate in the first round, and thus the pivot probabilities in that round. Second-round strategies allow first round voters to compute their expected utility for the different possible second rounds (A vs. B, A vs. C, and B vs. C). For a type t voter, the expected utility of a second round opposing i to j is given by

$$U(i,j|t) = \Pr(i|\{i,j\})\,U(i|t) + \Pr(j|\{i,j\})\,U(j|t),$$

where $\Pr(i|\{i,j\})$ is the probability that candidate i wins the second round if opposed to candidate j and $\Pr(j|\{i,j\}) = 1 - \Pr(i|\{i,j\})$.

For a type t, the expected gain of playing action i in the first round is:

$$\begin{aligned} G^1(i|t) = {} & \Pr\left(piv^1_{ki/kj}\right)[U(k,i|t) - U(k,j|t)] + \Pr\left(piv^1_{ji/jk}\right)[U(j,i|t) - U(j,k|t)] + \\ & \Pr\left(piv^1_{i/j}\right)[U(i|t) - U(j|t)] + \Pr\left(piv^1_{i/k}\right)[U(i|t) - U(k|t)] + \\ & \Pr\left(piv^1_{i/ij}\right)[U(i|t) - U(i,j|t)] + \Pr\left(piv^1_{i/ik}\right)[U(i|t) - U(i,k|t)] + \\ & \Pr\left(piv^1_{ji/j}\right)[U(j,i|t) - U(j|t)] + \Pr\left(piv^1_{jk/j}\right)[U(j,k|t) - U(j|t)] + \\ & \Pr\left(piv^1_{ki/k}\right)[U(k,i|t) - U(k|t)] + \Pr\left(piv^1_{kj/k}\right)[U(k,j|t) - U(k|t)], \end{aligned} \tag{7}$$

where $i, j, k \in \Psi^1$ and $i \neq j \neq k$. The first line in (7) reads as follows: if a ballot in favor of i is second-rank pivotal ki/kj, then the second round opposes k to i instead of k to j; if a ballot in favor of i is second-rank pivotal ji/jk, then the second round opposes j to i instead of j to k. The second line refers to the gains when the ballot is above-threshold pivotal and the three last lines refer to the gains when the ballot is threshold pivotal.

4 Voting Behavior: Equilibrium Analysis

Equilibrium multiplicity is inherent to multicandidate elections. Runoff elections are not an exception. In this section, I focus on pure strategy equilibria in runoff elections. I identify three equilibrium properties of runoff electoral systems when voters are strategic. First, I show that runoff electoral systems generally produce equilibria in which only two candidates receive votes. These *Duverger's Law equilibria* are shown to exist for any first-round victory threshold $\zeta \in \left[\frac{1}{3}, 1\right)$. I demonstrate that the existence of these Duverger's Law equilibria may prevent the election of the Condorcet winner. Second, I show that the sincere voting equilibrium does not always exist. Together, these two results prove that the Duverger's Law equilibria may be the only pure strategy equilibria in runoff elections. Lastly, I show that when the threshold for first-round victory is below 50%, the Condorcet loser may be the only likely winner in equilibrium. Indeed, she may win the election outright in the first round because all majority voters vote for the candidate they prefer instead of coordinating their votes behind one candidate.

For the sake of simplicity, I perform the equilibrium analysis under the simplifying assumption that the electorate is composed of only three types of voters: t_{AB}, t_{BA}, and t''_{CA}. Except for the first part of Theorem 2, all results extend to the general setup with more types of voters (see subsection 4.3 for details).

Types t''_{CA} are called the *minority voters*: in expected terms, they represent a minority of the electorate, i.e. $r\left(t''_{CA}\right) < 1/2$. They strictly prefer candidate C to the other candidates, about whom they are indifferent:

$$U\left(C|t''_{CA}\right) = 1 > U\left(A|t''_{CA}\right) = U\left(B|t''_{CA}\right) = 0. \tag{8}$$

These voters always vote for C.

Together, types t_{AB} and t_{BA} are called the *majority voters*: in expected terms, they represent a majority of the electorate, i.e. $r(t_{AB}) + r(t_{BA}) > 1/2$. Types t_{AB} and t_{BA} all identify candidate C as being the worst option but have different opinion about A and B. Types t_{AB} prefer A to B whereas types t_{BA} prefer B to A:[19]

$$U\left(A|t_{AB}\right) = 1 > U\left(B|t_{AB}\right) = 0 > U\left(C|t_{AB}\right) = -1$$

and

$$U\left(B|t_{BA}\right) = 1 > U\left(A|t_{BA}\right) = 0 > U\left(C|t_{BA}\right) = -1.$$

To be sure that the results do not hinge on any form of symmetry, I assume that, in expected terms, types t_{AB} represent a larger (or equal) fraction of the electorate than types t_{BA}: $r(t_{AB}) \geq r(t_{BA})$. Note that the particular values of $U\left(A|t\right)$, $U\left(B|t\right)$, and $U\left(C|t\right)$ are not necessary for my results.

I start with the analysis of the second-round voting behavior.

4.1 Second Round

Being a two-candidate election, the analysis of voters' behavior in the second round is straightforward. From (6), it immediately follows that:

[19] In a previous version of the paper, I considered a different source of divisions among voters: voters were divided because of information instead of preferences. I proved that the main results hold under that assumption.

Proposition 1 *In the second round, voters always vote for the candidate they prefer. Thus, the expected results of the second round depends on the identity of the candidates participating in that round:*
(i) when $\{P,Q\} = \{A,C\}$ *or* $\{C,A\}$*:* $\tau_A^2 = r(t_{AB}) + r(t_{BA}) > \tau_C^2 = r(t''_{CA})$,
(ii) when $\{P,Q\} = \{B,C\}$ *or* $\{B,C\}$*:* $\tau_B^2 = r(t_{AB}) + r(t_{BA}) > \tau_C^2 = r(t''_{CA})$,
(iii) when $\{P,Q\} = \{A,B\}$ *or* $\{A,B\}$*:* $\tau_A^2 = r(t_{AB}) + \sigma^2(A|t''_{CA})r(t''_{CA})$ *and* $\tau_B^2 = r(t_{BA}) + \sigma^2(B|t''_{CA})r(t''_{CA})$.

When C participates in the second round, majority voters coordinate their votes on the participating majority candidate. This ensures that the majority candidate, either A or B, defeats C with a probability that tends to 1 when n becomes large. When A and B are opposed, the result depends on the fractions of types t_{AB} and t_{BA}, $r(t_{AB})$ and $r(t_{BA})$, as well as on type t''_{CA} strategies, $\sigma^2(A|t''_{CA})$ and $\sigma^2(B|t''_{CA})$. For the sake of simplicity, I assume that types t''_{CA} abstain if C does not participate in the second round, i.e. $\sigma^2(\varnothing|t''_{CA}) = 1$ if $\{P,Q\} = \{A,B\}$. Therefore, when opposed to B, except if $r(t_{AB}) = r(t_{BA})$, A wins with a probability that tends to 1 when n becomes large. I will make clear that this assumption is not central to my results.

4.2 First Round

4.2.1 Duverger's Law Equilibria

The game theoretic version of Duverger's Law states that, in plurality elections with strategic voters, only two candidates obtain a positive faction of the votes. The game theoretic version of Duverger's Hypothesis states that, in the first round of a runoff election with strategic voters, at least three candidates obtain a positive fraction of the votes.

Definition 1 *A* Duverger's Law equilibrium *is a voting equilibrium in which only two candidates obtain a positive fraction of the votes. A* Duverger's Hypothesis equilibrium *is a voting equilibrium in which three candidates obtain a positive fraction of the votes.*

In a three candidate setup, regardless of the distribution of preferences among majority voters, plurality elections always produce at least two Duverger's Law equilibria (Myerson and Weber 1993, Fey 1997, Piketty 2000 and Bouton and Castanheira 2009): either all majority voters vote for candidate A or all majority voters vote for candidate B. The existence of multiple Duverger's Law equilibria highlights that voters may fail to coordinate on the "correct" candidate. For instance, even if majority voters all prefer A over B, i.e. $r(t_{BA}) = 0$, there is an equilibrium in which they all vote for B. Even if a Condorcet winner exists, she is not sure to win.

According to the Duverger's Hypothesis, runoff elections should not feature such an undesirable property: in the first round of a three-candidate runoff election, all three candidates obtain a positive fraction of the votes and the Condorcet winner is the only likely winner (Cox 1997, Piketty 2000 and Martinelli 2002). Nonetheless, the following Theorem shows that this feature does not hold when voters participating in the two rounds are not necessarily the same:

Theorem 1 (Duverger's Law equilibria) *In the first round of a runoff election, there exist two stable Duverger's Law equilibria in which all majority voters play $\psi = A$ (resp. B). For $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right)$, these equilibria exist for any $r(t''_{CA}) \in \left(0, \frac{1}{2}\right)$. For $\zeta = \frac{1}{2}$, these equilibria exist for any $r(t''_{CA}) \in \left[0.067, \frac{1}{2}\right)$. For $\zeta \in \left(\frac{1}{2}, 1\right)$, these equilibria exist for any $r(t''_{CA}) \in \left[z, \frac{1}{2}\right)$ where $z < 0.067$.*

The reason for Duverger's Law equilibria in runoff elections is the following. Consider the first-round strategy profile $\sigma^1(B|t_{BA}) = 1$ and $\sigma^1(B|t_{AB}) = 1 - \omega$ with $\omega \to 0$, for which alternative A's expected vote share is vanishingly small. What is the best response of a t_{AB} voter? If he votes for B and is pivotal in electing B in the first round, he saves himself either from a victory of C in the first round when $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right)$, i.e. if above-threshold pivotal B/C, or from the risk of an upset victory of C in the second round when $\zeta \in \left[\frac{1}{2}, 1\right)$, i.e. if threshold pivotal B/BC. In comparison, voting for A is valuable for a t_{AB} voter if a second round is held and his ballot is pivotal in bringing A to that round, i.e. if second-rank pivotal BA/BC. Comparing the probabilities of each of these events shows that, when the expected fraction of type t''_{CA} is not too small, the risk of a C victory (in either round) is too high in comparison with the likelihood of having A participating in the second round.

The specificities of the conditions on the size of the minority depend on the Poisson distribution of voters. Nevertheless, the trade-off is self-explanatory. A majority voter has an incentive to abandon a trailing candidate (A in the above example) if the risk of C's victory is too high compared to the first-round chances of bringing the trailing majority candidate to the second round. Typically, the larger C's vote share, the higher the risk of C's victory, and the lower the probability that one vote may bring the trailing majority candidate to the second round. This makes clear that the risk of an upset victory is crucial for the Duverger's Law equilibria to exist when $\zeta \in \left[\frac{1}{2}, 1\right)$. (See Section 5 for more details).

4.2.2 Sincere Voting Equilibrium

Theorem 1 does not show that Duverger's Law equilibria are the only equilibria in runoff elections. Actually, there might exist a Duverger's Hypothesis equilibrium in which all voters are *sincere*, i.e. $\sigma^1(A|t_{AB}) = \sigma^1(B|t_{BA}) = \sigma^1(C|t''_{CA}) = 1$. Nonetheless, this sincere voting equilibrium does not always exist.

Theorem 2 (Sincere voting) *In the first round of a runoff election, the sincere voting equilibrium may exist. Nonetheless, it does not exist when voters want to avoid the risk of an upset victory in the second round.*

Together Theorems 1 and 2 prove that the Duverger's Law equilibria may be the only pure strategy equilibria in runoff elections. This strongly qualifies Duverger's Hypothesis and extend Duverger's Law to runoff elections.

The intuition for the existence of the sincere voting equilibrium is as follows: majority voters vote to influence the identity of the majority candidate who will oppose C in the second round, i.e. they are second-rank pivotal CA/CB. The event $piv^1_{CA/CB}$ is relatively most probable when A and B are likely to tie for the second place and C is unlikely to pass the threshold. Since the probability of defeating C in the second round is the same for both majority candidates, majority voters vote for their most preferred candidate: t_{AB}-voters vote A and t_{BA}-voters vote B. As mentioned above, this part of Theorem 2 does not necessarily extend to the general setup (see subsection 4.3 for details).

The sincere voting equilibrium does not always exist. Indeed, some majority voters do not want to vote for their most preferred candidate when one majority candidate is unlikely to participate in the second round. In such a case, conditional on being pivotal, the election essentially boils down to a contest between one of the majority candidates and C. Some majority voters then abandon their most preferred candidate in order to ensure an outright victory of the other majority candidate in the first round. As in Duverger's Law equilibria, they do so to avoid the risk of an upset victory of C in the second round.

Theorem 2 is in stark contrast with previous results in the literature. Indeed, Cox (1997) argues that the sincere voting equilibrium does not exist in three-candidate runoff elections because of "push-over tactics": some supporters of the strongest candidate in the first round vote for an unpopular candidate in order to ensure the victory of their preferred candidate in the second round. Theorem 2 shows that (i) the sincere voting equilibrium may exist and supporters of the strongest candidate do not "push over" because there is a possibility of an outright victory in the first round,[20] and (ii) "push-over" is not the only reason that may prevent the existence of the sincere voting equilibrium: there is also the desire to avoid the risk of an upset victory in the second round.

4.2.3 The Ortega Effect

In this subsection, I prove that C, the Condorcet loser, may be the only likely winner in equilibrium of runoff elections with a threshold $\zeta \in \left(\frac{1}{3}, \frac{1}{2}\right)$. If C participates in the second round, majority voters coordinate behind the participating majority candidate (Proposition 1). Thus, $r(t''_{CA}) < 1/2$ implies that C cannot win the second round with a probability that tends to 1 as n becomes large. I thus focus on the possibility of an outright victory of C in the first round. In that round, C is the only likely winner if her expected vote share is above both the threshold and the expected vote shares of candidates A and B:

$$\tau_C^1 > \zeta, \text{ and} \tag{9}$$

$$\tau_C^1 > \max\{\tau_A^1, \tau_B^1\}. \tag{10}$$

[20] Actually, Bouton and Gratton (2011) show that, when the possibility of an outright victory in the first round is taken into account in the analysis, "push over tactics" cannot be supported in equilibrium.

From $r\left(t''_{CA}\right) < 1/2$, I have that $\tau^1_C > \max\{\tau^1_A, \tau^1_B\}$ is possible if and only if majority voters divide their votes, i.e. $\sigma^1(A|t_{AB}) > 0$ and $\sigma^1(B|t_{BA}) > 0$. In equilibrium,

$$\begin{aligned} G^1(A|t_{AB}) - G^1(B|t_{AB}) &\geq 0, \text{ and} \\ G^1(A|t_{BA}) - G^1(B|t_{BA}) &\leq 0, \end{aligned} \tag{11}$$

must then be satisfied. From (7) and Property 2, a sufficient condition for (11) to be strictly satisfied is[21]

$$\begin{aligned} mag\left(piv^1_{CA/CB}\right) > \max\{&mag(piv^1_{BA/BC}), mag\left(piv^1_{AB/AC}\right), mag(piv^1_{A/C}), \\ &mag(piv^1_{B/C}), mag(piv^1_{A/AC}), mag(mag^1_{B/BC})\}. \end{aligned} \tag{12}$$

In such a situation:

$$\begin{aligned} \frac{G^1(A|t_A) - G^1(B|t_A)}{\Pr(piv^1_{CA/CB})} \underset{n\to\infty}{\to} U(C,A|t_{AB}) - U(C,B|t_{AB}) &= \Pr\left(A|\{A,C\}\right) > 0, \\ \frac{G^1(A|t_B) - G^1(B|t_B)}{\Pr(piv^1_{CA/CB})} \underset{n\to\infty}{\to} U(C,A|t_{BA}) - U(C,B|t_{BA}) &= -\Pr\left(A|\{A,C\}\right) < 0. \end{aligned}$$

To prove that C may be elected in equilibrium, it is therefore sufficient to prove that (12) may be true when (9) and (10) are satisfied. This then follows:

Theorem 3 (Ortega effect) *For any runoff election with a threshold $\zeta < 0.5$, there are $\varepsilon > 0$ and $\kappa > 0$ such that, if $|r(t_{AB}) - r(t_{BA})| \leq \varepsilon$ and $\zeta < r(t''_{CA}) < \zeta + \kappa$, there exists an equilibrium with the following two properties:*
(i) $\sigma^1(A|t_{AB}) = \sigma^1(B|t_{BA}) = \sigma^1(C|t''_{CA}) = 1$, and
(ii) $\tau^1_C > \zeta > \max[\tau^1_A, \tau^1_B]$.

Why would majority voters divide their votes when C is expected to win? Consider the first-round choice of a majority voter who prefer B to A. If he expects C to pass the threshold (which is below 50%) and then to win outright, his main objective is to prevent C's victory. There are two ways to achieve this goal: (i) vote for the strongest majority candidate, say A, to defeat C directly, or (ii) vote for his preferred majority candidate and hence increase the number of votes necessary to pass the threshold for first

[21] From Proposition 1, $\Pr\left(A|\{A,C\}\right) > \frac{1}{2}$ $\forall n$ and $\Pr\left(A|\{A,C\}\right) \underset{n\to\infty}{\to} 1$.

round victory (which is a percentage of the total number of votes). If a second round is then held, C is almost certainly defeated since all majority voters support the remaining majority candidate. The second option has the advantage that it does not require the majority voter to abandon B, his most preferred candidate, to fight C. Actually, it allows him to hit two birds with one stone: preventing C outright victory and qualifying B for the second round.

The t_{BA}-voter chooses option (i) and abandon B, his most preferred candidate, if the above-threshold pivotability A/C is relatively likely. He chooses option (ii) and vote for B if the threshold pivotability CA/C and the second-rank pivotability CB/CA are relatively more likely. Thus, even if C is expected to win, it may be individually rational for majority voters to vote for the candidate they prefer. This is what I call the *Ortega effect*, which may allow the Condorcet Loser to be the only likely winner in equilibrium of a runoff election with a threshold below 50%. Roughly speaking, the Ortega effect happens when C is unlikely to be defeated by a majority candidate in the first round, and the two majority candidates tie for the second place.

Theorem 3 can be illustrated through a numerical example. Suppose that $\zeta = 0.4$, $r(t_{AB}) = 0.30$, $r(t_{BA}) = 0.29$, and $r(t''_{CA}) = 0.41$. With these parameter values, as for the Nicaraguan case discussed in the introduction, the Condorcet Loser, C, would asymptotically be ensured of a first-round victory if all voters vote for their most preferred candidate. First-round vote shares would be: $\tau_C^1 = 0.41 > \zeta > \tau_A^1 = 0.30 > \tau_B^1 = 0.29$. For these parameter values, $\sigma^1(A|t_{AB}) = \sigma^1(B|t_{BA}) = \sigma^1(C|t''_{CA}) = 1$ is an equilibrium strategy profile since, as illustrated in Table 2, (12) is satisfied: majority voters assess that if they are pivotal it is (i) to ensure that a second round is held (i.e. $piv^1_{CA/C}$) and (ii) to choose whom of A and B will participate to the second round with C (i.e. $piv^1_{CA/CB}$). Majority voters know that if they influence the outcome of the election, then a second round will be held. Therefore, types t_{AB} vote for A to ensure her participation in the second round and types t_{BA} vote for B to ensure her participation in the second round. Although majority voters correctly anticipate an outright victory of C in the first round, they divide their votes between the two majority candidates.

Table 2: equilibrium magnitudes

Threshold mag.*	**Above-Threshold mag.****	**Second-rank mag.*****
$mag(piv^1_{A/AC}) = -0.0223$	$mag(piv^1_{A/C}) = -0.0304$	$mag(piv^1_{BA/BC}) = -0.0125$
$mag(piv^1_{B/BC}) = -0.0273$	$mag(piv^1_{B/C}) = -0.0370$	$\mathbf{mag(piv^1_{CA/CB}) = -0.0003}$
$\mathbf{mag(piv^1_{CA/C}) = -0.0002}$	$mag(piv^1_{A/B}) = -0.0953$	$mag(piv^1_{AB/AC}) = -0.0125$
$mag(piv^1_{CB/C}) = -0.0003$		

* Threshold pivotal $(piv^1_{i/ij})$ if $x^1_i+1 > \zeta(x^1_i+x^1_j+x^1_k)\geq x^1_i\geq x^1_j\geq x^1_k$

** Above-threshold pivotal $(piv^1_{i/j})$ if $x^1_i= x^1_j-1 \geq \zeta(x^1_i+x^1_j+x^1_k)\geq x^1_k$

*** Second-rank pivotal $(piv^1_{ki/kj})$ if $x^1_i= x^1_j-1$ & $\zeta(x^1_i + x^1_j + x^1_k) \geq x^1_k> x^1_j$

The existence of the Ortega effect does not rely on any remaining uncertainty about the distribution of preferences in the electorate after the first round. On te contrary, it is reinforced if the risk of an upset victory of C is null. Preventing an outright victory of C in the first round by forcing the organization of a second round becomes more appealing if the majority candidate participating in the second round is sure to defeat C.

The Ortega effect does not exist in plurality elections. Indeed, majority voters' only way to defeat C when she is expected to win is to vote for the strongest majority candidate: a ballot cannot be threshold pivotal CA/C since, by definition, there is no possibility of a second round in a plurality election. Therefore, the Condorcet Loser can be the only likely winner in equilibrium of a plurality election if and only if there is no strongest majority candidate (this could also happen in a runoff election). In that case, majority voters divide their votes because they do not know on which candidate to coordinate. As shown by Fey (1997) this equilibrium is not very stable.

4.3 General setup

In this section, I show that, except for the first part of Theorem 2, the main results of the equilibrium analysis hold in the general setup with more than three types of voters.

The most striking feature of Theorem 1 is that there generally exists a Duverger's Law equilibrium in which the Condorcet winner does not receive any votes. One might wonder if such an eviction of the Condorcet winner from the electoral race relies on the relatively simple structure of preferences assumed in the previous section. The following Lemma shows that this is not the case (and implicitly extends Theorem 1 to the general setup):

Lemma 3 *When $r(t) > 0$ $\forall t \in T \backslash \{t''_{AB}, t''_{BA}\}$, there exists a Duverger's Law equilibrium in which the Condorcet winner does not receive any vote as long as the fraction of supporters of the least popular candidate, ϕ, is sufficiently large. For $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right)$, this equilibrium exists for any $\phi \in \left(0, \frac{1}{2}\right)$. For $\zeta = \frac{1}{2}$, this equilibrium exists for any $\phi \in \left[0.067, \frac{1}{2}\right)$. For $\zeta \in \left(\frac{1}{2}, 1\right)$, this equilibrium exist for any $\phi \in \left[z, \frac{1}{2}\right)$ where $z < 0.067$.*

Only the second part of Theorem 2, that the sincere voting equilibrium does not always exist, extends to a setup with more types of voters. Indeed, Bouton and Gratton (2011) explore the properties of the sincere voting equilibrium and of Duverger's Hypothesis equilibria in the first round of a runoff elections and show that the former does not exist when preferences in the electorate are sufficiently diverse, i.e. most (or all) types in T are represented. They also show that Duverger's Hypothesis equilibria may exist when one allows for mixed strategies. From Lemma 3, it follows that the second part of Theorem 2 holds when there are more than three types of voters.

Theorem 3 extends to the general model quite straightforwardly. First, knowing that $mag(piv^1_{C/CA})$ is (among) the largest magnitude helps to understand why it holds when types t''_{CA} are replaced by types t_{CA} and t_{CB}. These voters want to vote for C to ensure her victory in the first round. They do not want to vote for A or B since this could prevent C's outright victory. Second, the behavior of types t_{AC} and t_{BC} might seem problematic since they would not necessarily vote sincerely. For instance, when the expected outcome of the first round is $0 \leq \tau^1_A - \tau^1_B < \epsilon$ and $\tau^1_A < \zeta < \tau^1_C < \zeta + \kappa$, t_{BC}-voters may vote for C in order to ensure her outright victory in the first round and therefore avoid the risk of a victory of A in the second round.[22] Such insincere behavior does not preclude the Ortega effect. Indeed, there is a constellation of $r(t)$ values such that (i) C is the Condorcet loser

[22] For ϵ and κ sufficiently small, the ranking of magnitude is such that $mag\left(piv^1_{C/CA}\right) > mag\left(piv^1_{C/CB}\right) \geq mag(piv^1_{CA/CB}) >$ other magnitudes. Therefore, I have that types t_{CA}, t''_{CA}, t_{CB}, and t'_{BC} vote for C, types t_{AB}, t'_{AB}, t_{AC}, and t'_{AC} vote for A and types t_{BA} and t''_{BA} vote for B.

and (ii) the conditions

$$\begin{aligned} 0 &\leq \tau_A^1 - \tau_B^1 < \epsilon, \text{ and} \\ \tau_A^1 &\leq \zeta < \tau_C^1 < \zeta + \kappa \end{aligned}$$

are satisfied when $\tau_C^1 = r(t_{CA}) + r(t''_{CA}) + r(t_{CB}) + r(t_{BC}) + r(t'_{BC})$, $\tau_A^1 = r(t_{AB}) + r(t'_{AB}) + r(t_{AC}) + r(t'_{AC})$, and $\tau_B^1 = r(t_{BA}) + r(t''_{BA})$.

5 On the Groups of First and Second-Round Voters

In this section, I discuss the assumption of a complete new draw of the population of voters between the two rounds. I show that my results do not rely on this particular feature of the model. The necessary feature is that there exists a high enough risk of upset victory in the second round. This risk exists when, conditional on being pivotal, the distribution of preferences in the electorate remains uncertain. I show that such an uncertainty appears as soon as the set of first-round and second-round voters are not exactly the same. Since, this risk of upset victory plays a crucial role for Theorem 1 and the second part of Theorem 2 but not for the first part of the latter nor for Theorem 3, the following discussion focuses on the two former results.

5.1 When All Voters Participate in the Two Rounds

When the set of first-round voters is exactly the same as the set of second-round voters, the final outcome of the election is perfectly known conditional on being pivotal in the first round. For instance, conditional on being threshold pivotal A/AC, the risk of an upset victory of C in the second round is null. Indeed, a ballot is threshold pivotal A/AC if

$$x_A^1 + 1 > \zeta\left(x_A^1 + x_B^1 + x_C^1\right) \geq x_A^1 \geq x_C^1 \geq x_B^1.$$

Therefore, voters know that, even if $x_B^1 = 0$, candidate A will have enough vote to defeat C in the second round.

Hence, if the set of voters is exactly the same in both rounds, Duverger's Law equilibria do not exist in runoff elections with a threshold $\zeta \in [\frac{1}{2}, 1)$. For majority voters, an outright victory of a majority candidate in the first round, say A, is payoff equivalent to a second round opposing A to C. Indeed, from $\Pr(A|\{A,C\},piv^1_{A/AC})=1$, I have that

$$\Pr(A|\{A,C\},piv^1_{A/AC})U(A|t)+\Pr(C|\{A,C\},piv^1_{A/AC})U(C|t)=U(A|t),\ \forall t\in\{t_{AB},t_{BA}\}.$$

The best response of a t_{BA} voter anticipating that all other majority voters are voting for A is now to vote for B. He has nothing to gain by voting for A whereas casting a B-ballot may allow B to participate to the second round with A (and then potentially win). Therefore, neither Theorem 1 nor the second part of Theorem 2 hold when the voters participating in the two rounds are exactly the same.

5.2 When Some Voters Participate in the Two Rounds

In practice, the group of first-round voters usually differs from the group of second-round voters. There are two basic reasons for this: (i) some first round voters do not participate in the second round, and (ii) some voters only participate in the second round. The assumption that there are no voters participating in both rounds is not entirely satisfying. Yet, a model allowing for (i) voters participating only in the first round, (ii) voters participating only in the second round, and (iii) voters participating in both rounds, is relatively intractable. I therefore focus on a case in which there are voters participating in both rounds and voters participating only in the second round. Note that my results hold for the case in which there are voters participating in both rounds and voters participating only in the first round.[23]

The set of first round voters may differ from the set of second round voters because some voters may participate only in the second round. Conditional on being pivotal, voters obtain information about the distribution of preferences in the electorate. First, they learn the distribution of preferences in the group of first-round voters (see previous subsection). Second, they update their beliefs about the expected distribution of preferences of the new second-round voters, $r^2(t)$.[24] Applying Bayes' Rule, I have that:

$$r^2(t)=\frac{1}{x^1_A+x^1_B+x^1_C}\left(x^1_A\frac{r^1(t)\,\sigma(A|t)}{\tau^1_A}+x^1_B\frac{r^1(t)\,\sigma(B|t)}{\tau^1_B}+x^1_C\frac{r^1(t)\,\sigma(C|t)}{\tau^1_C}\right). \qquad (13)$$

[23] The proof is available upon request.

[24] I obtain similar results without Bayesian updating, i.e. assuming that $r^2(t)=r^1(t)\ \forall t$.

Considering the Duverger's Law equilibrium in which all majority voters vote for A, I am interested in the probability of an upset victory of C conditional on a ballot being threshold pivotal A/AC in the first round. This conditional probability is:

$$\Pr(C|\{A,C\}, piv^1_{A/AC}) = \Pr\left(x^1_C + x^2_C > x^1_A + x^1_B + x^2_A|\{A,C\}, piv^1_{A/AC}\right)$$

Where x^2_A and x^2_C are distributed according to Poisson distribution of mean $n^2\left(r^2(t_{AB}) + r^2(t_{BA})\right)$ and $n^2 r^2(t''_{CA})$ respectively. Since $x^1_B = 0$ when $\sigma^1(A|t_{AB}) = 1 = \sigma^1(A|t_{BA})$, this reduces to

$$\Pr(C|\{A,C\}, piv_{A/AC}) = \Pr\left(x^1_C\left(\frac{1-2\zeta}{1-\zeta}\right) > x^2_A - x^2_C\right). \tag{14}$$

I am now in position to prove that $\sigma^1(A|t_{AB}) = 1 = \sigma^1(A|t_{BA})$ (and $\sigma^1(C|t''_{CA}) = 1$) is an equilibrium in runoff elections with a threshold $\zeta \in [\frac{1}{2}, 1)$.[25] I divide the proof into two parts: (i) $\zeta = 1/2$ and (ii) $\zeta > 1/2$.

For $\zeta = 1/2$, I have from (14) that

$$\Pr(C|\{A,C\}, piv^1_{A/AC}) = \Pr\left(x^2_C > x^2_A\right).$$

Since all (other) majority voters vote A and all minority voters vote C, I have from Lemma 1 that

$$mag\left(\Pr(C|\{A,C\}, piv^1_{A/AC})\right) \geq -\left(\sqrt{1-r^2(t_C)} - \sqrt{r^2(t_C)}\right)^2.$$

Knowing from (13) that $r^2(t_{AB}) = \frac{1}{2}\frac{r^1(t_{AB})}{1-r^1(t''_{CA})}$, $r^2(t_{BA}) = \frac{1}{2}\frac{r^1(t_{BA})}{1-r^1(t''_{CA})}$, and $r^2(t''_{CA}) = \frac{1}{2}$, this becomes

$$mag\left(\Pr(C|\{A,C\}, piv^1_{A/AC})\right) = 0.$$

From the proof of Theorem 1, this directly implies that for runoff elections with a threshold $\zeta = 1/2$, Duverger's Law equilibria exist for any $r^1(t_C) \in (0, 1/2)$. This also shows that the updating of beliefs about the expected distribution of preferences in the group of second-round voters may weaken the conditions for the existence of Duverger's Law equilibria.

For $\zeta > 1/2$, the consequences of learning and beliefs updating are ambiguous. On the one hand, since $x^1_C(\frac{1-2\zeta}{1-\zeta}) < 0$, an upset victory of C in the second round requires

[25] For $\zeta \in (\frac{1}{3}, \frac{1}{2})$, the probability of an upset victory of C in the second round does not influence the behavior of majority voters: they are influenced by an above-threshold pivotability against C.

a larger number of new t''_{CA}-voters than with $\zeta = 1/2$, i.e. $x_C^2 > x_A^2 - x_C^1(\frac{1-2\zeta}{1-\zeta})$. This reduces the risk of an upset victory. On the other hand, if the threshold for first round victory is lower than the expected size of the majority, then, conditional on being pivotal, voters realize that the majority is smaller than expected: if $1 - r^1(t''_{CA}) > \zeta$, then $1/2 > r^2(t''_{CA}) > r^1(t''_{CA})$. This increases the risk of upset victory. The conditions under which Duverger's Law equilibria exist may thus be more demanding. Nonetheless, as long as $mag\left(\Pr(C|\{A, C\}, piv^1_{A/AC})\right)$ is sufficiently large, the Duverger's Law equilibria will exist.

I can prove in a similar fashion that the second part of Theorem 2 holds when the assumption of a complete new draw of voters is relaxed.

6 Victory Margin Requirements

In this section, I analyze runoff electoral systems that impose an extra condition for first-round victory: a victory margin requirement. In these electoral systems, a candidate wins outright in the first round if she receives more than a fraction ζ of the votes and if she has a β-points lead over the nearest competitor. I prove that the previous results hold.

Imposing a victory margin requirement has two consequences for pivot probabilities. First, there is an additional condition for a ballot to be threshold pivotal and above-threshold pivotal. For instance, without a victory margin requirement, a ballot is threshold pivotal i/ij if candidate i lacks one vote to pass the threshold of first round victory ζ and if the ranking is i then j then k, i.e. if $x_i^1 + 1 > \zeta\left(x_i^1 + x_j^1 + x_k^1\right) \geq x_i^1 \geq x_j^1 \geq x_k^1$. Now, in addition to these conditions, candidate i must have a lead over the other candidates larger than a fraction β of the votes, i.e. $x_i^1 - x_j^1 > \beta\left(x_i^1 + x_j^1 + x_k^1\right)$. Since the magnitude of a threshold pivot probability and an above threshold pivot probability can only be affected negatively by this new constraint, it is clear that

$$\begin{aligned} mag(piv^{1,VM}_{i/ij}) &\leq mag(piv^1_{i/ij}),\ \forall i, j \in \{A, B, C\}, i \neq j, \\ mag(piv^{1,VM}_{i/j}) &\leq mag(piv^1_{i/j}),\ \forall i, j \in \{A, B, C\}, i \neq j, \end{aligned} \tag{15}$$

where the superscript VM refers to a runoff election with a victory margin requirement. For instance, $mag(piv^{1,VM}_{i/ij})$ denotes the threshold-pivot probability i/ij when a victory margin is required.

The second consequence is that there is a new pivotal event in the first-round. In runoff electoral systems with victory margin requirements, a ballot can allow a candidate to win outright in the first round if she has enough votes to pass the threshold but lacks one vote to have the β-points lead over her nearest competitor, i.e. if $x_i^1 - x_j^1 = \beta\left(x_i^1 + x_j^1 + x_k^1\right)$ and $x_i^1 > \zeta\left(x_i^1 + x_j^1 + x_k^1\right) \geq x_j^1 \geq x_k^1$. In such a situation, I say that a ballot is *margin pivotal* $i-j$, denoted $piv_{i-j}^{1,VM}$. When margin pivotal $i-j$, not casting a ballot in favor of candidate i leads to a second round opposing i to j. From Property 1 (in Appendix A1),

$$\begin{aligned} mag(piv_{i-j}^{1,VM}) &= mag(piv_{j-i}^{1,VM}) = 0 \text{ if } \begin{cases} \tau_i^1 - \tau_j^1 = \beta \\ \tau_i^1 \geq \zeta \\ \tau_j^1 \geq \tau_k^1 \end{cases} \\ mag(piv_{i-j}^{1,VM}) &= mag(piv_{j-i}^{1,VM}) < 0 \text{ otherwise.} \end{aligned} \tag{16}$$

The trade-off underlying the existence of Duverger's Law equilibria (and the non-existence of the sincere voting equilibrium) is the same as before: majority voters vote for the strong majority candidate in order to avoid the risk of an upset victory of C in the second round. The difference is that majority voters may now have to ensure that the stronger majority candidate obtains a large enough margin of victory. This new requirement can influence majority voters incentives in two ways. On the one hand, this requirement may strengthen the incentives of majority voters to coordinate. Indeed, it may be more likely that the strong majority candidate falls short of one vote to pass the margin of victory than she falls short of one vote to rank above C (i.e. $mag(piv_{B-C}^{1,VM}) > mag(piv_{B/BC}^{1})$. On the other hand, if the victory margin requirement is so demanding that it is almost impossible to satisfy, i.e. β is too high, then the new requirement weakens the incentives of majority voters to coordinate. Majority voters prefer not to coordinate if it is improbably that the strong majority candidate will win outright.[26] The extension of the second part of Theorem 2 to runoff elections with victory margin requirement follows directly from this argument.

The reason explaining the Ortega effect is also the same as without a victory margin

[26] Details on the conditions under which Duverger's law equilibria exist in runoff elections with victory margin requirements are available upon request.

requirement: majority voters divide their votes in the first round because they realize that if C does not win outright in the first round this is because a second round is held (and not because one majority candidate defeats her directly). Nonetheless, by definition the victory margin requirement imposes an additional constraint for an outright victory of C in the first round: $\tau_C^1 - \max\{\tau_A^1, \tau_C^1\} > \beta$. This new constraint restricts the set of parameters for which the Ortega effect exist: $r(t''_{CA}) - r(t_{AB}) > \beta$ has to be satisfied.[27]

7 Conclusion

This paper analyzed the voting equilibria in three-candidate runoff elections. I proposed a new model of three-candidate runoff elections which included two new features. First, voters participating in the two rounds are not necessarily the same. This implies a positive and endogenous risk of upset victory in the second round. Second, the model allowed for many different types of runoff systems: any threshold for first round victory between 0% and 100% as well as more sophisticated rules, e.g. moving thresholds and victory margin requirements. I demonstrated three main results: (i) runoff elections produce multiple *Duverger's Law* equilibria in which only two candidates obtain a positive fraction of the votes, (ii) the sincere voting equilibrium does not always exist, and (iii) the *Ortega effect* may lead to the systematic victory of the Condorcet loser in runoff elections with a threshold below 50%.

Though relatively general, the analysis is arguably stylized on two dimensions. First, voters of a same type have the same preference intensities. Relaxing this assumption should not change the main results. In Duverger's Law equilibria, as long as the risk of upset victory is large enough, some voters are willing to abandon their most preferred candidate in order to avoid the victory of their least preferred candidate. For a sufficiently large electorate, this is true no matter the intensity of their preferences.[28] In the equilibrium sustaining the Ortega effect, all majority voters vote for the candidate they prefer. Preferring a candidate more intensely cannot affect such strategies nor the outcome they

[27] Proof available upon request.

[28] Actually, in a setup with heterogenous preferences, Bouton and Gratton (2011) prove the existence of Duverger's law equilibria in runoff elections with a threshold at 50%.

imply. Second, there are "only" three candidates. With respect to voters' behavior and the number of serious candidates in equilibrium, i.e. candidates receiving a positive fraction of votes, this assumption should be innocuous. Indeed, voters' strategic incentives imply that there are at most three serious candidates in runoff elections: the candidate expected to rank fourth would indeed be abandoned by her supporters given that she could not qualify for the second round.[29] My model can thus be seen as a "reduced form" of a model with more candidates. Nonetheless, by doing so I exclude the possibility of analyzing how the three serious candidate are selected out of a set of more candidates. This selection certainly suffers from coordination problems that might lead to inefficient outcomes. This is an interesting avenue for future research.

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Appendices

Appendix A1 provides a reminder of some fundamental properties of Poisson games (Myerson 2000 and 2002). Appendices A2 and A3 demonstrate the claims made in Sections 3 and 4 respectively.

Appendix A1: Large Poisson Games in Runoff Elections

In a Poisson game, population size follows a Poisson distribution of mean n. Since types are attributed by i.i.d. draws, the number of voters of each type also follows a Poisson distribution of mean $n\, r(t)$, and, as shown by Myerson (2000), the number of ψ-votes in round ρ follows a Poisson distribution of mean $n\ \tau_\psi^\rho$:

$$\Pr\left(x_\psi^\rho\right) = \exp\left(-\tau_\psi^\rho\ n\right)\ \frac{\left(\tau_\psi^\rho\ n\right)^{x_\psi^\rho}}{x_\psi^\rho!}. \tag{17}$$

The *action profile* of a group of players is the vector that lists, for each action ψ, the number of players in the group who are choosing action c. I denote by x^ρ an action profile in round ρ. The set of possible action profiles for the players in round ρ is $Z\left(\Psi^\rho\right)$, i.e. $Z\left(\Psi^\rho\right)$ is the set of vectors $x^\rho = \left\{x_\psi^\rho\right\}_{\psi\in\Psi^\rho}$. From (17), the probability that the action profile is x^ρ is:

$$\Pr\left(x^\rho\right) = \prod_{\psi\in\Psi^\rho}\left(\frac{\exp\left(-\tau_\psi^\rho\ n\right)\left(\tau_\psi^\rho\ n\right)^{x_\psi^\rho}}{x_\psi^\rho!}\right). \tag{18}$$

An event A^ρ in round ρ is a set of action profiles that satisfy given constraints, i.e. it is a subset of $Z\left(\Psi^\rho\right)$. As shown by Myerson (2000, Theorem 1), it follows from (18) that:

Property 1 *For a large population of size n, the probability of an event A^ρ is such that*

$$mag\left(A^\rho\right) \equiv \lim_{n\to\infty}\frac{\log\left[\Pr\left(A^\rho\right)\right]}{n} = \max_{x^\rho\in A}\sum_\psi \frac{x_\psi^\rho}{n}\left(1-\log\left(\frac{x_\psi^\rho}{n\tau_\psi^\rho}\right)\right)-1.$$

That is, the probability that event A^ρ occurs is exponentially decreasing in n. $mag\left(A^\rho\right)\in[-1,0]$ is called the magnitude *of event A^ρ. Its absolute value represents the “speed” at which the probability decreases towards 0: the more negative is the magnitude, the faster the probability goes to 0.*

Myerson (2000, Corollary 1) shows that:

Property 2 *Compare two events with different magnitudes: mag* $(A^{\rho}) <$ *mag* $(A^{\rho\prime})$*. Then, the probability ratio of the former over the later event goes to zero as n increases:*

$$mag\left(A^{\rho}\right) < mag\left(A^{\rho\prime}\right) \Longrightarrow \frac{\Pr\left(A^{\rho}\right)}{\Pr\left(A^{\rho\prime}\right)} \underset{n\to\infty}{\longrightarrow} 0.$$

Together, Properties 1 and 2 have been called the *magnitude theorem* by Myerson (2000). The intuition is that the probabilities of different events do not converge towards zero at the same speed. Hence, unless two events have the same magnitude, their likelihood ratio converges either to zero or to infinity when the electorate grows large.[30] Proofs in this paper rely extensively on these two properties.

Appendix A2: Proofs for Section 3

Proof. As detailed in Property 1 (in Appendix A1), the magnitude of the event that candidates P and Q have exactly the same number of vote is:

$$\begin{aligned} \lim_{n\to\infty} \tfrac{\log\left[\Pr\left(x_P^2 = x_Q^2\right)\right]}{n} &= \max_{x^2} \sum_{\psi} \frac{x_\psi^2}{n}\left(1 - \log\frac{x_\psi^2}{n\tau_\psi^2}\right) - 1 \\ & \quad s.t.\ x_P^2 = x_Q^2 \end{aligned} \tag{19}$$

If we denote $x_P^{\rho} = x_Q^{\rho} = x$, we find that this is maximized in $x^* = n\sqrt{\tau_P^2 \tau_Q^2}$. Substituting for x^* in (19) thus yields:

$$\lim_{n\to\infty} \tfrac{\log\left[\Pr\left(x_P^2 = x_Q^2\right)\right]}{n} = -\left(\sqrt{\tau_P^2} - \sqrt{\tau_Q^2}\right)^2.$$

The event that candidates P and Q have exactly the same number of vote is the pivotability QP, i.e. piv_{QP}^2. The event that candidate P trails behind candidate Q by exactly one vote is the pivotability PQ, i.e. piv_{PQ}^2. (Notice the difference between piv_{PQ}^2 and piv_{QP}^2 which follows from the alphabetical order tie breaking rule).

From Myerson (2000, Theorem 2), we have that $mag(piv_{PQ}^2)$ and $mag(piv_{QP}^2)$ are equal:

$$\lim_{n\to\infty} \tfrac{\log\left[\Pr\left(x_P^2 = x_Q^2 - 1\right)\right]}{n} = \lim_{n\to\infty} \tfrac{\log\left[\Pr\left(x_P^2 = x_Q^2\right)\right]}{n}.$$

■

Proof of Lemma 2. There are three types of magnitudes to compute. I only present the details for the magnitude of the threshold pivot probabilities i/ij and ij/i. The other cases are derived in a similar fashion (and available upon request).

A ballot is threshold pivotal i/ij when $x_i^1 + 1 > \zeta\left(x_i^1 + x_j^1 + x_k^1\right) \geq x_i^1 \geq x_j^1 \geq x_k^1$. From Myerson (2000,

[30] These properties are quite general and not specific to the Poisson distribution. This is the reason why most of the results extend directly to the multinomial distribution.

Theorem 2), I know that I can focus on the case $\zeta\left(x_i^1+x_j^1+x_k^1\right) = x_i^1 \geq x_j^1 \geq x_k^1$ without loss of generality. Applying Property 1 (in Appendix A1), I have:

$$mag(piv_{i/ij}^1) = \max_x \sum_\psi \frac{x_\psi^1}{n}\left(1-\log(\frac{x_\psi^1}{n\tau_\psi^1})\right)-1 \tag{20}$$

$$s.t. \begin{cases} x_i^1 = \zeta\left(x_i^1+x_j^1+x_k^1\right) \\ x_i^1 \geq x_j^1 \geq x_k^1 \end{cases} \tag{21}$$

If I denote $x_j^1+x_k^1 = x_{i/ij}^1$, $x_j^1 = \alpha_{i/ij}x_{i/ij}^1$, and $x_k^1 = \left(1-\alpha_{i/ij}\right)x_{i/ij}^1$, and if I abstract from the second constraint (or if it is not binding) in (21), I find that this is maximized in

$$\begin{aligned} x_{i/ij}^{1*} &= \left(\frac{(1-\zeta)\tau_i^1}{\zeta}\right)^\zeta\left(\frac{\tau_j^1}{\alpha_{i/ij}}\right)^{\alpha_{i/ij}}\left(\frac{\tau_k^1}{1-\alpha_{i/ij}}\right)^{1-\alpha_{i/ij}}, \\ \alpha_{i/ij}^* &= \frac{\tau_j^1}{\tau_k^1+\tau_j^1}. \end{aligned}$$

Substituting for $x_{i/ij}^{1*}$ and $\alpha_{i/ij}^*$ in (20) yields what I call the *unconstrained* magnitude (denoted by the superscript *):

$$mag(piv_{i/ij}^{1,*}) = \left(\frac{\tau_j^1+\tau_k^1}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau_i^1}{\zeta}\right)^\zeta - 1. \tag{22}$$

The magnitude of the threshold pivot probability i/ij is unconstrained if

$$\frac{\zeta}{1-\zeta} \geq \frac{\tau_j^1}{\tau_j^1+\tau_k^1} \geq \frac{1}{2}. \tag{23}$$

From (22) and (23), I have that

$$mag(piv_{i/ij}^1) = mag(piv_{i/ij}^{1*}) \text{ if } \frac{\zeta}{1-\zeta} \geq \frac{\tau_j^1}{\tau_j^1+\tau_k^1} \geq \frac{1}{2}.$$

I still have to compute $mag(piv_{i/ij}^1)$ when (23) is not satisfied. From $\zeta \in \left[\frac{1}{3},1\right)$, I have that $\frac{\zeta}{1-\zeta} \geq \frac{1}{2}$ and then there are two other possible cases: (i) $\frac{\tau_j^1}{\tau_j^1+\tau_k^1} > \frac{\zeta}{1-\zeta} \geq \frac{1}{2}$, and (ii) $\frac{\zeta}{1-\zeta} \geq \frac{1}{2} > \frac{\tau_j^1}{\tau_j^1+\tau_k^1}$.

In case (i), the constraint $x_i^1 \geq x_j^1$ is binding. I thus bind the constraint, i.e. set $\alpha_{i/ij} = \frac{\zeta}{(1-\zeta)}$, and maximize the same problem as in (20). This yields:

$$mag(piv_{i/ij}^1) = \left(\frac{\sqrt{\tau_i^1\tau_j^1}}{\zeta}\right)^{2\zeta}\left(\frac{\tau_k^1}{1-2\zeta}\right)^{1-2\zeta} - 1 \text{ if } \frac{\tau_j^1}{\tau_j^1+\tau_k^1} > \frac{\zeta}{1-\zeta} \geq \frac{1}{2}.$$

In case (ii), the constraint $x_j^1 \geq x_k^1$ is binding. I thus bind the constraint, i.e. set $\alpha_{i/ij} = 1/2$, and maximize the same problem as in (20). This yields:

$$mag(piv_{i/ij}^1) = \left(\frac{2\sqrt{\tau_j^1\tau_k^1}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau_i^1}{\zeta}\right)^\zeta - 1 \text{ if } \frac{\zeta}{1-\zeta} \geq \frac{1}{2} > \frac{\tau_j^1}{\tau_j^1+\tau_k^1}.$$

I have then proven that $mag(piv_{i/ij}^1)$ is as defined in (3). From Myerson (2000, Theorem 2), I have that $mag(piv_{ij/i}^1) = mag(piv_{i/ij}^1)$. ■

Appendix A3: Proofs for Section 4

Proof of Theorem 1. The proof is in three parts. I identify sufficient conditions for Duverger's Law equilibria to exist when (i) $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right)$, (ii) $\zeta = \frac{1}{2}$, and (iii) $\zeta \in \left[\frac{1}{2}, 1\right)$. Since the proofs of existence of the two Duverger's Law equilibria, i.e. $\sigma(A|t_{AB}) = \sigma(A|t_{BA}) = \sigma(C|t''_{CA}) = 1$ and $\sigma(B|t_{AB}) = \sigma(B|t_{BA}) = \sigma(C|t''_{CA}) = 1$, are similar, I only produce the proof for the case in which all majority types vote for A.

A sufficient condition for the Duverger's Law equilibrium $\sigma(A|t_{AB}) = \sigma(A|t_{BA}) = \sigma(C|t''_{CA}) = 1$ to exist is that $\forall t \in \{t_{AB}, t_{BA}\}$

$$G^1(A|t)/G^1(B|t) \underset{n\to\infty}{\to} \infty \text{ and} \tag{24}$$

$$G^1(A|t)/G^1(C|t) \underset{n\to\infty}{\to} \infty. \tag{25}$$

(i) Duverger's Law equilibria when $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right)$:

From (7) and Property 2, I have that a sufficient condition for (24) and (25) to be satisfied is

$$\begin{aligned} mag(piv^1_{A/C}) &> mag(piv^1_{i/j})\ \forall\{i,j\} \neq \{A,C\} \text{ and } \{C,A\}, \\ mag(piv^1_{A/C}) &> mag(piv^1_{ki/kj})\ \forall i,j, \\ mag(piv^1_{A/C}) &> mag(piv^1_{i/ij})\ \forall i,j. \end{aligned}$$

For $\sigma(A|t_{AB}) = \sigma(A|t_{BA}) = \sigma(C|t''_{CA}) = 1$, I have that $\tau^1_A = 1 - r(t''_{CA})$ and $\tau^1_B = 0$. From Lemma 2, this implies that:

$$mag\left(piv^1_{A/C}\right) = mag\left(piv^1_{C/A}\right) = \left(\frac{\tau^1_B + \tau^1_C}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_A}{\zeta}\right)^{\zeta} - 1 = \left(\frac{r(t''_{CA})}{1-\zeta}\right)^{1-\zeta}\left(\frac{1-r(t''_{CA})}{\zeta}\right)^{\zeta} - 1$$

is the only magnitude that can be larger than -1. Therefore, I have that (24) and (25) are both satisfied. This is true for any $0 < r(t''_{CA}) < 1/2$.

(ii) Duverger's Law equilibria when $\zeta = \frac{1}{2}$:

For $\sigma(A|t_{AB}) = \sigma(A|t_{BA}) = \sigma(C|t''_{CA}) = 1$, I have $\tau^1_A = 1 - r(t''_{CA})$ and $\tau^1_B = 0$. From Lemma 2, this implies that:

$$mag\left(piv^1_{A/AC}\right) = mag\left(piv^1_{AC/A}\right) = -\left(\sqrt{\tau^1_A} - \sqrt{\tau^1_C}\right)^2 = -\left(\sqrt{1 - r(t''_{CA})} - \sqrt{r(t''_{CA})}\right)^2$$

is the only magnitude that can be larger than -1.[31] Remind that a ballot cannot be above-threshold pivotal when $\zeta \geq \frac{1}{2}$.

Nonetheless, (24) and (25) are not necessarily satisfied when $mag\left(piv^1_{A/AC}\right)$ is the largest magnitude. Indeed, when threshold pivotal A/AC, the expected payoff of a type t voter is

$$[U(A|t) - U(A,C|t)] = \Pr(C|\{A,C\}) \underset{n\to\infty}{\to} 0.$$

[31] Note that $mag\left(piv^1_{C/CA}\right) = mag\left(piv^1_{CA/C}\right) = -1$ but this has no influence on the choice of whether voting for A or B for t_{AB} and t_{BA} voters.

Indeed, if the second round opposes A to C, and A wins that round, then being first-round pivotal has no value. This is why being first-round pivotal is only valuable with probability $\Pr(C|\{A,C\})$.

Since the magnitude of all pivot probabilities other than $piv^1_{A/AC}$ are equal to minus one, a sufficient condition for (24) and (25) to be satisfied is

$$mag(piv^1_{A/AC}\cdot \Pr(C|\{A,C\})) > -1.$$

Since the distribution of A and C votes are identical in the first and second round, I have that:

$$\Pr(C|\{A,C\}) \geq \Pr\left(piv^1_{A/AC}\right)$$

and then that

$$mag(piv^1_{A/AC}\cdot \Pr(C|\{A,C\})) \geq 2mag(piv^1_{A/AC}).$$

Therefore, no voter deviates from $\sigma^1(A|t_{AB}) = \sigma^1(A|t_{BA}) = \sigma^1(C|t''_{CA}) = 1$ if:

$$2mag\left(piv^1_{A/AC}\right) > -1.$$

From $mag\left(piv^1_{A/AC}\right) = -\left(\sqrt{1-r(t''_{CA})} - \sqrt{r(t''_{CA})}\right)^2$, this condition boils down to:

$$\sqrt{1-r(t''_{CA})} - \sqrt{r(t''_{CA})} < \sqrt{1/2},$$

or: $r(t''_{CA}) > 0.06699$.

(iii) Duverger's Law equilibria when $\zeta \in \left(\frac{1}{2},1\right)$:

For $\sigma(A|t_{AB}) = \sigma(A|t_{BA}) = \sigma(C|t''_{CA}) = 1$, , I have $\tau^1_A = 1 - r(t''_{CA})$ and $\tau^1_B = 0$. From Lemma 2 this implies that:

$$\begin{aligned} mag\left(piv^1_{A/AC}\right) &= mag\left(piv^1_{AC/A}\right) = \left(\frac{\tau^1_B+\tau^1_C}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_A}{\zeta}\right)^{\zeta} - 1 \\ &= \left(\frac{r(t''_{CA})}{1-\zeta}\right)^{1-\zeta}\left(\frac{1-r(t''_{CA})}{\zeta}\right)^{\zeta} - 1 \end{aligned} \tag{26}$$

is the only magnitude that can be larger than -1.[32] Remind that a ballot cannot be above the threshold pivotal when $\zeta \geq \frac{1}{2}$.

As in point (ii), since the magnitude of all pivot probabilities other than $piv^1_{A/AC}$ are equal to minus one, a sufficient condition for $\sigma^1(A|t_{AB}) = \sigma^1(A|t_{BA}) = \sigma^1(C|t''_{CA}) = 1$ to be an equilibrium is that

$$mag\left(\Pr(C|\{A,C\})\right) + mag\left(piv^1_{A/AC}\right) > -1.$$

From (26) and knowing that $mag\left(\Pr(C|\{A,C\})\right) \geq 2\sqrt{(1-r(t''_{CA}))\, r(t''_{CA})} - 1$, this condition is satisfied when:

$$2\sqrt{(1-r(t''_{CA}))\, r(t''_{CA})} + \left(\frac{r(t''_{CA})}{1-\zeta}\right)^{1-\zeta}\left(\frac{1-r(t''_{CA})}{\zeta}\right)^{\zeta} \geq 1. \tag{27}$$

[32] Note that $mag\left(piv^1_{C/CA}\right) = mag\left(piv^1_{CA/C}\right) = -1$ but this has no influence on the choice of whether voting for A or B for t_{AB} and t_{BA} voters.

Knowing that

$$\frac{\partial\left(\left(\frac{1-r(t''_{CA})}{1-\zeta}\right)^{1-\zeta}\left(\frac{r(t''_{CA})}{\zeta}\right)^{\zeta}\right)}{\partial\zeta}=\left(\frac{r(t''_{CA})}{1-\zeta}\right)^{1-\zeta}\left(\frac{1-r(t''_{CA})}{\zeta}\right)^{\zeta}\log\left(\frac{1-\zeta}{\zeta}\frac{r(t''_{CA})}{1-r(t''_{CA})}\right)$$

I have that

$$\begin{aligned}\frac{\partial\left(\left(\frac{1-r(t''_{CA})}{1-\zeta}\right)^{1-\zeta}\left(\frac{r(t''_{CA})}{\zeta}\right)^{\zeta}\right)}{\partial\zeta} &> 0 \text{ if } \zeta<1-r(t''_{CA}),\\ &= 0 \text{ if } \zeta=1-r(t''_{CA}),\\ &< 0 \text{ if } \zeta>1-r(t''_{CA}),\end{aligned}$$

and then that

$$\min_{\zeta}\left(\frac{r(t''_{CA})}{1-\zeta}\right)^{1-\zeta}\left(\frac{1-r(t''_{CA})}{\zeta}\right)^{\zeta}>\min\left\{\underbrace{2\sqrt{(1-r(t''_{CA}))\,r(t''_{CA})}}_{\text{for }\zeta=1/2},\underbrace{1-r(t''_{CA})}_{\text{for }\zeta=1}\right\}.$$

There are then two cases to consider: (i) $2\sqrt{(1-r(t''_{CA}))\ r(t''_{CA})}<1-r(t''_{CA})$ and (ii) $2\sqrt{(1-r(t''_{CA}))\,r(t''_{CA})}\geq 1-r(t''_{CA})$. In case (i) I have that $\min\left\{2\sqrt{(1-r(t''_{CA}))\,r(t''_{CA})},1-r(t''_{CA})\right\}=2\sqrt{(1-r(t''_{CA}))\,r(t''_{CA})}$ and then that

$$\left(\frac{1-r(t''_{CA})}{1-\zeta}\right)^{1-\zeta}\left(\frac{r(t''_{CA})}{\zeta}\right)^{\zeta}>2\sqrt{(1-r(t''_{CA}))\,r(t''_{CA})}. \tag{28}$$

Knowing that if $r(t''_{CA})>0.06699$, then $2\sqrt{(1-r(t''_{CA}))\,r(t''_{CA})}>\frac{1}{2}$, I have from (28) that (27) is satisfied if $r(t''_{CA})>Z$ with $Z<0.06699$. In case (ii) I have that $\min\left\{2\sqrt{(1-r(t''_{CA}))\,r(t''_{CA})},1-r(t''_{CA})\right\}=1-r(t''_{CA})$. Since $(1-r(t''_{CA}))>\frac{1}{2}$, both $\left(\frac{1-r(t''_{CA})}{1-\zeta}\right)^{1-\zeta}\left(\frac{r(t''_{CA})}{\zeta}\right)^{\zeta}$ and $2\sqrt{(1-r(t''_{CA}))\,r(t''_{CA})}$ are larger than $\frac{1}{2}$. Therefore, (27) is always strictly satisfied. ■

Proof of Theorem 2. The proof is in two parts: (i) I show that the sincere voting may exist, and (ii) that it does not always exist.

(i) Existence of the sincere voting equilibrium:

For $\sigma^1(A|t_{AB})=\sigma^1(B|t_{BA})=\sigma^1(C|t''_{CA})=1$, I have from Lemma 2 that $mag(piv_{CA/CB})=0$ when $r(t_{AB})=r(t_{BA})<r(t''_{CA})\leq\zeta$. From Property 1, I know that the magnitude of any event A^1 is bounded by -1 and 0: $mag(A^1)\in[-1,0]\ \forall A^1$. Together with $\Pr(A|\{A,C\})=\Pr(B|\{B,C\})\underset{n\to\infty}{\to}1$, (7) and Property 2, this implies that there is a n sufficiently large such that $G^1(A|t_{AB})-G^1(B|t_{AB})>0>G^1(A|t_{BA})-G^1(B|t_{BA})$. Since t''_{CA}-voters always vote for C in the first round, we have that $\sigma^1(A|t_{AB})=\sigma^1(B|t_{BA})=\sigma^1(C|t''_{CA})=1$ is an equilibrium. From the continuity of pivot probabilities (and then of $G^1(A|t)-G^1(B|t)$) in the expected vote shares, I have that the sincere voting equilibrium is not non-generic.

(ii) Non-existence of the sincere voting equilibrium:

Straightforward from Theorem 1 and the continuity of the pivot probabilities (hence the payoffs) in the expected vote shares. ■

Proof of Theorem 3. First, I show that $mag(piv^1_{CA/CB}) = 0$ when $\tau^1_A = \tau^1_B < \tau^1_C = \zeta$, where $\zeta \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Second, I show that $\sigma^1(A|t_{AB}) = \sigma^1(B|t_{BA}) = \sigma^1(C|t''_{CA}) = 1$ are equilibrium strategies when $r(t_{AB}) = r(t_{BA})$, $r(t''_{CA}) = \zeta$ and $\frac{r(t_{AB})+r(t_{BA})}{2} < \zeta$. Third, I show that there always exist $\varepsilon > 0$ and $\kappa > 0$ such that, if $r(t_{AB}) - r(t_{BA}) < \varepsilon$ and $\zeta < \tau_C < \zeta + \kappa$ then $\sigma^1(A|t_{AB}) = \sigma^1(B|t_{BA}) = \sigma^1(C|t''_{CA}) = 1$ is an equilibrium.

From $\tau^1_C = \zeta > \tau^1_A = \tau^1_B$, I have that

$$\tau^1_C \geq 2\frac{\zeta}{1-\zeta}\sqrt{\tau^1_A\tau^1_B} \geq \sqrt{\tau^1_A\tau^1_B}.$$

From this and Lemma 2, I have:

$$mag(piv^1_{CA/CB}) = \left(\frac{2\sqrt{\tau^1_A\tau^1_B}}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_C}{\zeta}\right)^{\zeta} - 1.$$

Substituting for $\tau^1_C = \zeta$ and $\tau^1_A = \tau^1_B$, I have that

$$mag(piv^1_{CA/CB}) = \left(\frac{1-\tau^1_C}{1-\zeta}\right)^{1-\zeta}\left(\frac{\tau^1_C}{\zeta}\right)^{\zeta} - 1 = 0.$$

For $\sigma^1(A|t_{AB}) = \sigma^1(B|t_{BA}) = \sigma^1(C|t''_{CA}) = 1$, I have $\tau^1_A = r(t_{AB})$, $\tau^1_B = r(t_{BA})$ and $\tau^1_C = r(t''_{CA})$. Thus, $mag(piv^1_{CA/CB}) = 0$ if

$$\begin{aligned} r(t_{AB}) &= r(t_{BA}), \\ r(t''_{CA}) &= \zeta, \text{ and} \\ \frac{r(t_{AB})+r(t_{BA})}{2} &< \zeta. \end{aligned} \tag{29}$$

From Property 1, I know that the magnitude of any event A^1 is bounded by -1 and 0: $mag(A^1) \in [-1, 0]$ $\forall A^1$. Therefore, $mag(piv^1_{CA/CB})$ is the largest magnitude when conditions in (29) are satisfied except if there are other magnitudes that equal 0. From Lemma 2, it can be checked easily (but tediously) that this is the case for two other magnitudes: $mag(piv^1_{C\backslash CA}) = 0$ and $mag(piv^1_{C\backslash CB}) = 0$. Nonetheless, from (7) I have that neither $\Pr\left(piv^1_{C\backslash CA}\right)$ nor $\Pr\left(piv^1_{C\backslash CB}\right)$ influence types-t_{AB} and -t_{BA} choice between A and B. Therefore, types t_{AB} prefer to vote for A and types t_{BA} prefer to vote for B, i.e. $G^1(A|t_{AB}) - G^1(B|t_{AB}) > 0 > G^1(A|t_{BA}) - G^1(B|t_{BA})$, when conditions in (29) are satisfied. Types-t''_{CA} always prefer to vote for C.

Since all magnitudes are continuous in τ^1_A, τ^1_B and τ^1_C, there always exist $\varepsilon > 0$ and $\kappa > 0$ such that, if $r(t_{AB}) - r(t_{BA}) < \varepsilon$ and $\zeta < r\left(t''_{CA}\right) < \zeta + \kappa$, then $\sigma(A|t_{AB}) = \sigma(B|t_{BA}) = \sigma(C|t''_{CA}) = 1$ are equilibrium strategies for which C wins outright in the first round with a probability that tends to 1 when $n \to \infty$. ■

Proof of Lemma 3. I only show the proof for $\zeta = 0.5$. The other cases follow directly from the detailed case and the proof of Theorem 1.

Assume without loss of generality that B is the Condorcet winner, i.e.

$$\begin{aligned} B \text{ vs. } A \quad &: \quad r(t_{BC}) + r(t_{BA}) + r(t_{CB}) + r(t'_{BC}) > \\ & \quad r(t_{AB}) + r(t_{AC}) + r(t_{CA}) + r(t'_{AC}), \\ B \text{ vs. } C \quad &: \quad r(t_{BC}) + r(t_{BA}) + r(t_{AB}) + r(t'_{AB}) > \\ & \quad r(t_{CB}) + r(t_{AC}) + r(t_{CA}) + r(t'_{AC}), \end{aligned} \tag{30}$$

and that C is the least popular candidate among A and C:

$$\begin{aligned} & r(t_{AB}) + r(t_{AC}) + r(t_{BA}) + r(t'_{AB}) + r(t'_{AC}) > \\ & r(t_{BC}) + r(t_{CA}) + r(t_{CB}) + r(t'_{BC}). \end{aligned} \tag{31}$$

Suppose that all voters who (weakly) prefer A to C, i.e. $t_{AB}, t_{AC}, t_{BA}, t'_{AB}, t'_{AC}$, vote A, and all voters who (strictly) prefer C to A, i.e. $t_{BC}, t_{CA}, t_{CB}, t'_{BC}, t''_{CA}$, vote C.[33] Then, since $\tau^1_C \in \left(0, \frac{1}{2}\right)$, I have from Lemma 2 that

$$mag\left(piv^1_{A/AC}\right) = 2\sqrt{\tau^1_A \tau^1_C} - 1 = mag\left(piv_{C/CA}\right)$$

are the only magnitudes that can be larger than -1.

From (7), a sufficient condition for this strategy profile to be an equilibrium is that

$$mag(piv^1_{A/AC} \cdot \Pr(C|\{A, C\})) > -1, \tag{32}$$

and

$$mag(piv^1_{C/CA} \cdot \Pr(A|\{A, C\})) > -1 \tag{33}$$

are simultaneously satisfied. Condition (32) ensures that all voters who prefer A to C vote for A and condition (33) ensures that all voters who prefer C to A vote for C. Since $\tau^1_A = \tau^2_A = r(t_{AB}) + r(t_{AC}) + r(t_{BA}) + r(t'_{AB}) + r(t'_{AC})$ and $\tau^1_C = \tau^2_C = r(t_{BC}) + r(t_{CA}) + r(t_{CB}) + r(t'_{BC})$, I know from the proof of Theorem 1 that a sufficient condition for (32) and (33)to be satisfied is:

$$\sqrt{1 - (r(t_{BC}) + r(t_{CA}) + r(t_{CB}) + r(t'_{BC}))} - \sqrt{r(t_{BC}) + r(t_{CA}) + r(t_{CB}) + r(t'_{BC})} < \sqrt{1/2}.$$

This boils down to: $r(t_{BC}) + r(t_{CA}) + r(t_{CB}) + r(t'_{BC}) > 0.06699$.

Finally, $r(t_{BC}) + r(t_{CA}) + r(t_{CB}) + r(t'_{BC}) > 0.06699$ is compatible with (30) and (31). Thus, there may exist a Duverger's law equilibrium in which the Condorcet winner does not receive any vote. ■

[33] Note that the same result could be proven for any strategy of types t'_{AC} not including B.