Swelling of Elastic Materials

Fluids Deforming Solids

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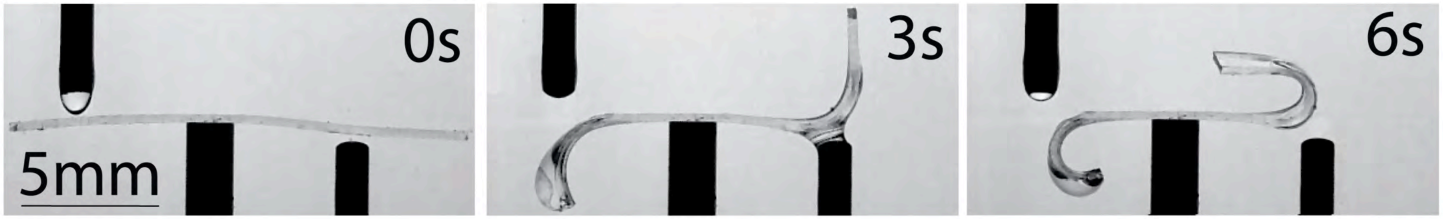


Fluids & Elasticity

Flow through porous medium

- Darcy's Law

Elastic deformation of medium

- Biot's Poroelasticity

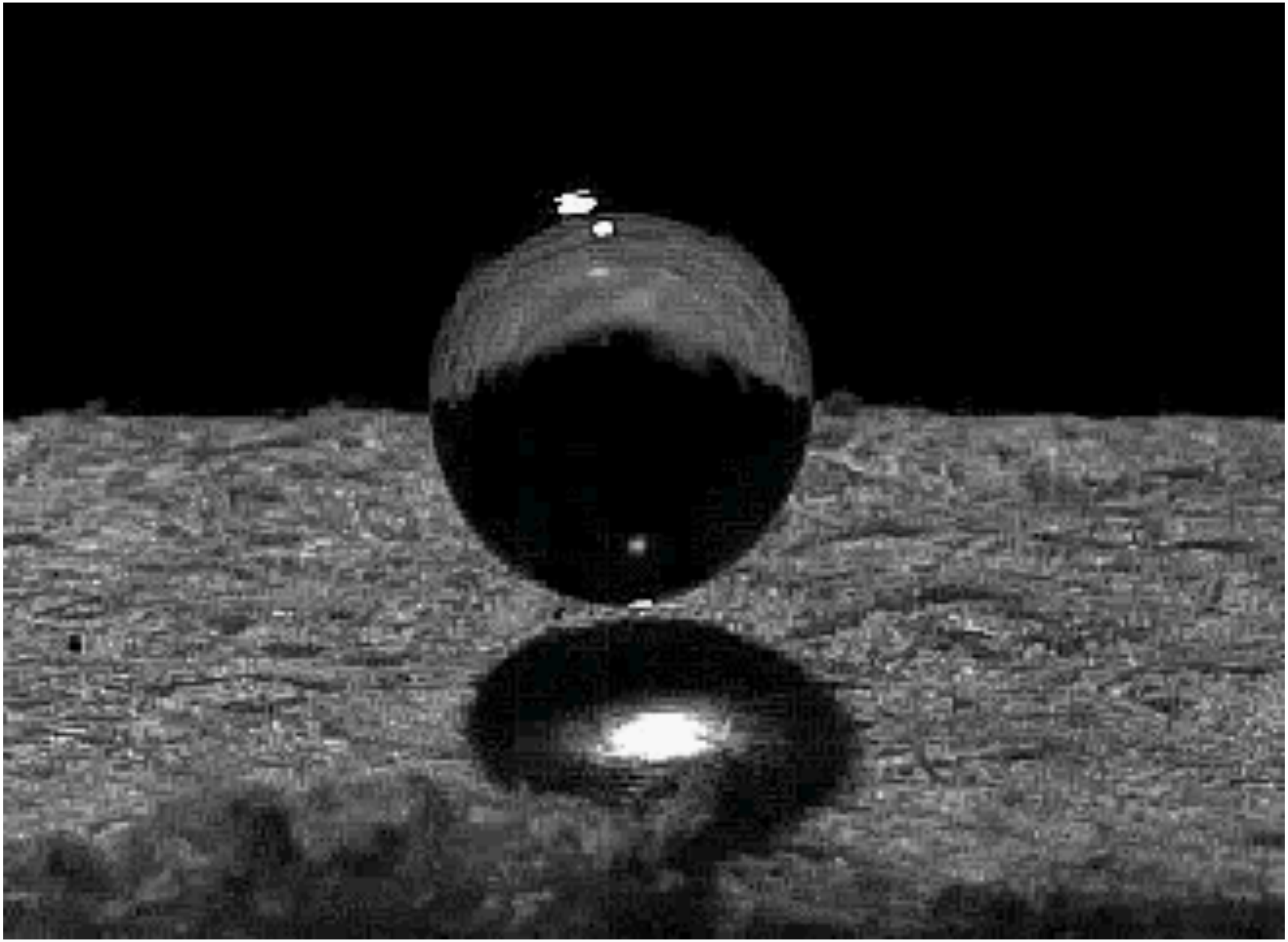
Swelling

- Polymers

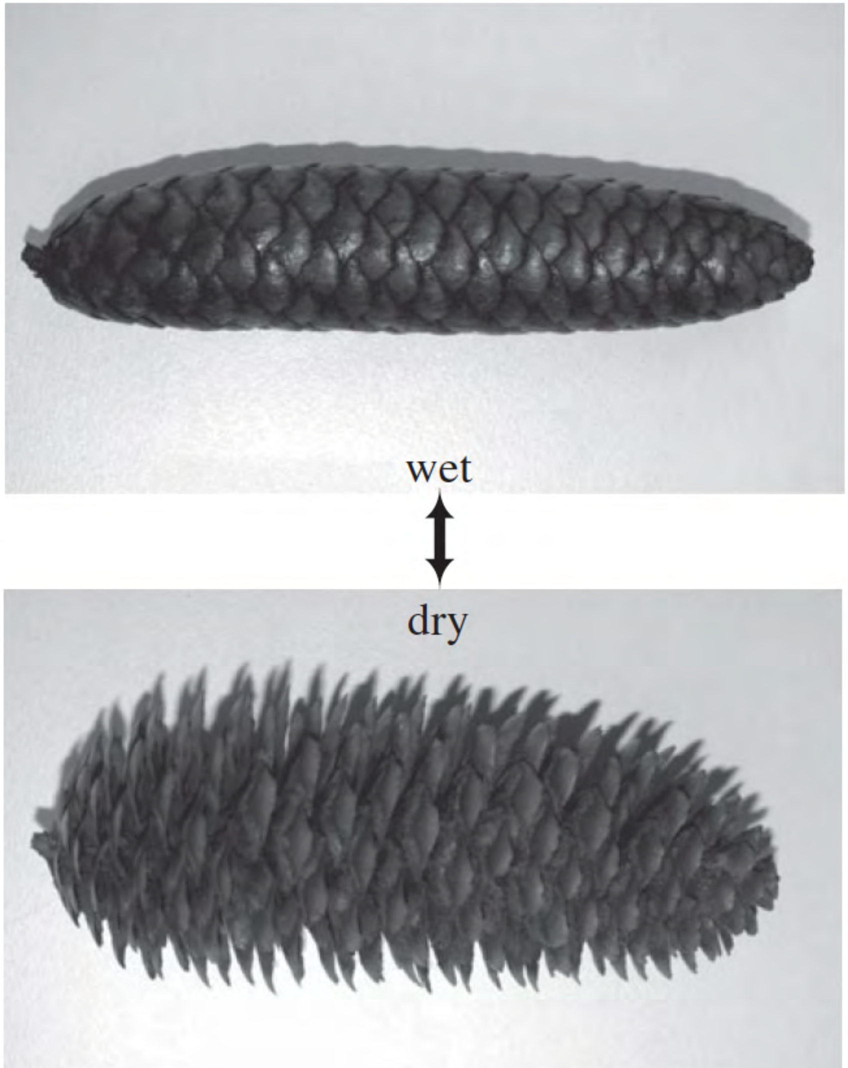
Fluids Deforming Solids

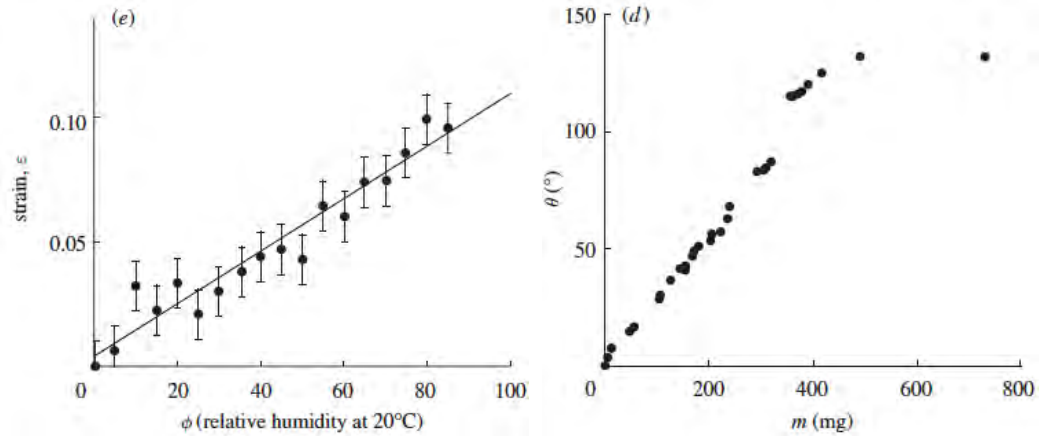
- Surface Tension – Elastocapillarity
- Swelling & Growth
- Maxwell Stresses

Swelling a Sponge



Pine Cones





Tree-bound pine cones:

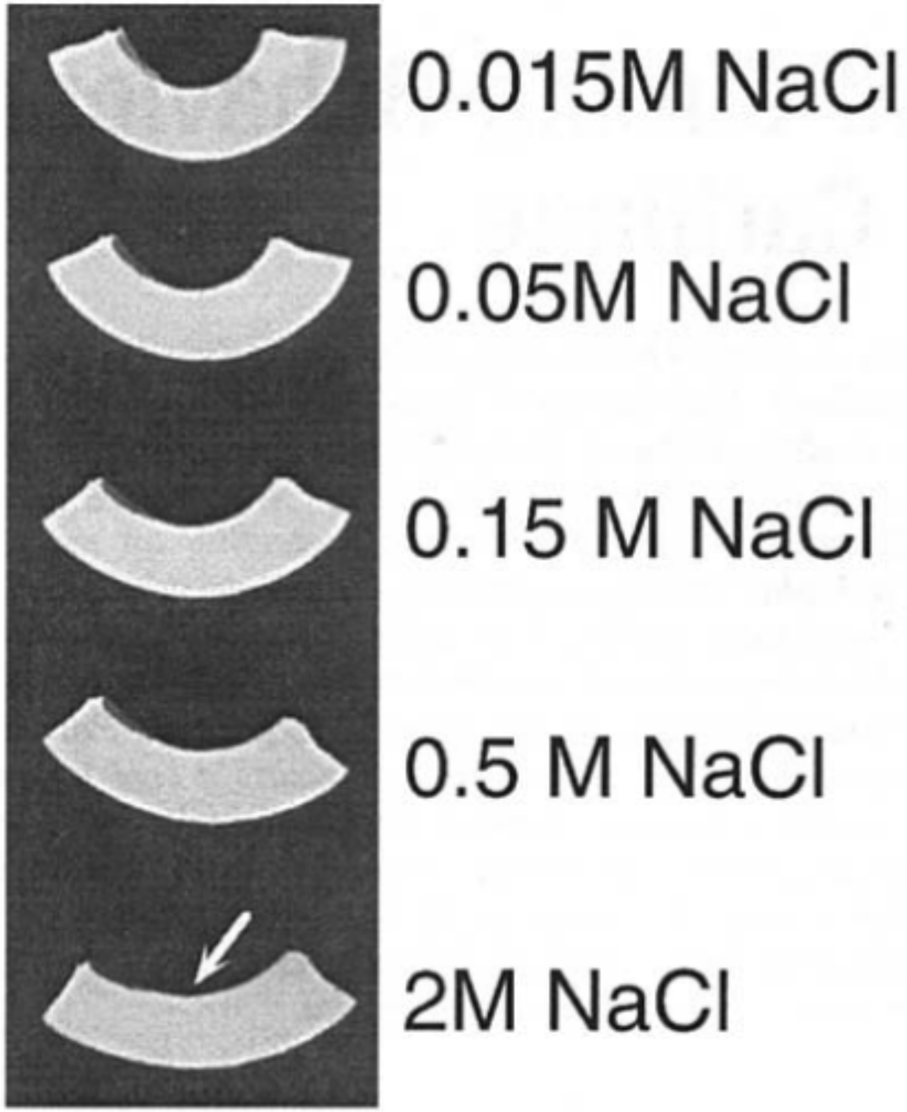
Hydrated & **closed**, protecting seeds

Fallen pine cones:

Dried out & **opened**, releasing seeds

E. Reyssat and L. Mahadevan. *Journal of the Royal Society Interface*, **6**, 951, 2009.

Articular Cartilage



Shape change caused by ion concentration.

Residual strain at physiological conditions: **3-15%**

Tensile prestress in cartilage protective against frequent compresses forces.

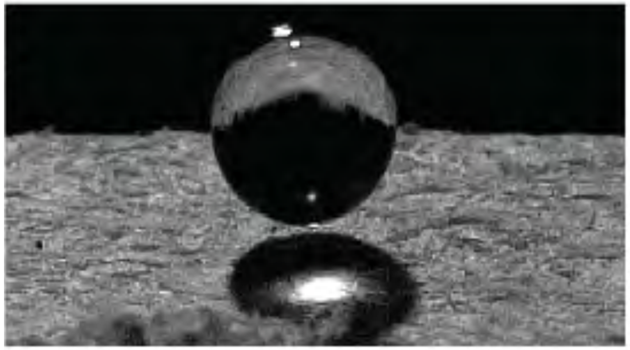
L. A. Setton, H. Tohyama, and V. C. Mow, *Journal of Biomedical Engineering*, **120**, 355, 1998.

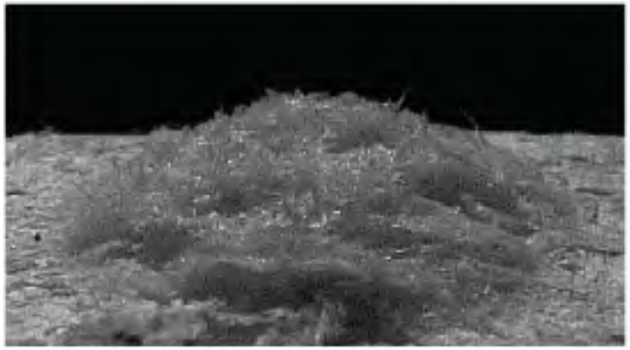
Lichens in the Rain

Francischeefilms, "Lichens time lapse", https://www.youtube.com/watch?v=FWfPMOKnW2M, (2015)

Swelling & Growth

Materials Science





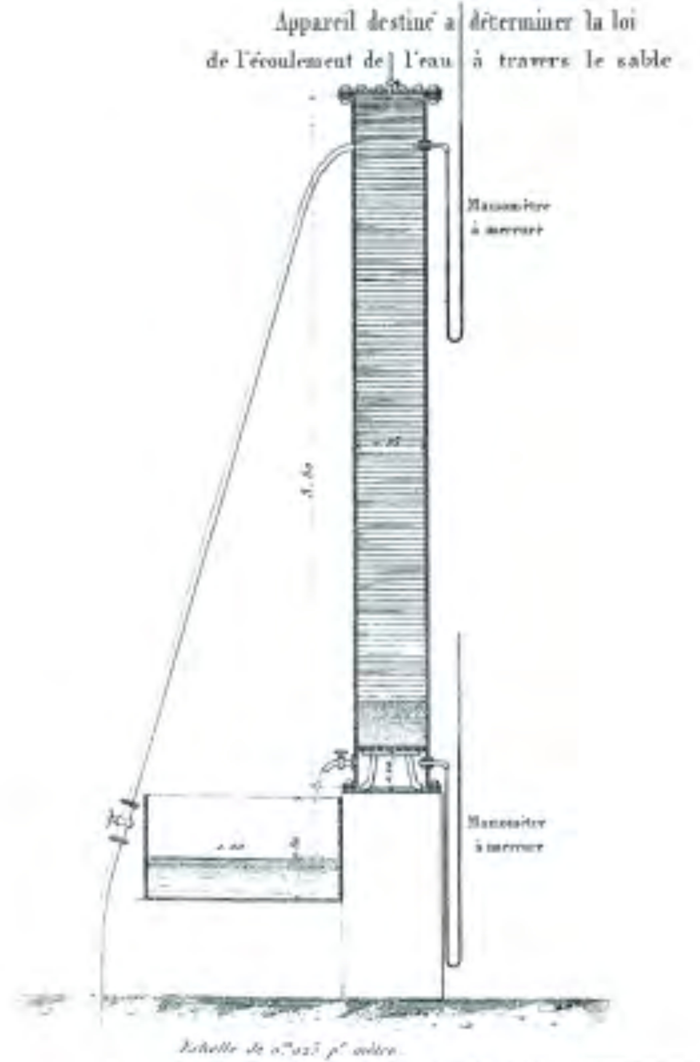
Swelling of a sponge.

Mechanics

An almond leaf which was attacked by *Taphrina Deformans*.

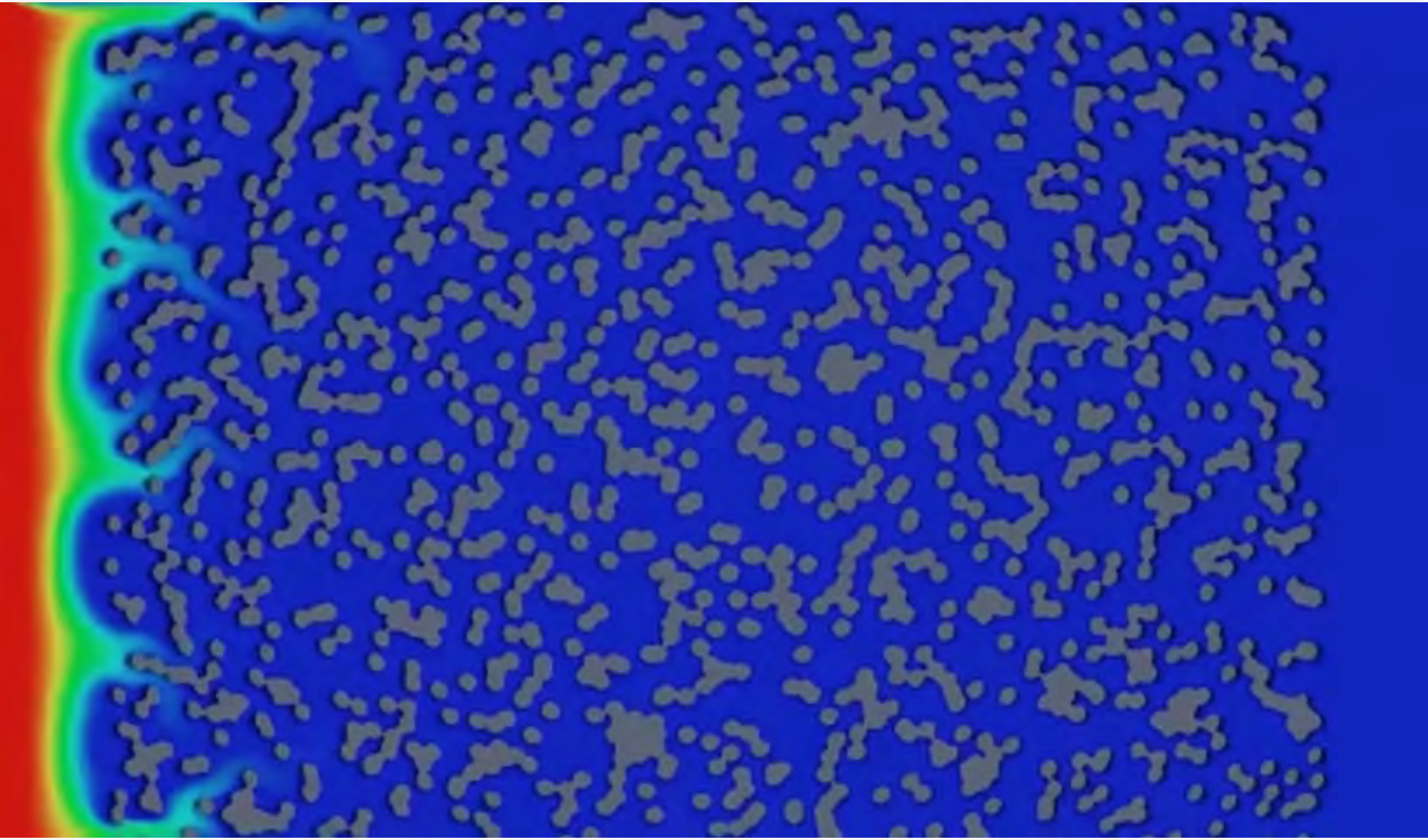
E. Sharon and E. Efrati. *Soft Matter*, **6**, 5693, 2010.

Flow in Porous Media



C. Soulaine, "On the Origin of Darcy's law.", http://web.stanford.edu/~csoulain/PORE_SCALE/Chap1_Darcy.pdf, 2015.

Flow in Porous Media



I. Moumine, https://www.youtube.com/watch?v=qTvHXRT9qt4, 2015.

Fluid Dynamics

Navier-Stokes Equations (momentum conservation)

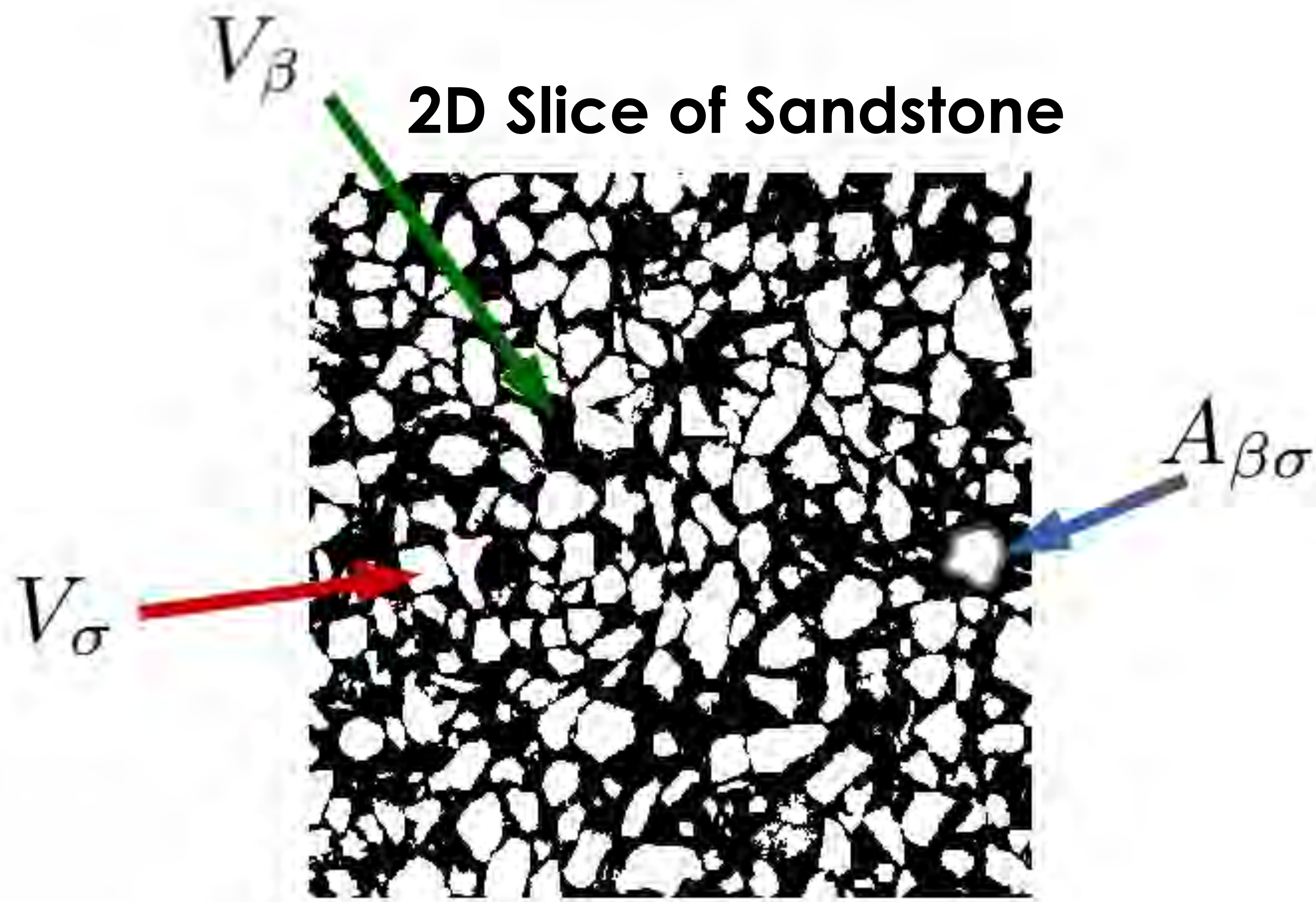
$$\overbrace{\rho\left(\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{variation}} + \underbrace{\mathbf{u}\cdot\nabla\mathbf{u}}_{\text{convection}}\right)}^{\text{Inertial acceleration}} = \overbrace{\underbrace{-\nabla p}_{\text{pressure}} + \underbrace{\mu\nabla^2\mathbf{u}}_{\text{diffusion}} + \underbrace{\mathbf{f}}_{\text{body forces}}}^{\text{Forces}}$$

Continuity Equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

Porous Media

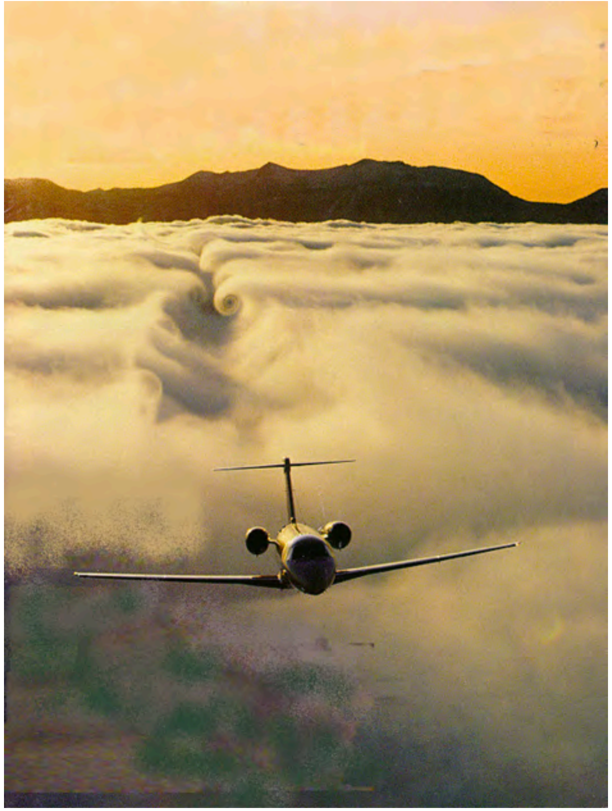
2D Slice of Sandstone



C. Soulaine, "On the Origin of Darcy's law.", http://web.stanford.edu/~csoulain/PORE_SCALE/Chap1_Darcy.pdf, 2015.

Fluid Behavior

Inertial Forces



Trailing airplane vortices

Viscous Forces

Coiling honey

Reynolds Number: inertial/viscous

Airplane: http://eis.bris.ac.uk/~glhmm/gfd/Airplane-ChrisWillcox.jpg
Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg

Fluid Behavior

Inertial Forces

Viscous Forces

Reynolds Number: inertial/viscous

$$\overbrace{\rho\left(\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{variation}}+\mathbf{u}\cdot\nabla\mathbf{u}\right)}^{\text{Inertial acceleration}}=\overbrace{-\nabla p+\underbrace{\mu\nabla^2\mathbf{u}}_{\text{diffusion}}+\mathbf{f}}^{\text{Forces}}$$

Steady inertial forcing due to the convective derivative:

- Time dependence arises from U_0

$$\rho\mathbf{u}\cdot\nabla\mathbf{u}$$

Linear unsteady term sets the inertial time scale to **establish steady flows**:

$$\rho\partial\mathbf{u}/\partial t$$

Time scale estimated by balancing unsteady **inertial force density** with **viscous force density**

$$f_u\sim\rho U_0/\tau_i \qquad f_v\sim\mu U_0/L_0^2$$

$$\tau_i\sim\frac{\rho L_0^2}{\mu}$$

Time required for a vorticity to diffuse a distance L_0, with a diffusivity $\nu=\mu/\rho$, $\tau_i \sim 10\text{ms}$

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
Airplane: http://eis.bris.ac.uk/~glhmm/gfd/Airplane-ChrisWillcox.jpg
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Fluid Behavior

Inertial Forces

Viscous Forces

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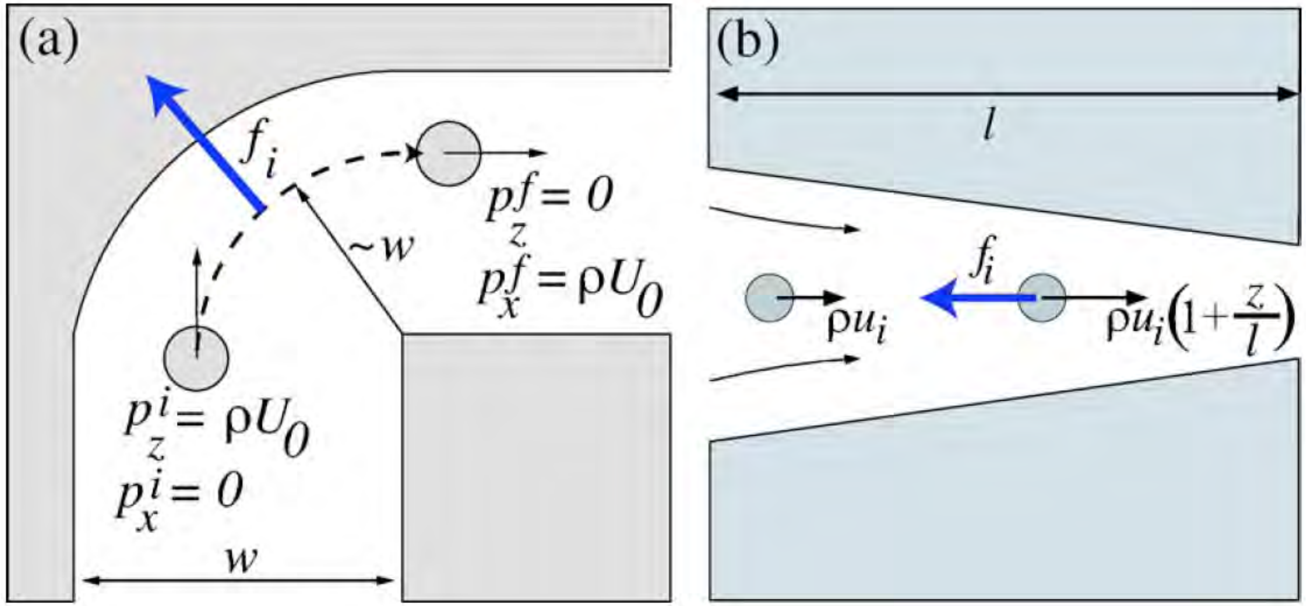

Fluid element accelerating around curve.

- During a turn time:
  $\tau_0 \sim w/U_0$
- Loss of momentum density:
  ρU_0
- By exerting an inertial centrifugal force density:

$$f_i \sim \rho U_0/\tau_0 = \rho U_0^2/w$$

Fluid element in a channel of contracting length.

- By mass conservation, velocity increases as:
  $u \sim U_0(1 + z/l)$
- Gain momentum at a rate:

$$f_i \sim \rho \frac{\mathrm{d}u}{\mathrm{d}t} = \rho U_0 \frac{\mathrm{d}u}{\mathrm{d}z} \sim \frac{\rho U_0^2}{l}$$

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Fluid Behavior

Inertial Forces

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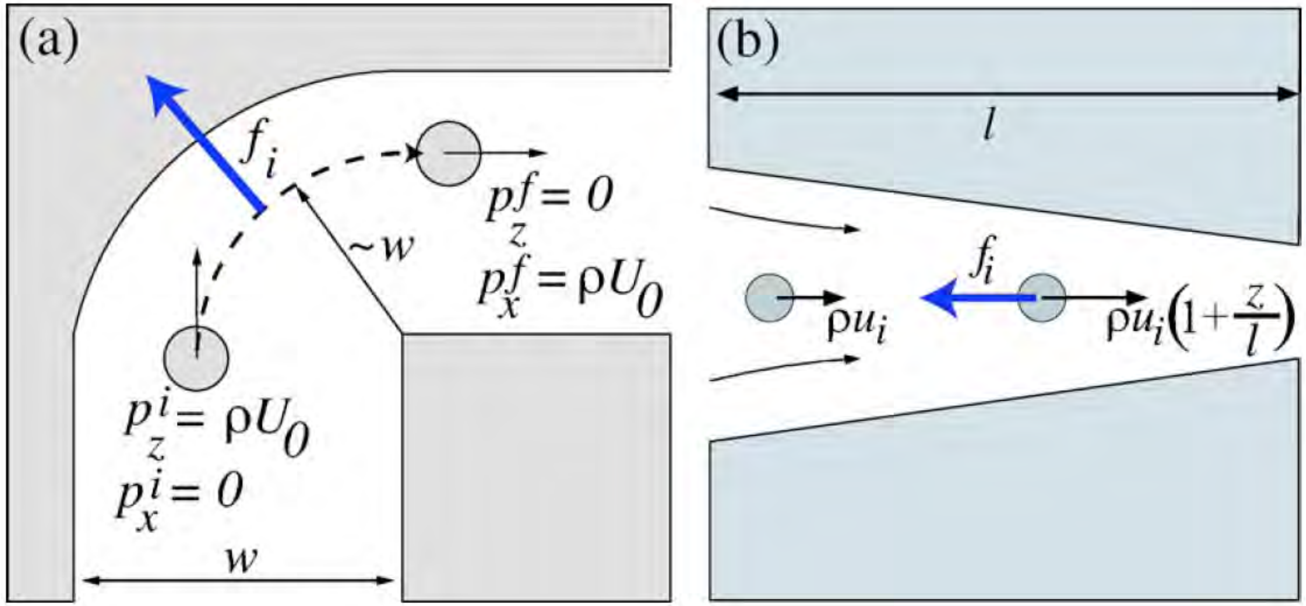

Inertial Forces $f_i \sim \rho U_0/\tau_0 = \rho U_0^2/w$ $\qquad f_i \sim \rho \frac{\mathrm{d}u}{\mathrm{d}t} = \rho U_0 \frac{\mathrm{d}u}{\mathrm{d}z} \sim \frac{\rho U_0^2}{l}$

Viscous Forces

- Viscous force densities result from gradients in viscous stress: $f_v \sim \mu U_0/L_0^2$

Ratio of these two force densities is the **Reynolds number**:

$$\frac{f_i}{f_v} = \frac{\rho U_0 L_0}{\mu} \equiv \mathscr{R}$$

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
Airplane: http://eis.bris.ac.uk/~glhmm/gfd/Airplane-ChrisWillcox.jpg
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Fluid Behavior

Inertial Forces

Viscous Forces

Reynolds Number: inertial/viscous $\frac{f_i}{f_v} = \frac{\rho U_0 L_0}{\mu} \equiv \mathscr{R}$

Estimation of Reynolds numbers for common microfluidic devices.

- Typical fluid – water
  - Viscosity: 1.025 cP @ 25°C
  - Density: 1 g/mL
- Typical channel dimensions
  - Radius/height (smaller than width): 1 – 100 μm
- Typical velocities
  - Average velocity: 1 μm/s – 1 cm/s

Typical Reynolds number:

$$\mathscr{R} \sim \mathcal{O}(10^{-6}) - \mathcal{O}(10^{1})$$

Low Reynolds number: viscous forces > inertial forces

- Flows are **linear.**
- Nonlinear terms in Navier-Stokes disappear
  - Linear, predictable **Stokes flow**

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
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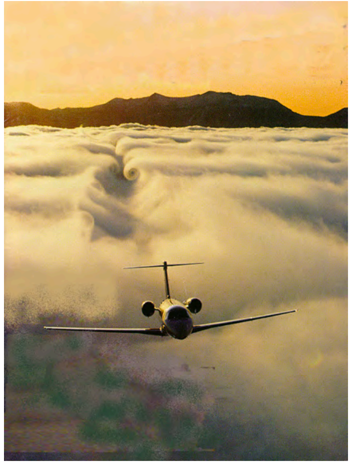
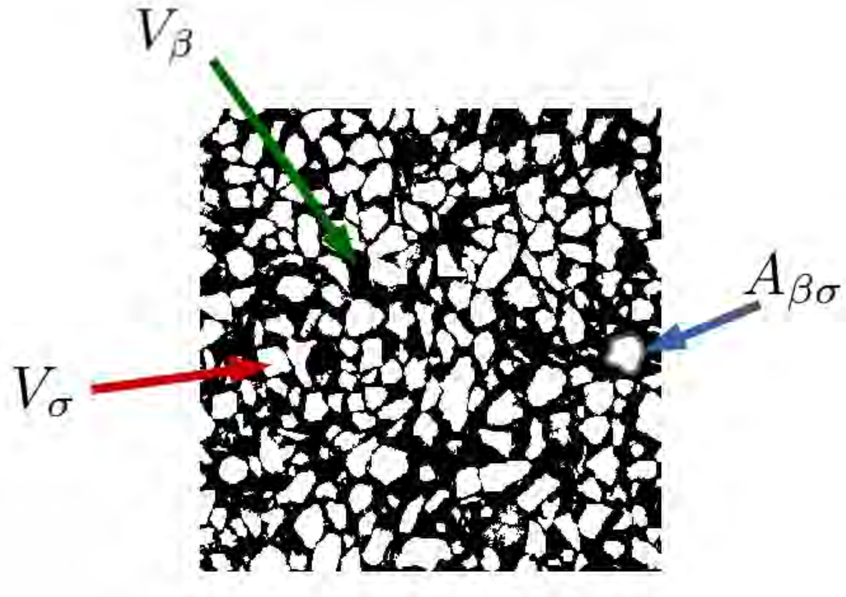
Fluid Behavior

Inertial Forces



Viscous Forces

Reynolds Number: inertial/viscous $\frac{f_i}{f_\nu} = \frac{\rho U_0 L_0}{\mu} \equiv \mathscr{R}$



Typical Reynolds number:

$$\mathcal{R} = \frac{\rho U_0 d_{30}}{\mu} \leq 1$$

Low Reynolds number: viscous forces > inertial forces

- Flows are **linear.**
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Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg

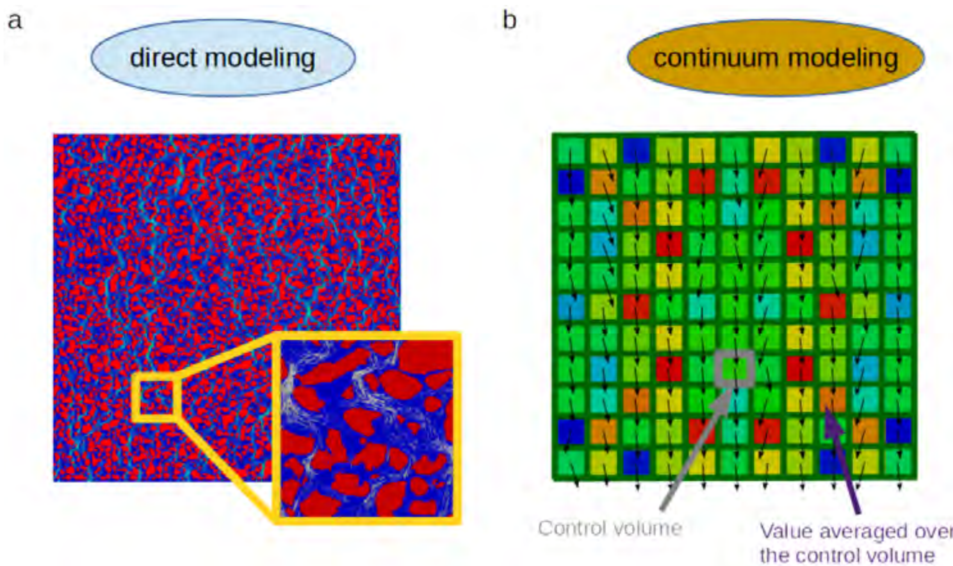
Stokes Equations

Stokes Equations (momentum & mass conservation)

$$0 = \nabla \cdot \mathbf{u}_\beta$$

$$0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{u}_\beta$$

…but, in order to proceed, we need to prescribe BC's on each grain…



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Porous Media

Stokes Equations (momentum & mass conservation)

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…but, in order to proceed, we need to prescribe BC's on each grain…

Darcy-Brinkman-Stokes Equations

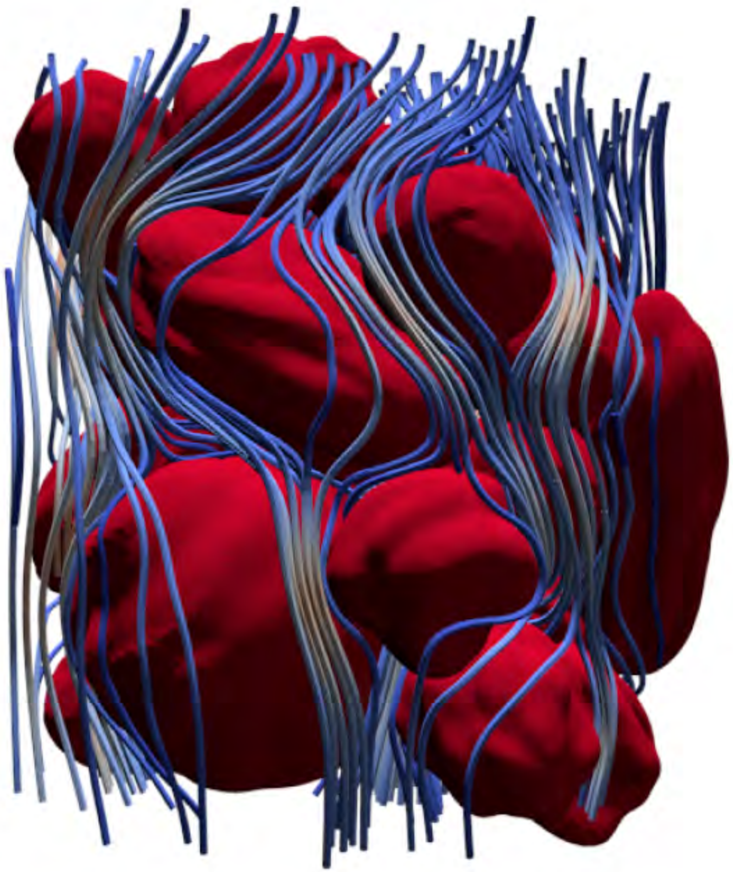
Averaging over pressures and velocities.

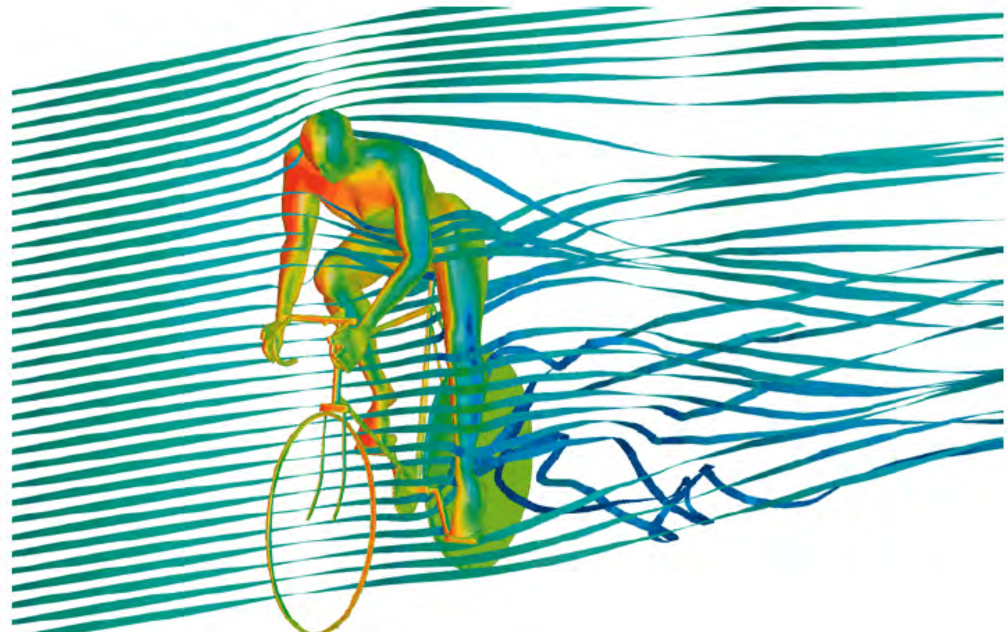
$$0 = -\nabla \langle p_\beta \rangle + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{u}_\beta \rangle - \underbrace{\mu_\beta k_{ij}^{-1} \cdot \langle \mathbf{u}_\beta \rangle}_{\text{Viscous friction}}$$

Void space: $\langle \phi_\beta \rangle = \frac{1}{V} \int_{V_\beta} \mathrm{d}V$

C. Soulaine, "On the Origin of Darcy's law.", http://web.stanford.edu/~csoulain/PORE_SCALE/Chap1_Darcy.pdf, 2015.

Viscous Drag





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Porous Media

Darcy-Brinkman-Stokes Equations

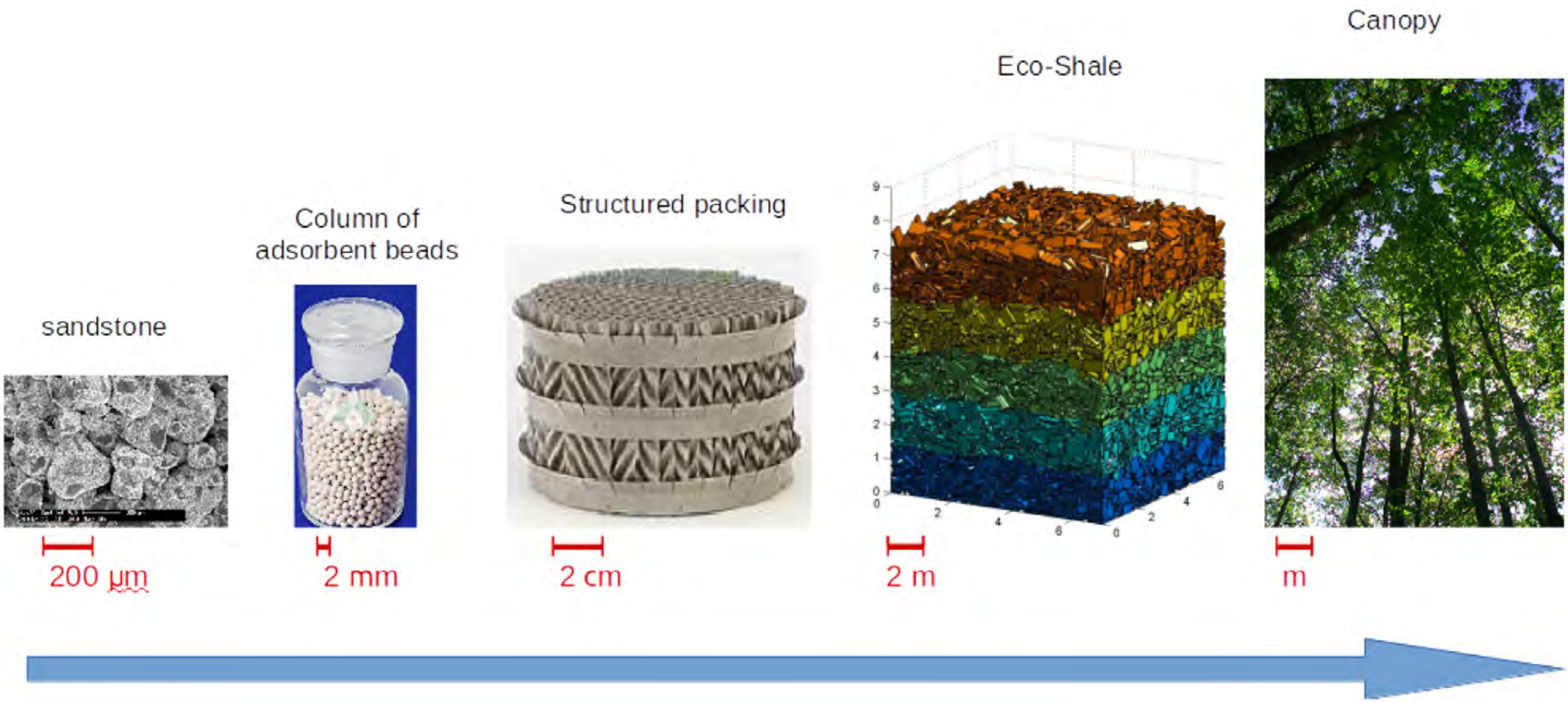
Averaging over pressures and velocities.

$$0 = -\nabla\langle p_\beta\rangle + \rho_\beta \mathbf{g} + \underset{\text{Often negligible}}{\cancel{\mu_\beta \nabla^2 \langle \mathbf{u}_\beta \rangle}} - \underbrace{\mu_\beta k_{ij}^{-1} \cdot \langle \mathbf{u}_\beta \rangle}_{\text{Viscous friction}}$$

Darcy's law (volumetric flux, isotropic medium)

$$\mathbf{q} = -\frac{k}{\mu}\left(\nabla\langle p_\beta\rangle - \rho_\beta \mathbf{g}\right)$$

Porous Media



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Biot Poroelasticity

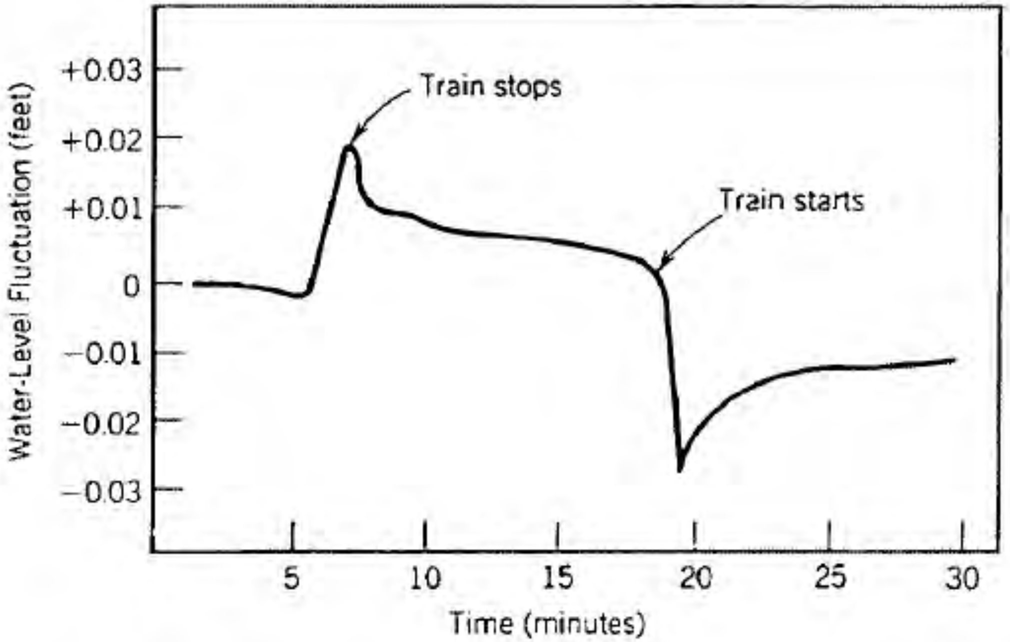


Figure 1.1: Water-level fluctuations due to a passing train. An approaching train compresses the aquifer, which quickly raises the pore pressure in the affected region. Fluid then flows away from the high-pressure region. As the train departs, the aquifer expands, thereby quickly reducing the pore pressure in the affected region. Fluid again flows in response to the pressure differences, but this time it builds up the pore pressure. The approximately equal and opposite behaviors demonstrate that the aquifer is elastic (Domenico and Schwartz, 1998, p. 65; Jacob, 1940).

Biot Poroelasticity

Coupled problem:

Pore pressure has time dependence, as does poroelastic stresses/strains.

Poroelasticity:

Cannot solve fluid flow problem independent of stress field.

- Stress changes in fluid-saturated porous media typically produce significant changes in pore pressure.

Increment in **total work** associated with **strain increment** and **fluid content**.

$$\mathrm{d}W = \sigma_{ij}\ \mathrm{d}\varepsilon_{ij} + p\ \mathrm{d}\zeta$$

Biot Poroelasticity

Time dependence work is related to the **fluid flux** through **Darcy's law**.

$$\alpha \frac{\partial \varepsilon}{\partial t} + S \frac{\partial p}{\partial t} = \frac{k}{\mu} \nabla^2 p$$

Squeeze the soil – how much water comes out?

Pressurize the water – how much water will go in the soil?

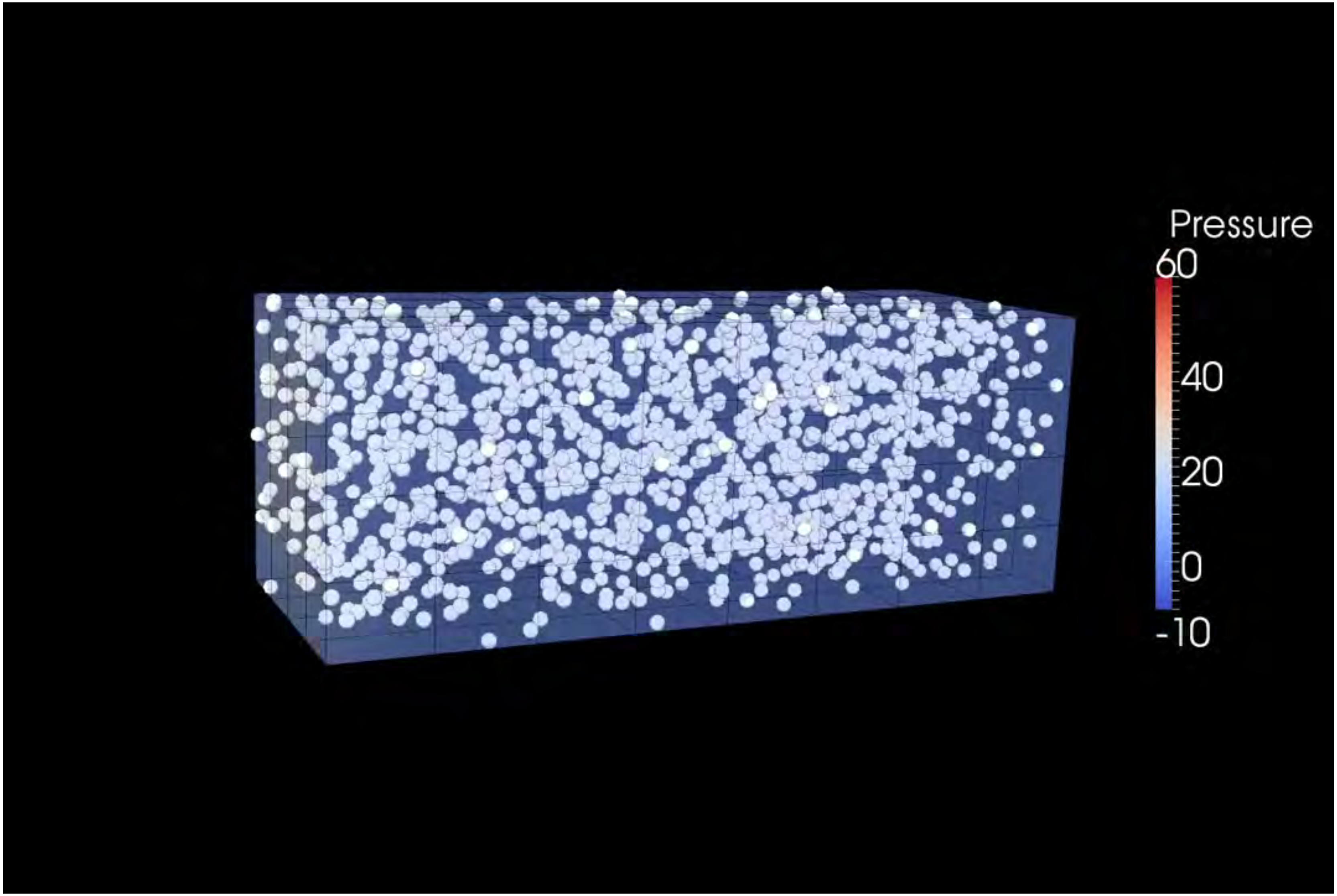
Compression of the medium (e.g. soil) includes compression of pore fluid and particles plus the fluid expelled from an element by flow.

Resistance of medium defined by bulk and shear moduli.

$$\left(K + \frac{4}{3} G\right) \nabla^2 \varepsilon = \alpha \nabla^2 p$$

Stress caused by (1.) hydrostatic pressure of water filling pores, and (2.) average stress in porous network. **Stresses** in the soil **carried** in part by the **fluid** and in part by **solid**.

Biot Poroelasticity



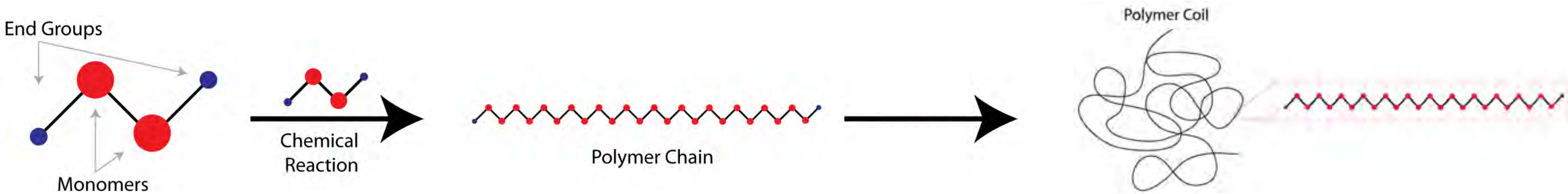
L. Berger, https://www.youtube.com/watch?v=qTvHXRT9qt4, 2012.

Swelling Spheres

Igor30, “Play with Water Balz Balls Jumbo Polymer Hydrogel”, https://www.youtube.com/watch?v=GX2PRQi6Tdk, 2014.

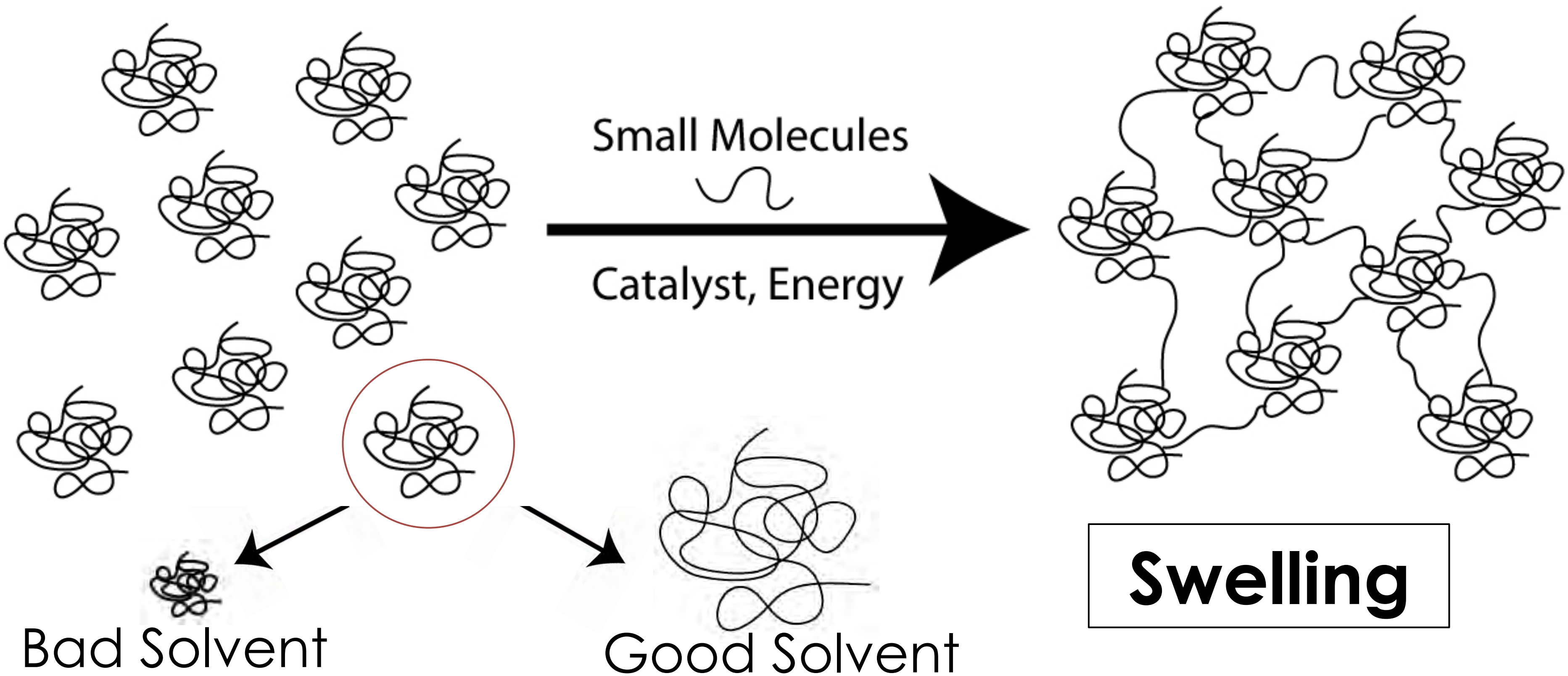
BOSTON UNIVERSITY

Polymers & Swelling



Polymer Chains

Crosslinked Polymer



Free Energy

Gibbs Free Energy of Dilution

$$\Delta\mathcal{G} = \underbrace{\Delta H}_{\text{Heat}} - T \underbrace{\Delta S}_{\text{Entropy}}$$

Equilibrium Swelling $\Delta\mathcal{G} = 0$

At constant pressure

Determination via Osmotic Pressure

Excess pressure required to keep mixed phase in equilibrium with the pure liquid.

$$\Pi = -\frac{RT}{V}\ln\left(\frac{p}{p_0}\right)$$

p: Vapor pressure of liquid in equilibrium with mixture.

$\mathbf{p_0}$: Saturation pressure

Free Energy

Gibbs Free Energy of Dilution

Enthalpy ~ Internal Energy

Entropy of dilution (Boltzmann)

Flory-Huggins Equation

Free, long polymer chains

$$\Delta\mathcal{G} = RT\left[\ln\left(1 - v_2\right) + v_2 + \chi v_2^2\right]$$

Flory-Huggins Chi parameter: dimensionless, polymer/fluid interactions.

Good solvents: $\chi \sim 0.1 - 0.5$

Flory-Rehner Equation

Crosslinked polymer networks

- The **entropy** change caused by **mixing** of polymer and solvent.
- The **entropy** change caused by **reduction** in numbers of possible chain **conformations** on swelling.
- The **heat of mixing** of polymer and solvent, which may be positive, negative, or zero.

Equilibrium swelling of a crosslinked network:

$$\ln(1 - v_2) + v_2 + \chi v_2^2 + \frac{\rho V_s}{M_c} v_2^{1/3} = 0$$

Approximate equilibrium stretch:

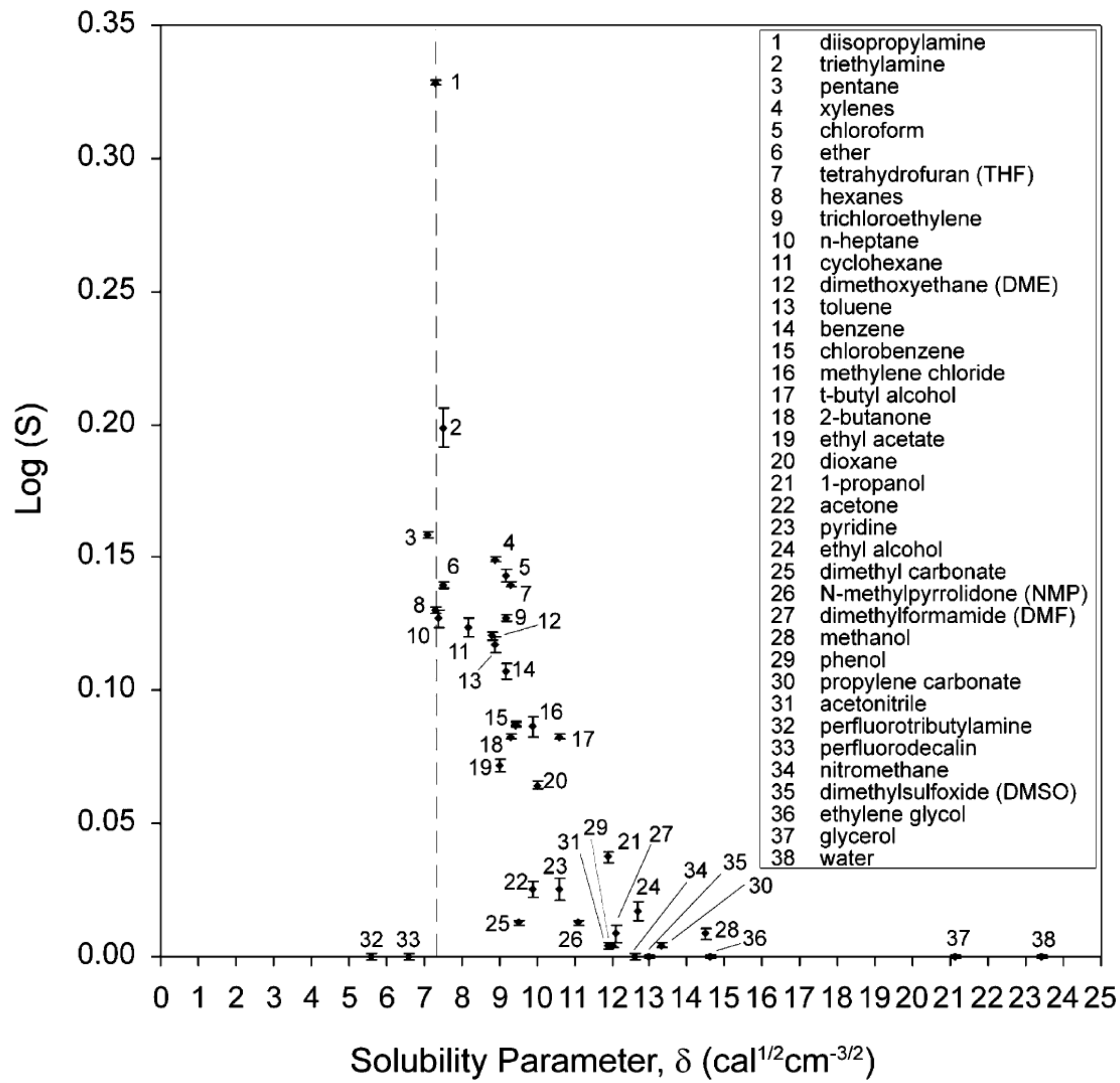
$$\lambda_{eq} \approx \left(\frac{RT}{V_s} \frac{1/2 - \chi}{G} \right)^{1/5}$$

PDMS & Swelling

Table 1. Solubility Parameters, Swelling Ratios, and Dipole Moments of Various Solvents Used in Organic Synthesis

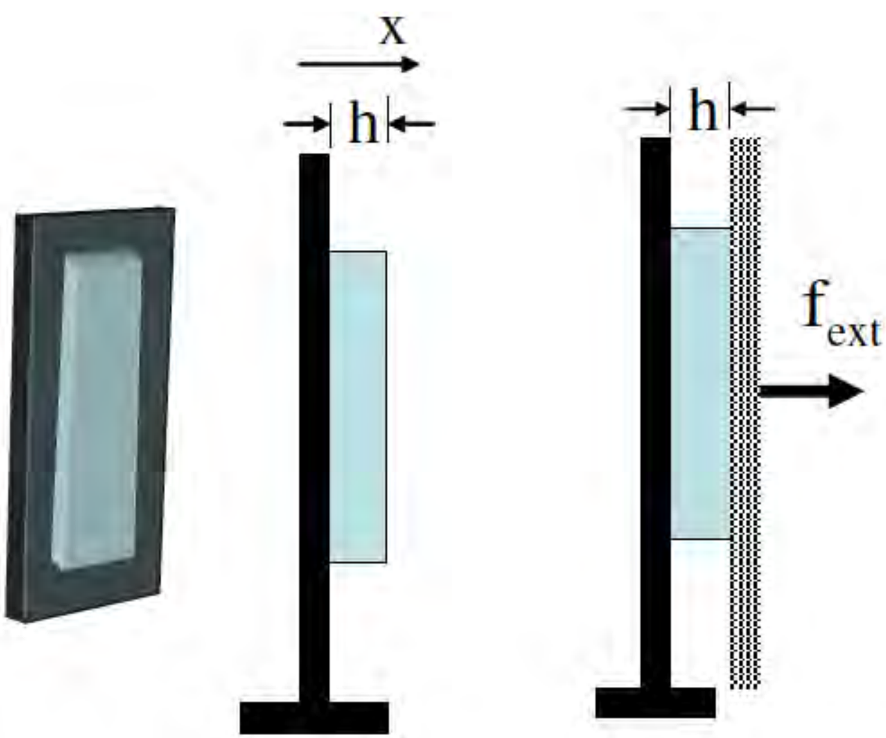
solvent	δ^a	S^b	μ (D)	ref[c]	rank[d]
perfluorotributylamine	5.6	1.00	0.0	10	32
perfluorodecalin	6.6	1.00	0.0	10	33
pentane	7.1	1.44	0.0	10	3
poly(dimethylsiloxane)	7.3	∞	0.6–0.9	8, 14	
diisopropylamine	7.3	2.13	1.2	10	1
hexanes	7.3	1.35	0.0	10	8
n-heptane	7.4	1.34	0.0	10	10
triethylamine	7.5	1.58	0.7	8,10	2
ether	7.5	1.38	1.1	10	6
cyclohexane	8.2	1.33	0.0	10	11
trichloroethylene	9.2	1.34	0.9	10	9
dimethoxyethane (DME)	8.8	1.32	1.6	10	12
xylenes	8.9	1.41	0.3	10	4
toluene	8.9	1.31	0.4	10	13
ethyl acetate	9.0	1.18	1.8	8,10	19
benzene	9.2	1.28	0.0	10	14
chloroform	9.2	1.39	1.0	10	5
2-butanone	9.3	1.21	2.8	10	18
tetrahydrofuran (THF)	9.3	1.38	1.7	10	7
dimethyl carbonate	9.5	1.03	0.9	8,10	25
chlorobenzene	9.5	1.22	1.7	10	15
methylene chloride	9.9	1.22	1.6	10	16
acetone	9.9	1.06	2.9	8,12	22
dioxane	10.0	1.16	0.5	10	20
pyridine	10.6	1.06	2.2	10	23
N-methylpyrrolidone (NMP)	11.1	1.03	3.8	10	26
tert-butyl alcohol	10.6	1.21	1.6	8,12	17
acetonitrile	11.9	1.01	4.0	10	31
1-propanol	11.9	1.09	1.6	8,10	21
phenol	12.0	1.01	1.2	8,12	29
dimethylformamide (DMF)	12.1	1.02	3.8	8,10	27
nitromethane	12.6	1.00	3.5	10	34
ethyl alcohol	12.7	1.04	1.7	8,12	24
dimethyl sulfoxide (DMSO)	13.0	1.00	4.0	10	35
propylene carbonate	13.3	1.01	4.8	10	30
methanol	14.5	1.02	1.7	8,12	28
ethylene glycol	14.6	1.00	2.3	8,12	36
glycerol	21.1	1.00	2.6	13,15	37
water	23.4	1.00	1.9	8,12	38

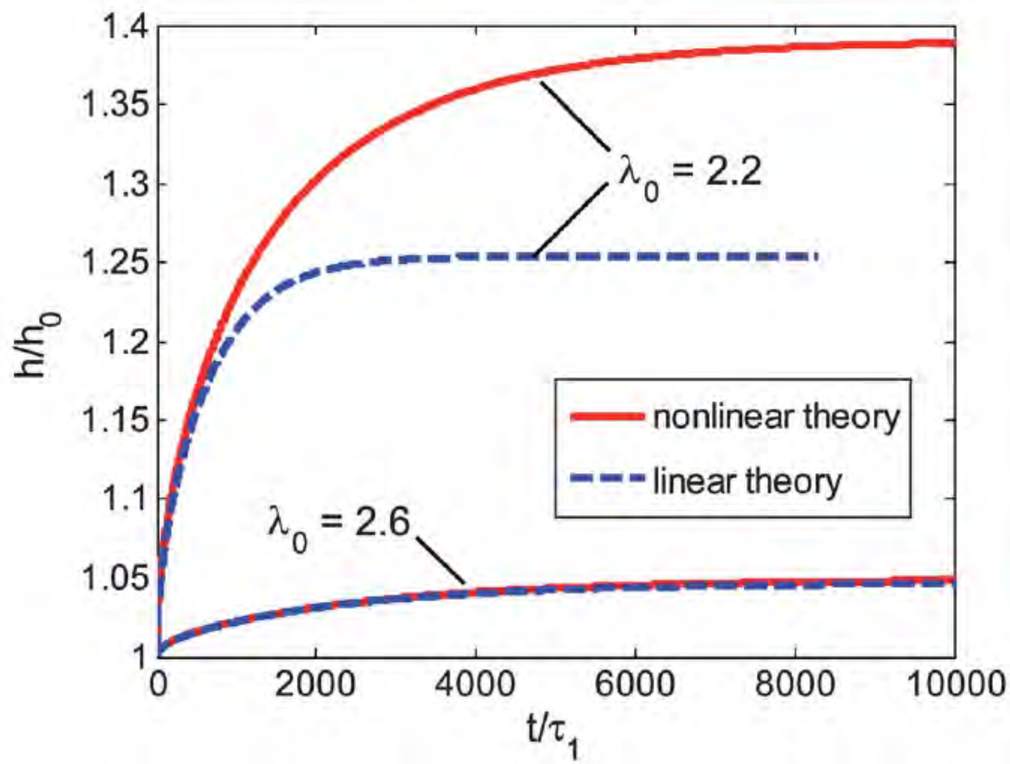
[a] δ in units of cal$^{1/2}$ cm$^{-3/2}$. [b] S denotes the swelling ratio that was measured experimentally; $S = D/D_0$, where D is the length of PDMS in the solvent and D_0 is the length of the dry PDMS. [c] References refer to literature values of δ and μ. [d] Rank refers to the order of the solvent in decreasing swelling ability (see Figure 1).



J.N. Lee, C. Park, G.M. Whitesides, "Solvent Compatibility of Poly(dimethylsiloane)-Based Microfluidic Devices", Anal. Chem. 75, 6544-6554, 2003.

Swelling a Disk

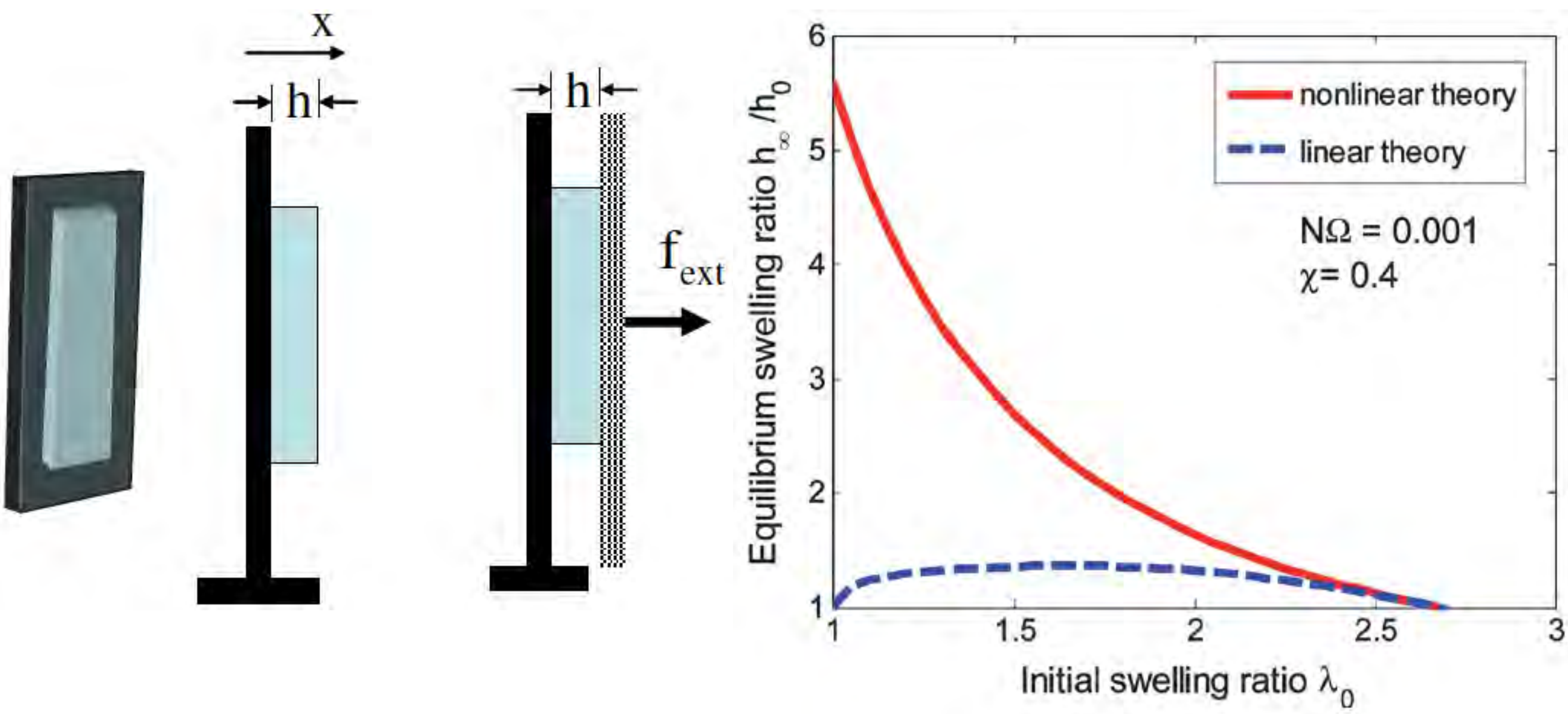




λ is the initial swelling ratio.

M. Doi, "Gel Dynamics," *Journal of the Physical Society of Japan*, 78(5), 052001, 2009.

Swelling a Disk



M. Doi, "Gel Dynamics," *Journal of the Physical Society of Japan*, 78(5), 052001, 2009.

Swelling a Sphere

Swelling Dynamics

Linearized, similar to poroelasticity

$$\left(K + \frac{4}{3}G\right)\nabla\nabla\cdot\mathbf{u} + G\nabla^2\mathbf{u} = \nabla^2 p$$

Incompressibility & Darcy's law

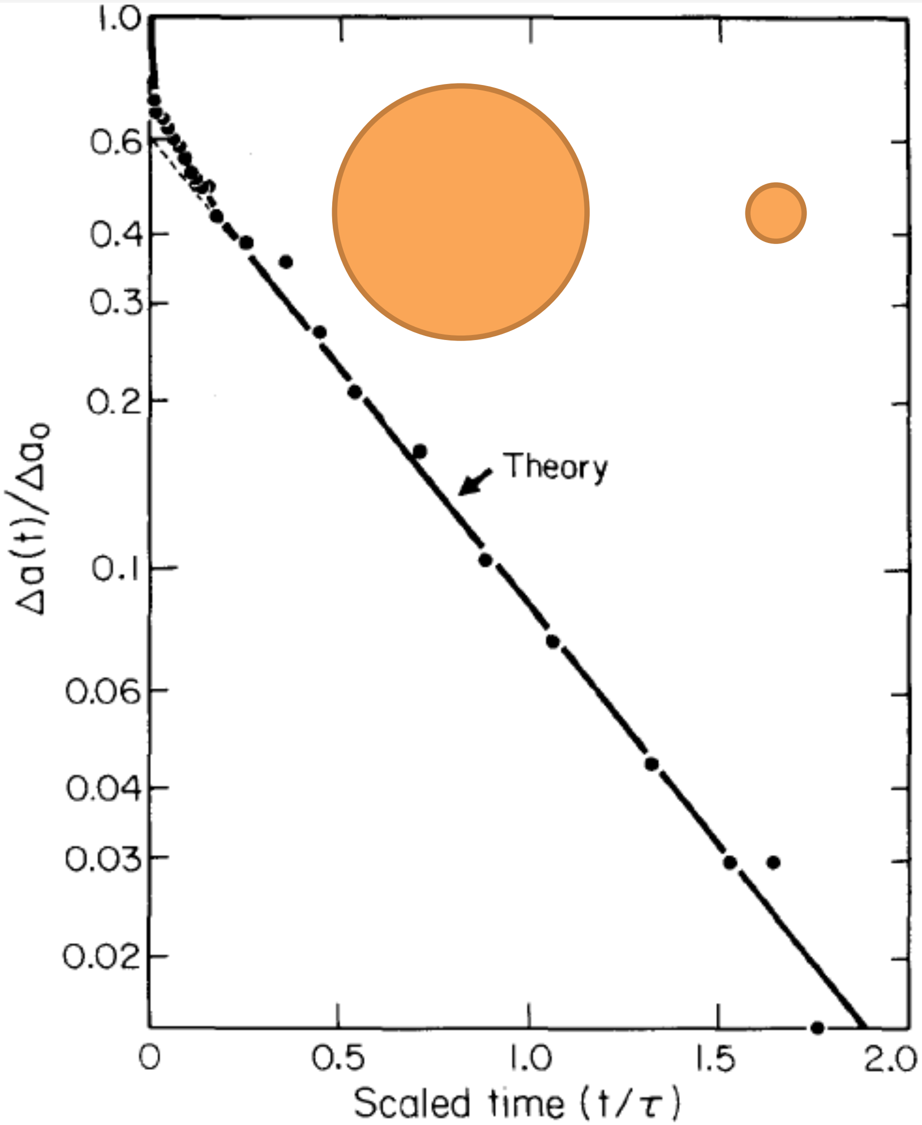
$$\nabla\cdot\mathbf{u} = k\nabla^2 p$$

Volume change (e.g. sphere)

$$\underbrace{\alpha(\mathbf{x},t)}_{\text{Volume change}} = \nabla\cdot\mathbf{u}$$

Satisfied by diffusion relation:

$$\frac{\partial\alpha}{\partial t} = D\nabla^2\alpha \qquad D \equiv \left(K + \frac{4}{3}G\right)k$$



M. Doi, "Gel Dynamics," *Journal of the Physical Society of Japan*, 78(5), 052001, 2009.

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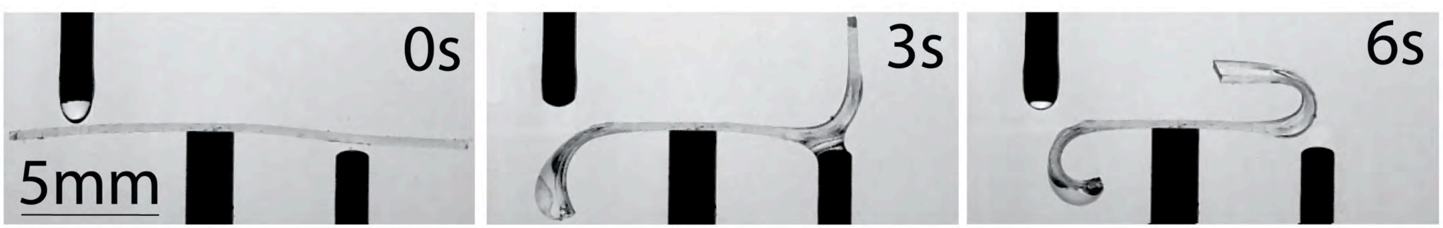
Swelling of Elastic Materials

Fluids Deforming Solids

Douglas P. Holmes

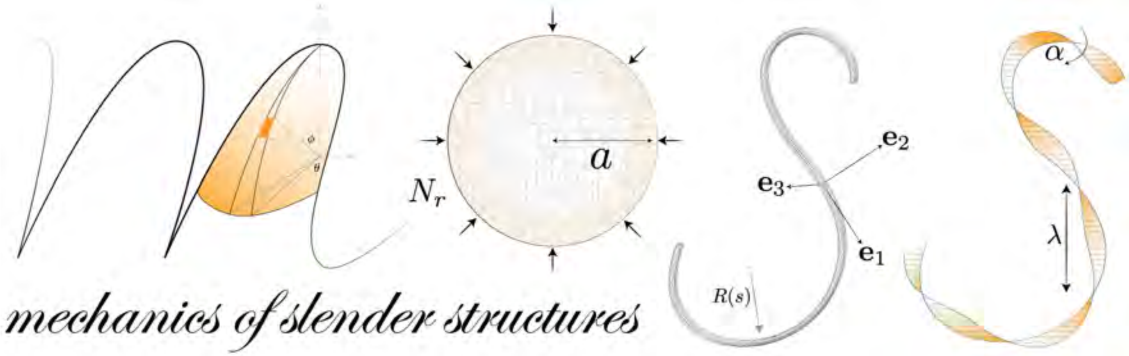
Mechanical Engineering
Boston University

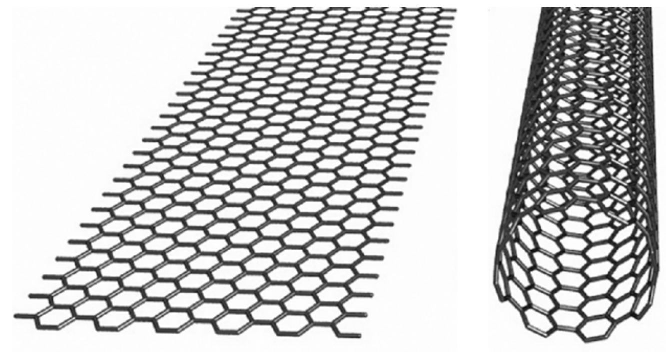


4U Summer School on Complex Motion in Fluids – Denmark (2015)

Gelatin cubes dropped onto solid surface High Speed Video 6200 fps

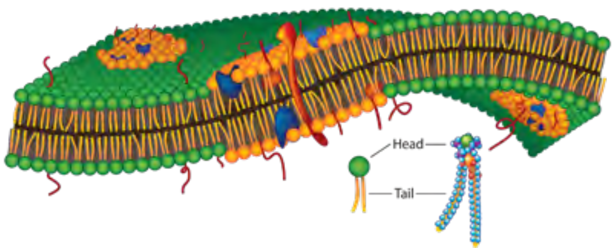
ModernistCuisine, "Gelatin cubes dropped onto solid surface High Speed Video 6200 fps", https://www.youtube.com/watch?v=4n5AfHYST6E, 2011.





(a) (b)

Source: Kreupl *et al.* (2004)



http://www.lanl.gov/science/1663/august2011/images/Cell-Wave-Final.png

http://isabelleteo.deviantart.com/art/Just-hair-292904304

http://www.contactlensescomparison.com/wp-content/themes/smallbiz/images/lens.jpg

Geometric Non-linearities:

- **Buckling**
- Wrinkling
- **Folding**
- Creasing
- **Snapping**

How do objects change shape?

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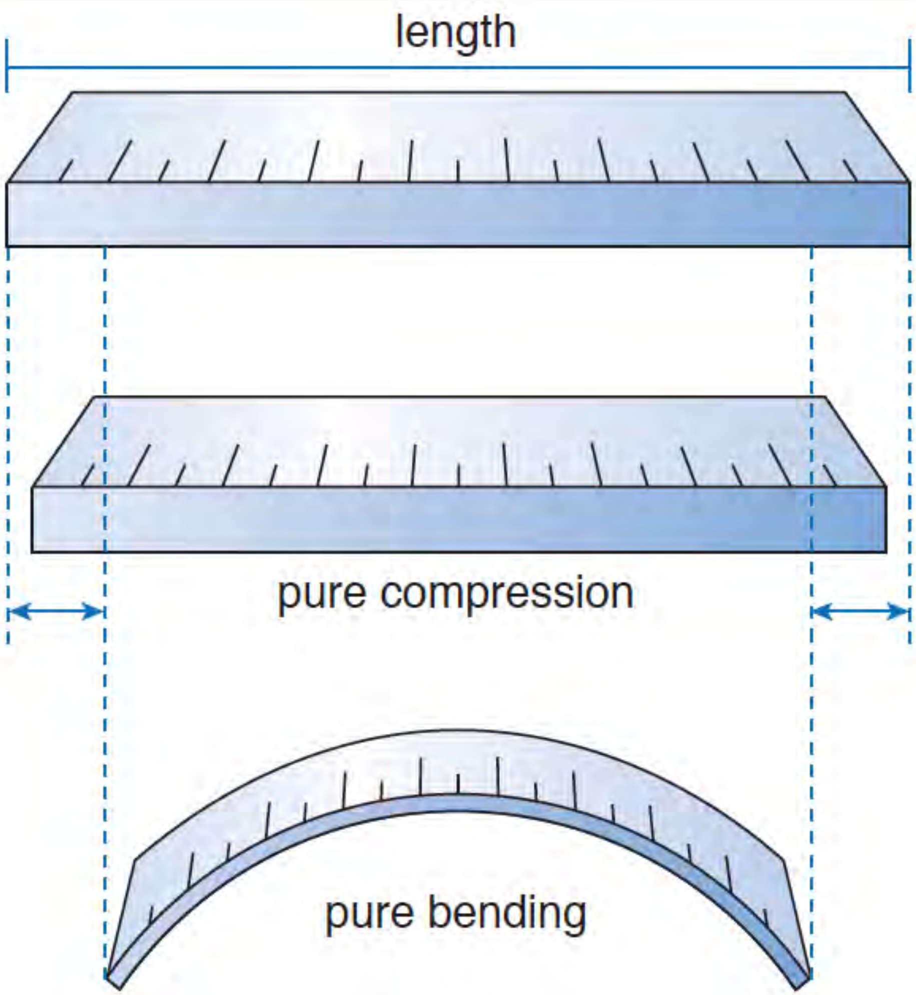
How do you "**grow**" a **structure** into a desired shape?

Thin Structures



Bending vs. **Stretching**

E – Elastic Modulus h – thickness
$\varepsilon_{\alpha\beta}$ – in-plane strain κ – curvature

$$U_m \sim E\, {\color{red}h}\, \varepsilon_{\alpha\beta}^2$$

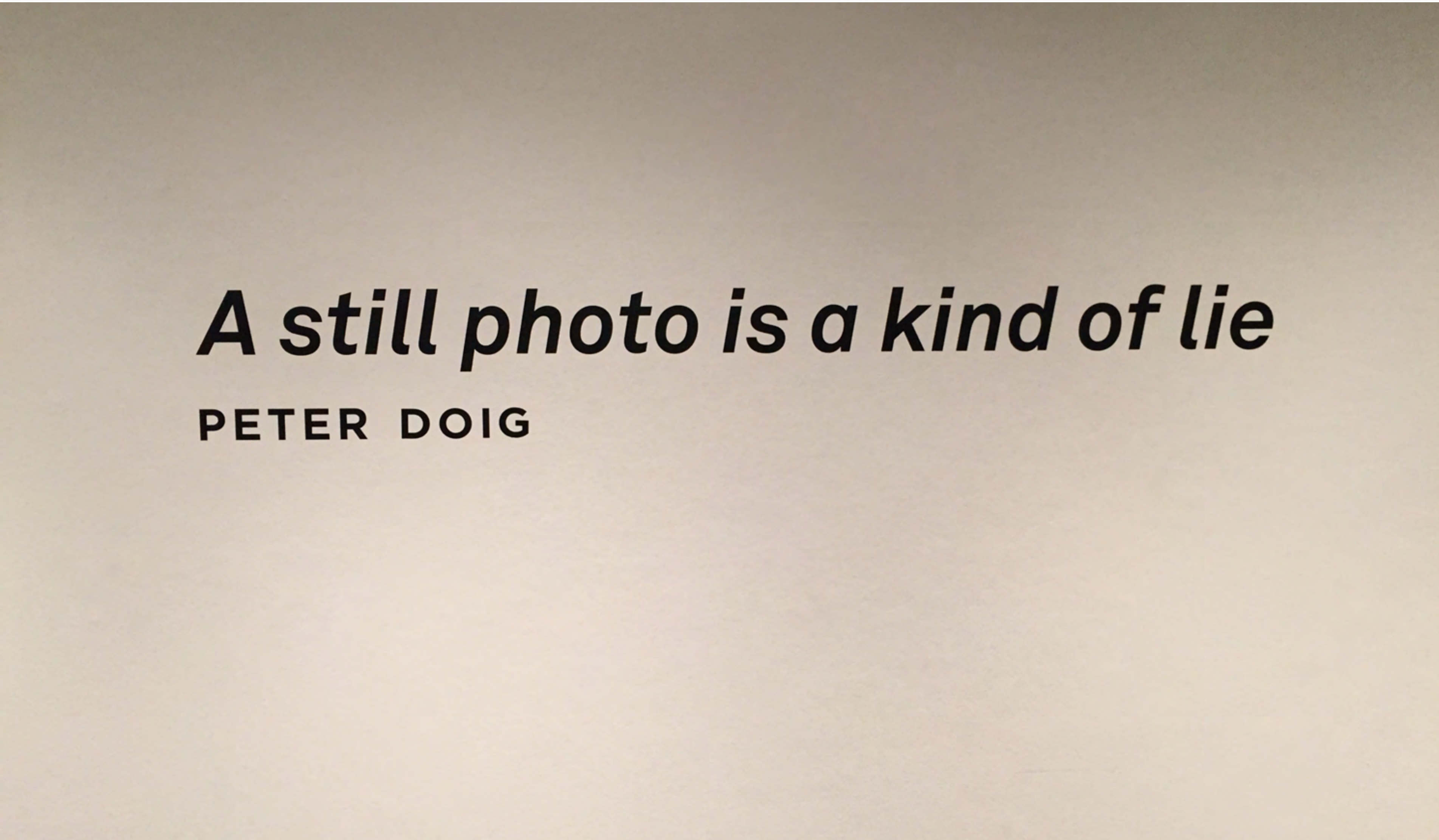
Energy in Compression $\sim$ thickness

$$U_b \sim E\, {\color{red}h^3}\, \kappa^2$$

Energy in Bending $\sim$ thickness^3

Thin structures deform by **bending** & avoid **stretching**

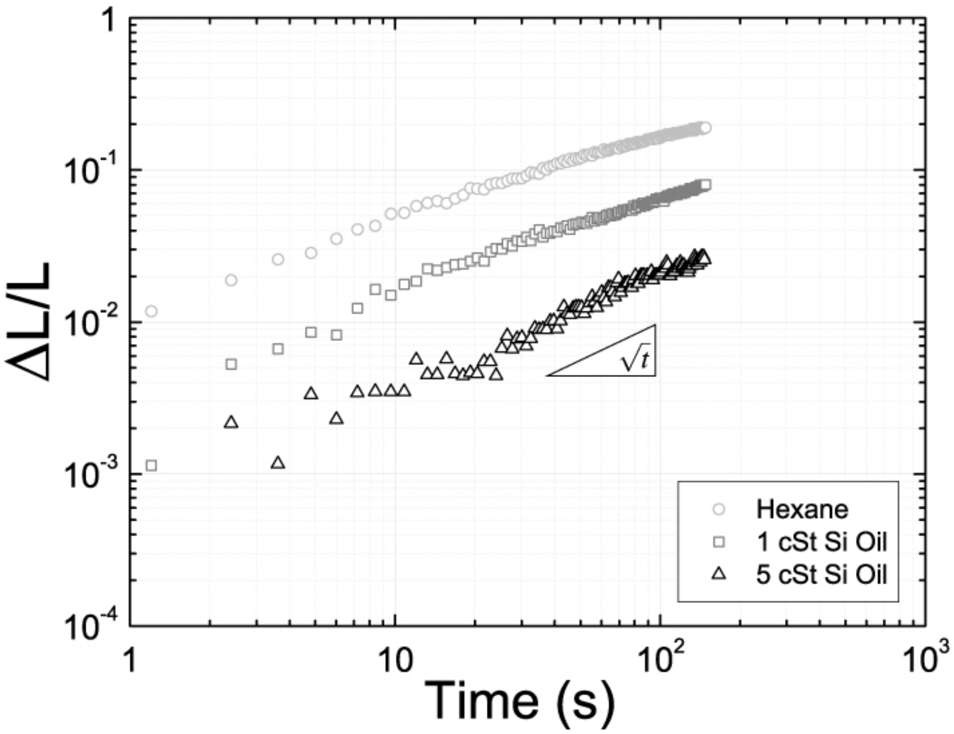
E. Sharon and M. Marder, "Leaves, flowers, and garbage bags: Making waves", *American Scientist*, 2004.



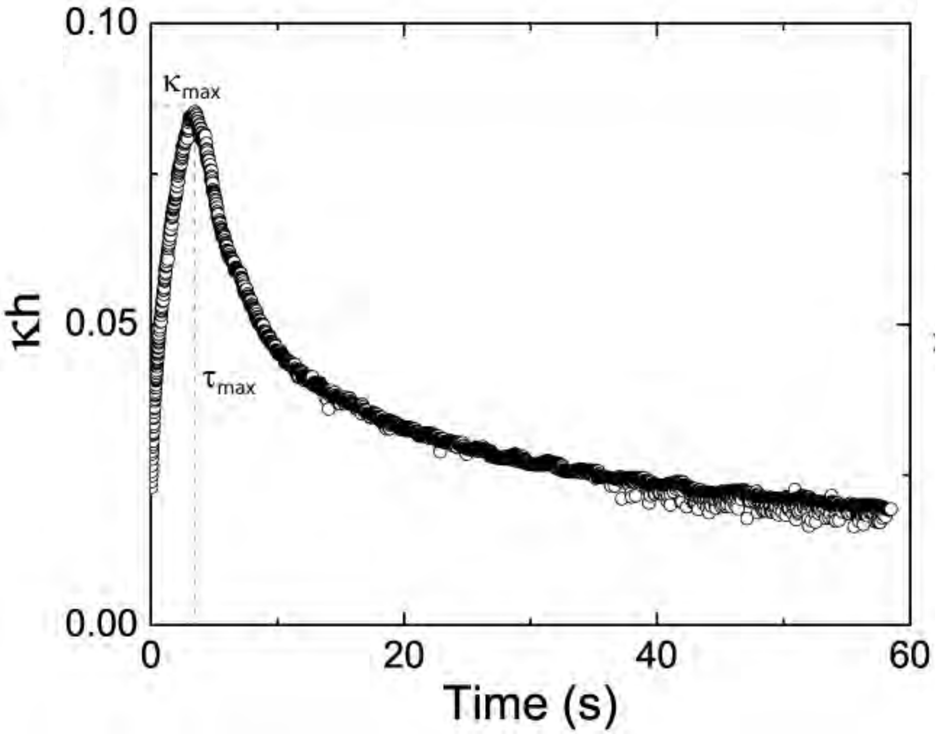
Louisiana Art Museum, 2015.

Swelling Dynamics

Diffusive-like Dynamics

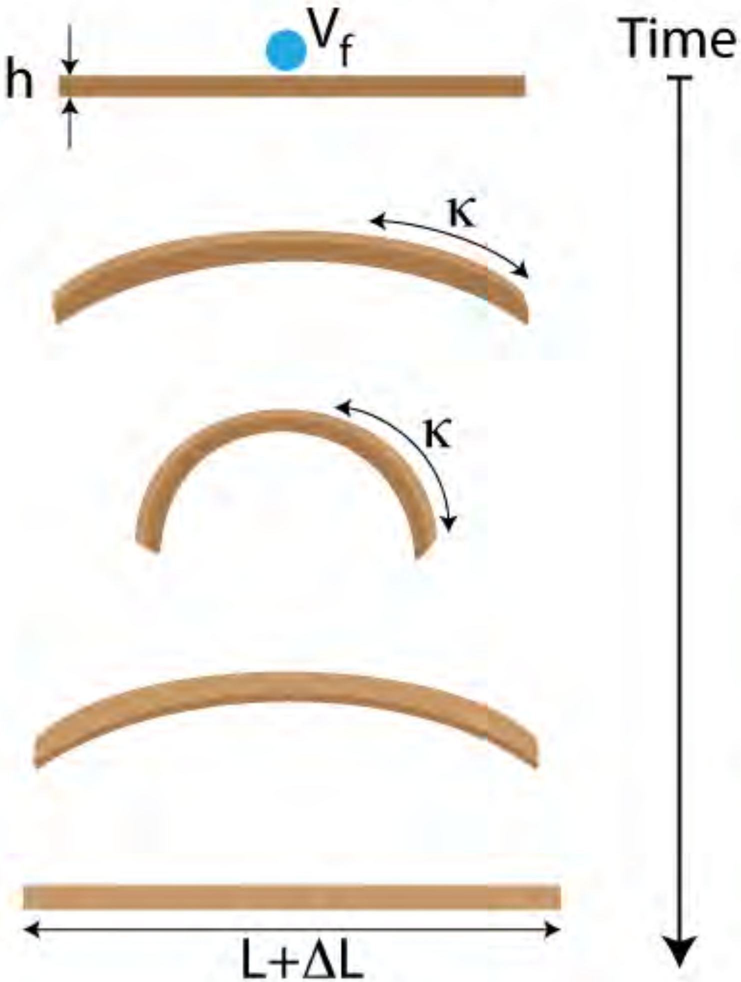


Beam Bending



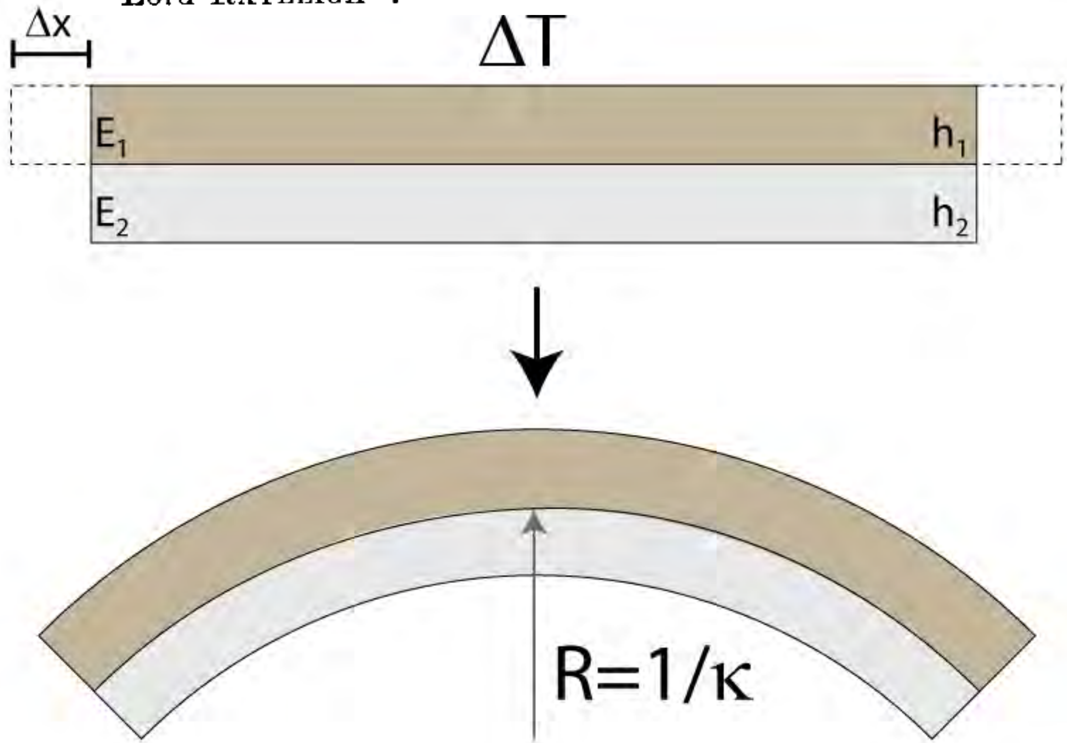
D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. "Bending and Twisting of Soft Materials by Non-homogenous Swelling" *Soft Matter*, **7**, 5188, 2011.

Beam Bending



FEBRUARY 1901.

XV. *On the Stresses in Solid Bodies due to unequal Heating, and on the Double Refraction resulting therefrom.* By Lord RAYLEIGH*.

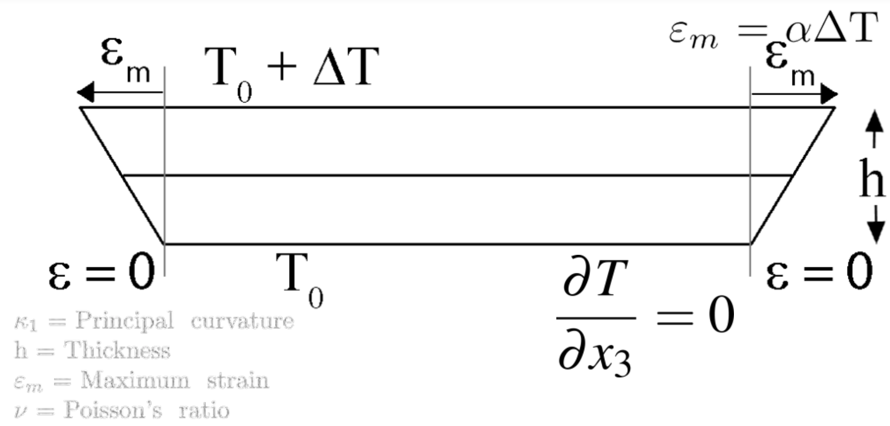


Dynamics of bending : Diffusion of temperature

D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. "Bending and Twisting of Soft Materials by Non-homogenous Swelling" *Soft Matter*, **7**, 5188, 2011.

Diffusion

- Thermal diffusion through the beam thickness.
- Shape obtained by minimizing the bending moment in the beam.
- Beam curvature as temperature diffuses.



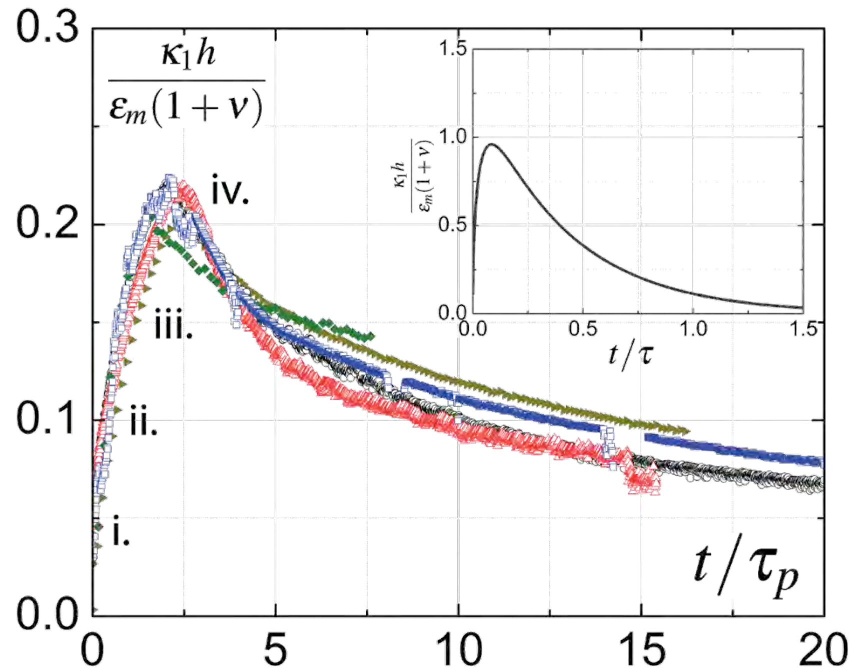
Beam curvature as solvent diffuses:

$$\frac{\kappa_1 h}{\varepsilon_m(1+\nu)} = 1.33e^{-\frac{\pi^2 t/\tau}{4}} - 0.77e^{-\frac{9\pi^2 t/\tau}{4}} + \ldots$$

Poroelastic time scale:

$$\tau_p \approx \frac{\mu h^2}{kE}$$

μ = Solvent viscosity
h = Thickness
k = Permeability (k ≈ 10^{-18} m^2/s)
E = Elastic modulus (E = 10^6 Pa)



D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. "Bending and Twisting of Soft Materials by Non-homogenous Swelling" *Soft Matter*, **7**, 5188, 2011.

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Swelling

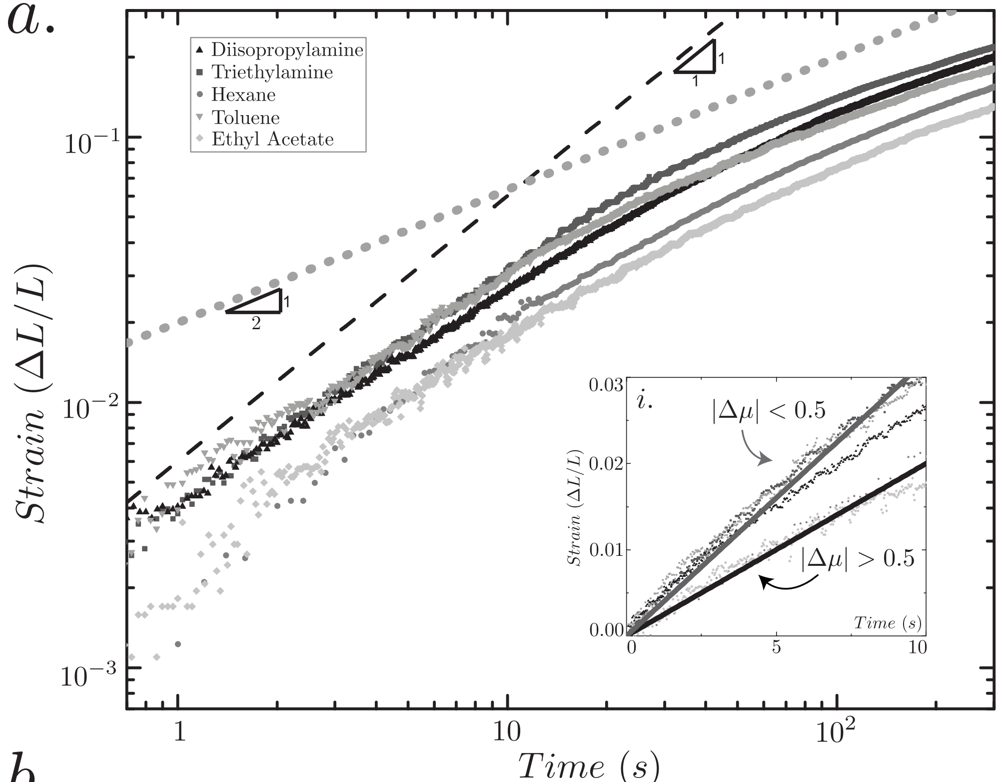
Material	δ_s $(cal^{1/2}cm^{-3/2})$	μ (D)	ε_{eq}
PDMS	7.3	0.6-0.9	–
Diisopropylamine	7.3	1.2	**1.13**
Triethylamine	7.5	0.7	**0.58**
Hexanes	7.3	0.0	**0.35**
Toluene	8.9	0.4	**0.31**
Ethyl acetate	9.0	1.8	**0.18**

$$\delta_s = \text{Solubility parameter}$$

$$\mu = \text{Solvent polarity}$$

$$\varepsilon_{eq} = \text{Strain at equilibrium}$$

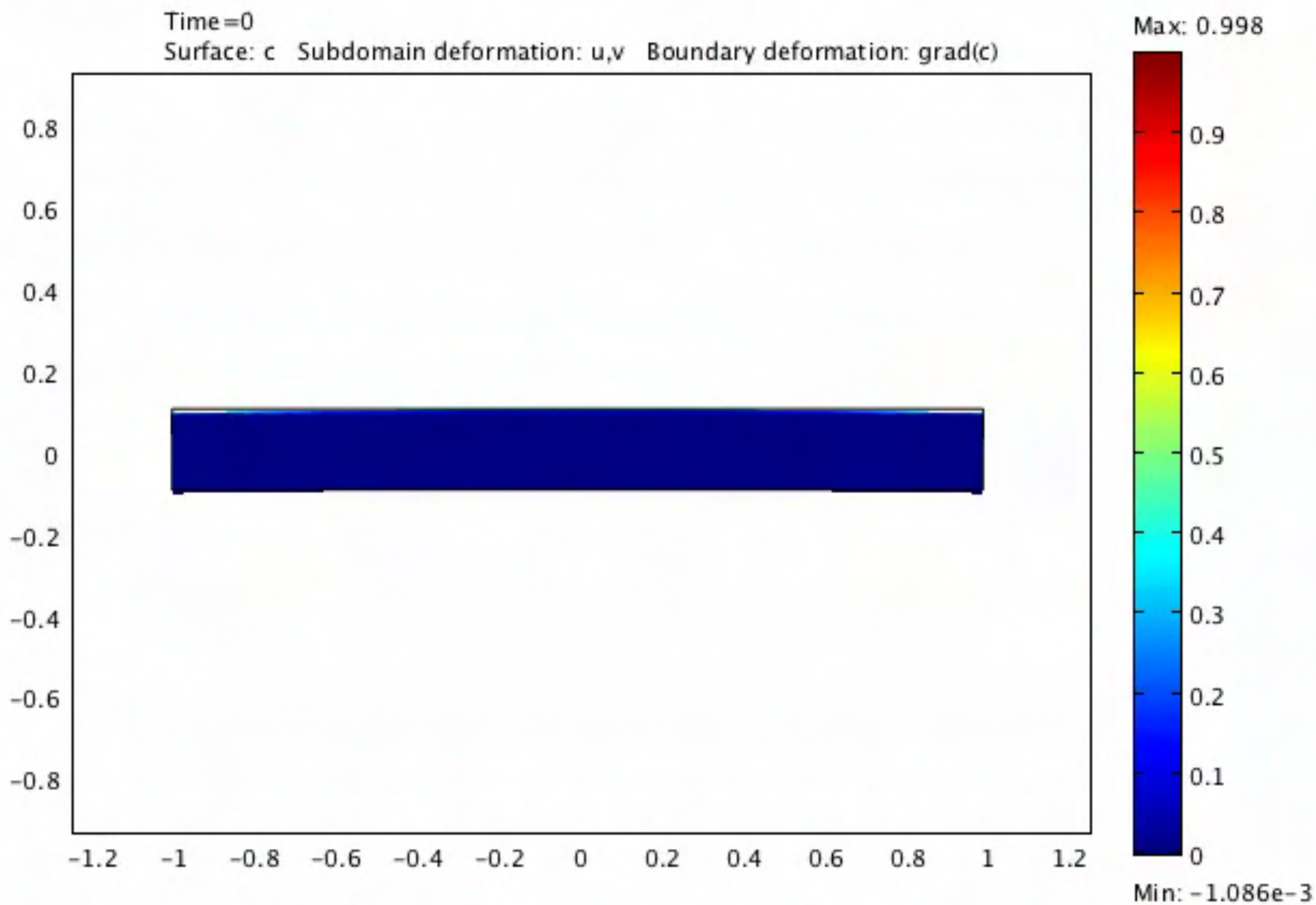
a.



b.

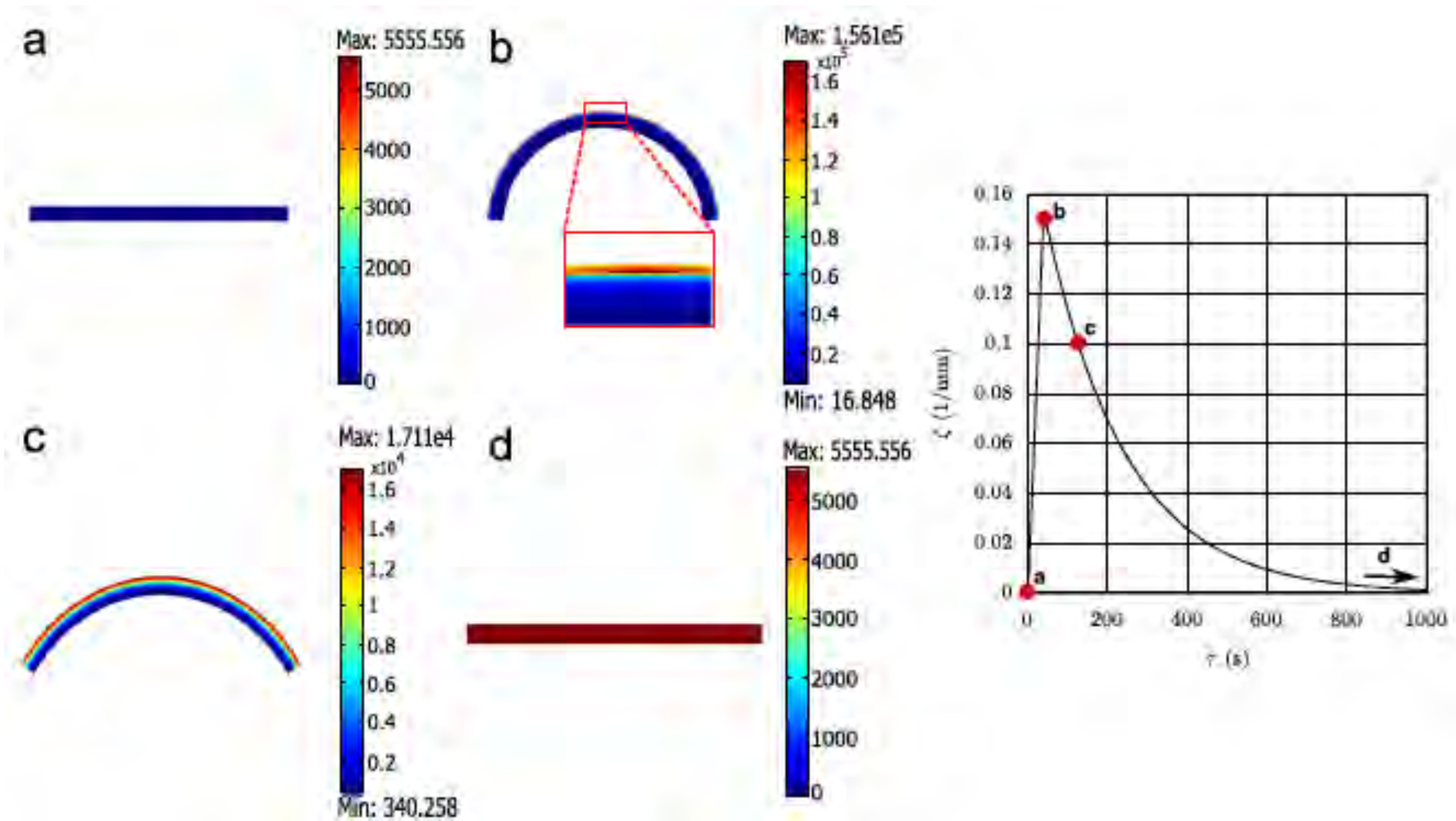
D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. “Bending and Twisting of Soft Materials by Non-homogenous Swelling” *Soft Matter*, **7**, 5188, 2011.

Nonlinear Swelling



A. Lucantonio, P. Nardinocchi, L. Teresi, "Transient analysis of swelling-induced large deformationsin polymer gels" *Journal of the Mechanics and Physics of Solids*, 61, 205-218, 2013.

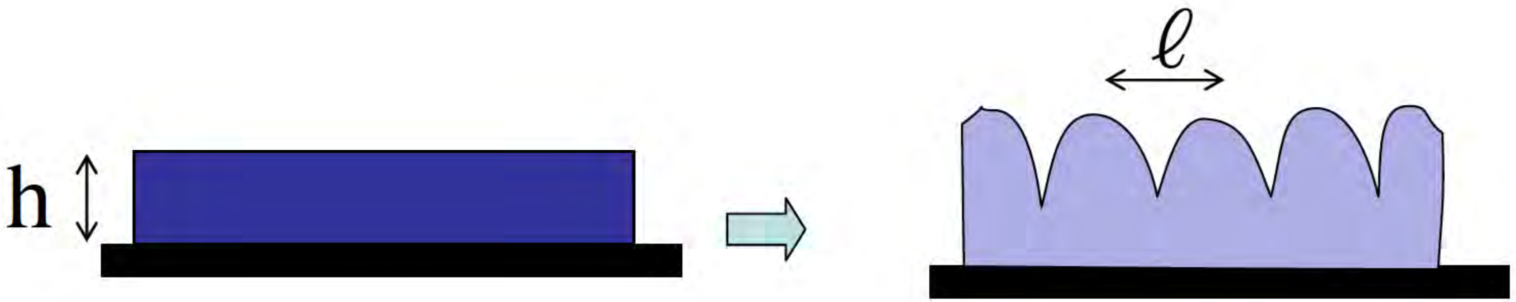
Nonlinear Swelling



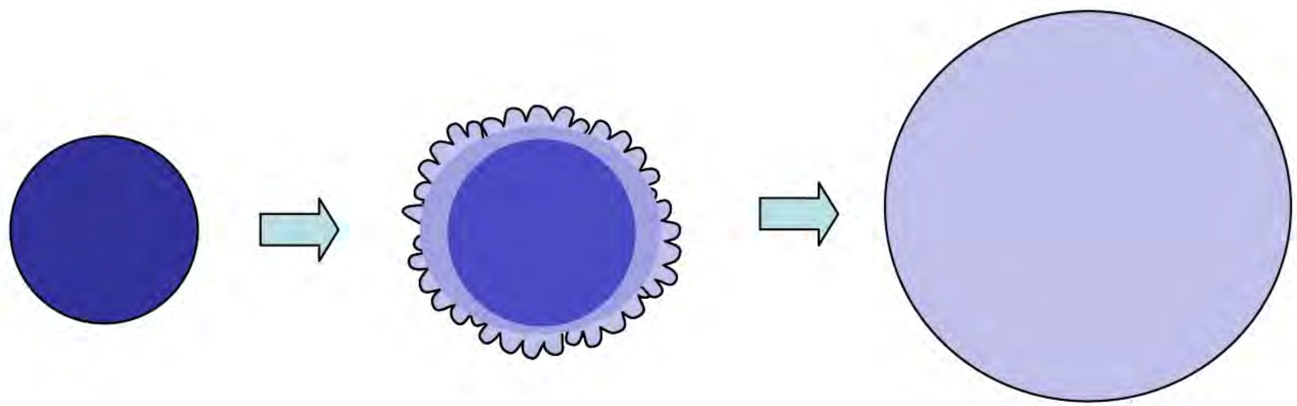
A. Lucantonio, P. Nardinocchi, L. Teresi, "Transient analysis of swelling-induced large deformationsin polymer gels" *Journal of the Mechanics and Physics of Solids*, 61, 205-218, 2013.

What happens when you swell a thicker beam?

Mechanical Instability



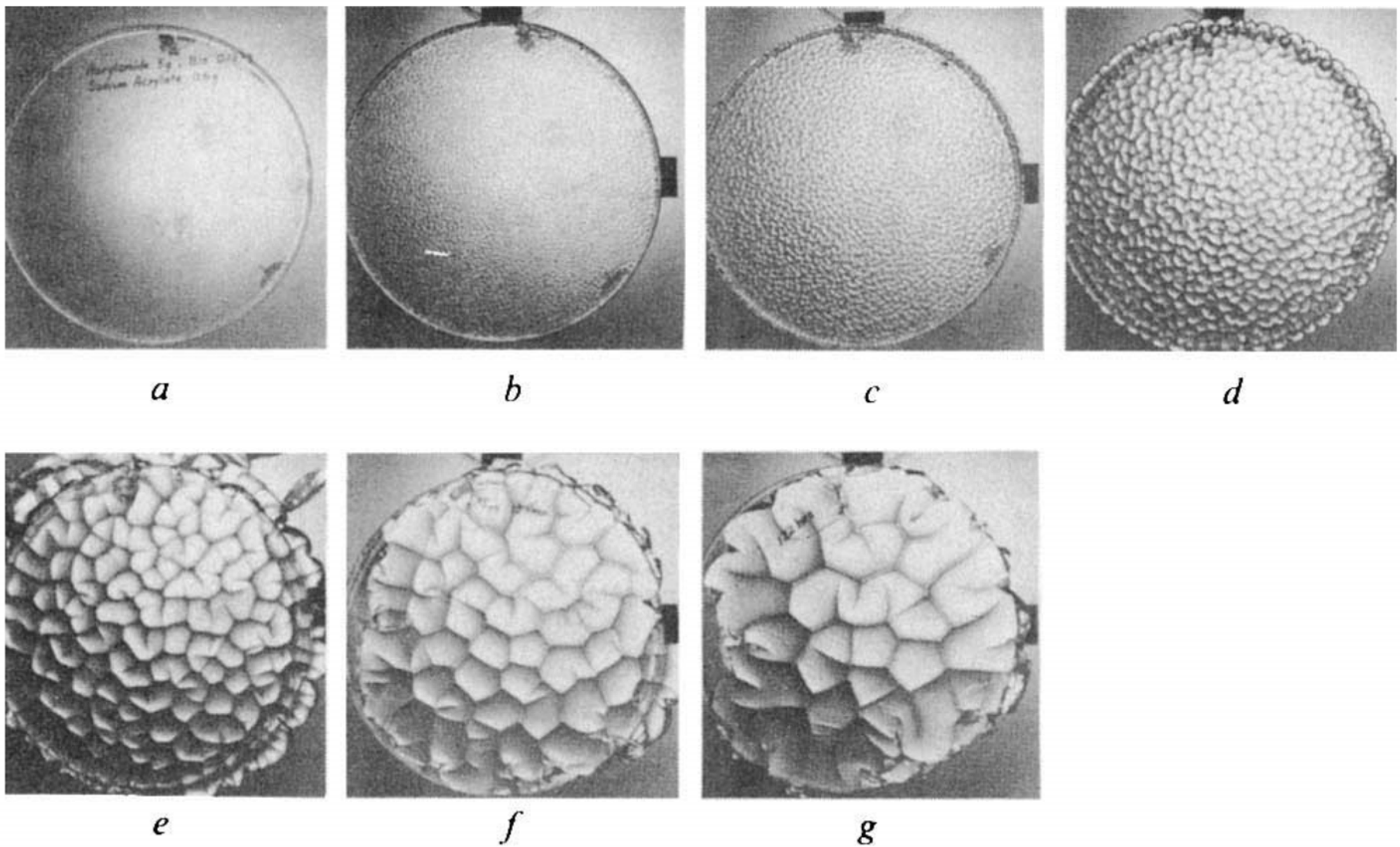
(a) (b)



(c) (d) (e)

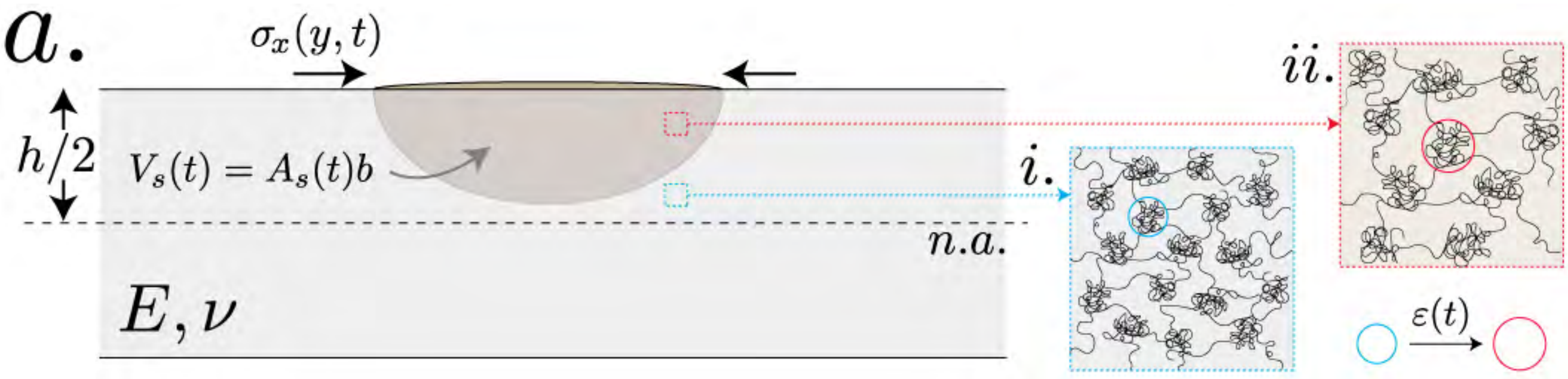
M. Doi, "Gel Dynamics," *Journal of the Physical Society of Japan*, 78(5), 052001, 2009.

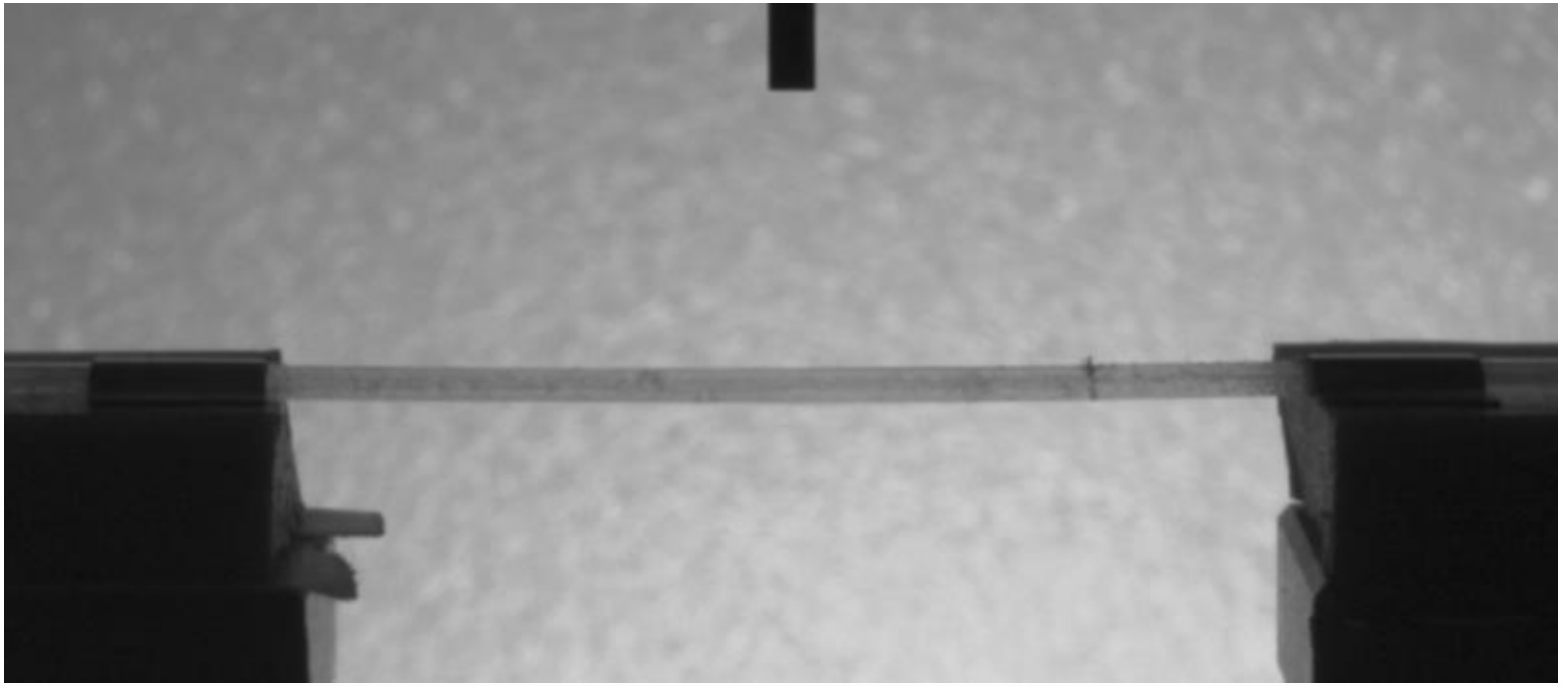
Mechanical Instability



H. Tanaka, H. Tomita, A. Takasu, T. Hayashi, and T. Nishi. *Physical Review Letters*, **68**, 18, 1992.

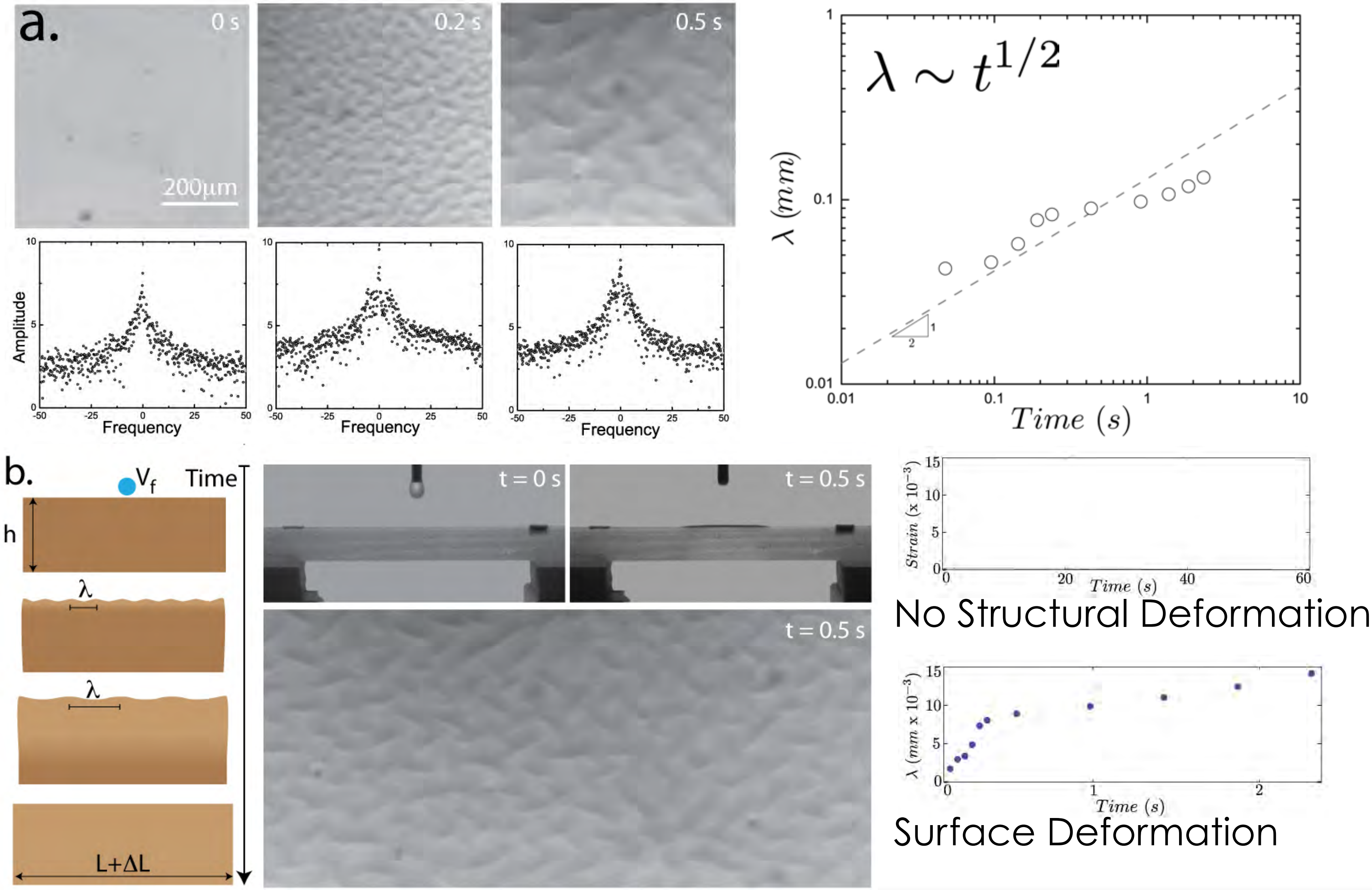
Bending and Buckling





A. Pandey and D.P. Holmes. "Swelling-Induced Deformations: A Materials-Defined Transition from Structural Instability to Surface Instability," Soft Matter, **9**, 5524, (2013).

Bending and Buckling



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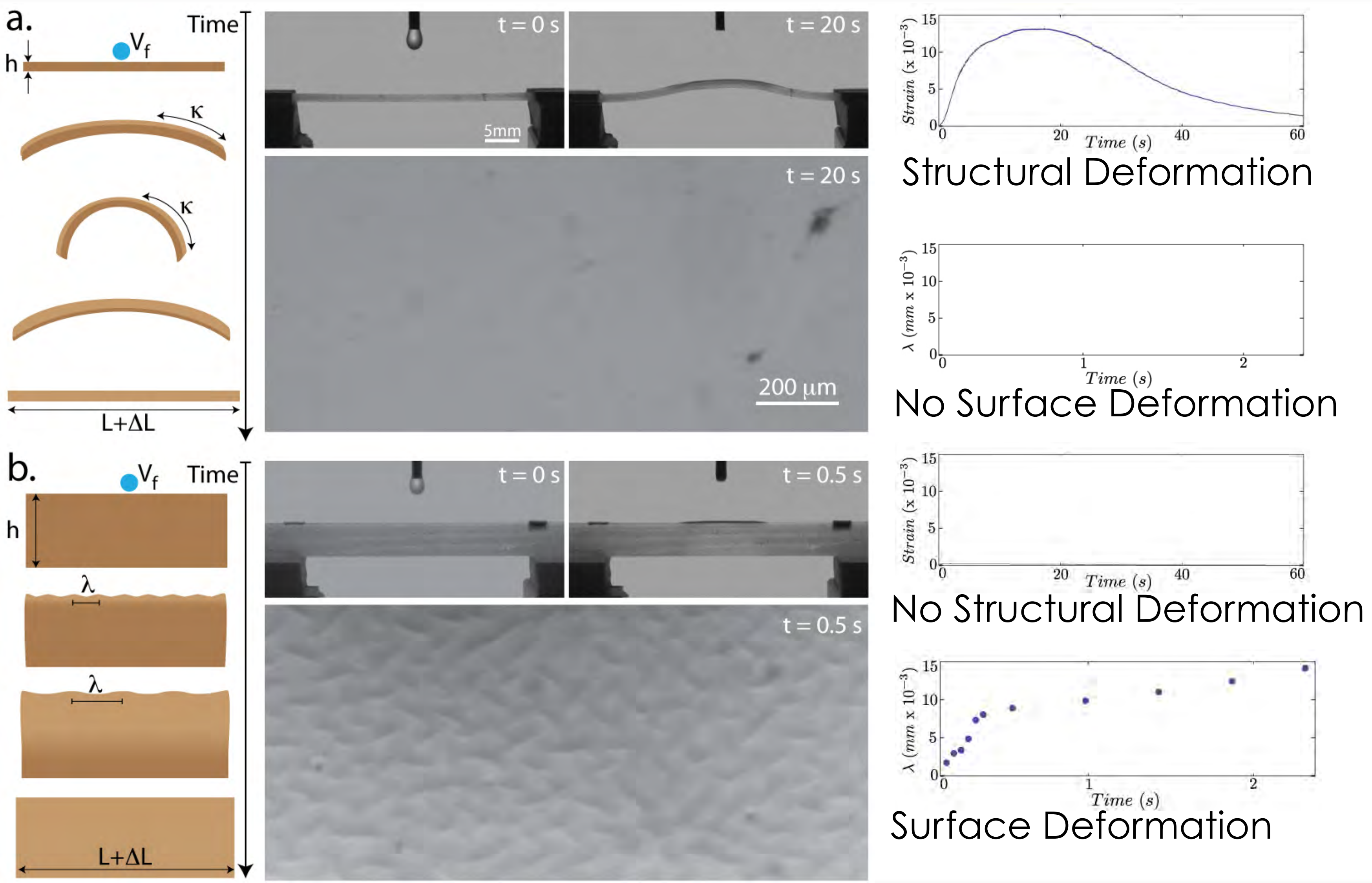
Bending and Buckling



A. Pandey and D.P. Holmes. "Swelling-Induced Deformations: A Materials-Defined Transition from Structural Instability to Surface Instability," Soft Matter, **9**, 5524, (2013).

Bending vs. Swelling

Can the fluid bend the structure?

Bending

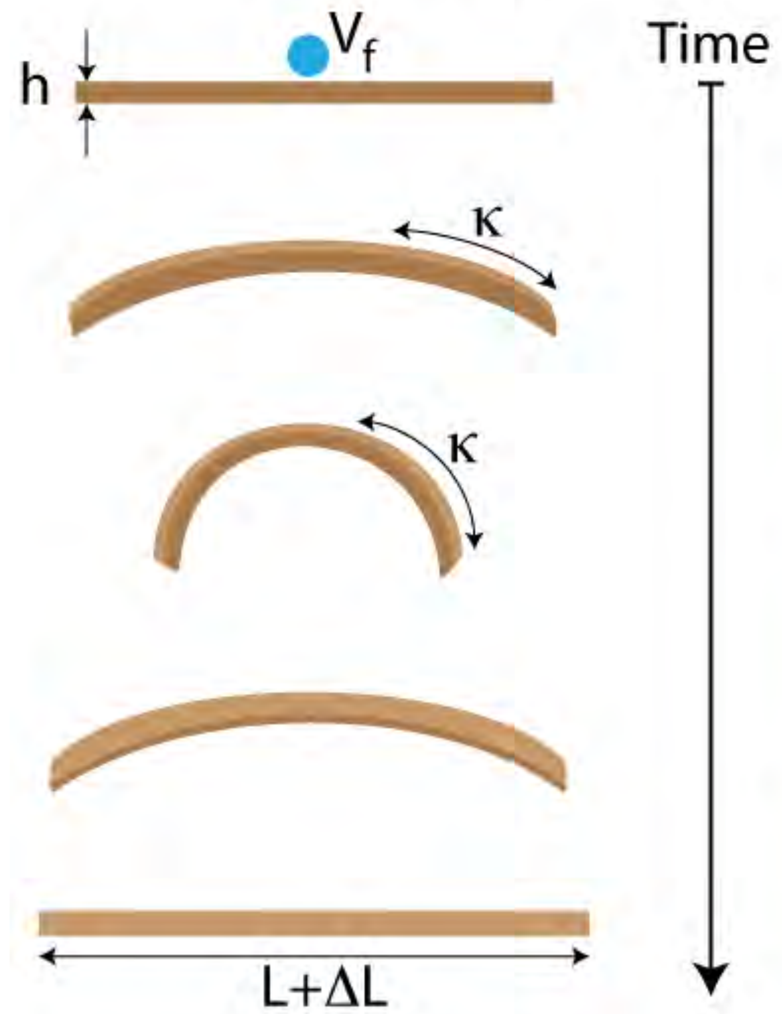
$$\mathcal{U}_b = \frac{B}{2}\int_L \theta'(s)^2 \mathrm{d}s \sim \overline{E}h^3$$

Swelling

$$\mathcal{U}_s = \int_{V_f} \sigma\varepsilon_{eq}\,\mathrm{d}V_f \sim E\varepsilon_{eq}^2 V_f$$

Length scale:

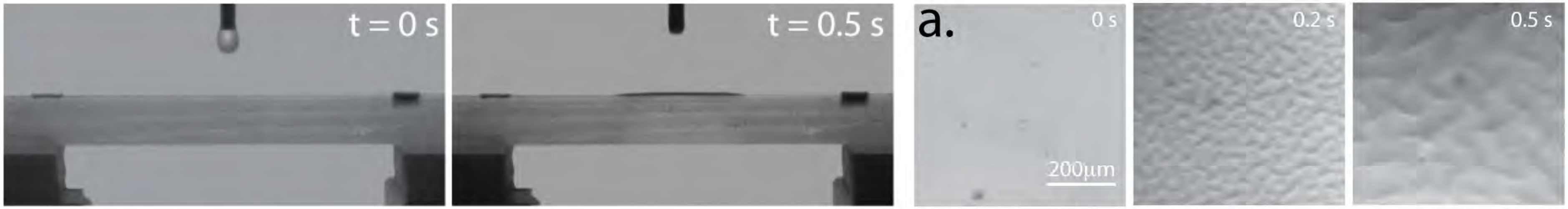
$$\ell_{es} \sim \left(\varepsilon_{eq}^2 V_f\right)^{1/3}$$



A. Pandey and D.P. Holmes. "Swelling-Induced Deformations: A Materials-Defined Transition from Structural Instability to Surface Instability," Soft Matter, **9**, 5524, (2013).

Thin structures bend…

Thick structures stay flat, while their surface creases…

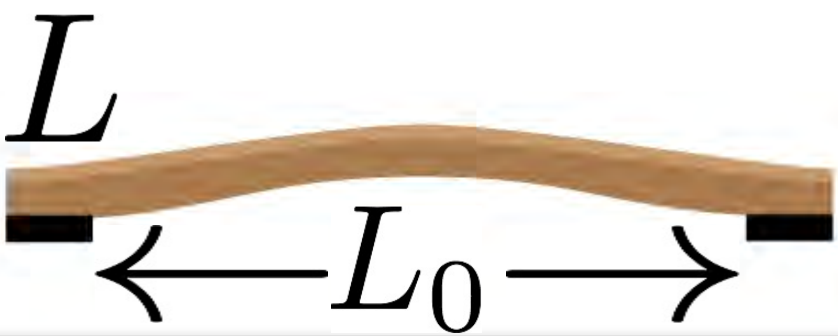


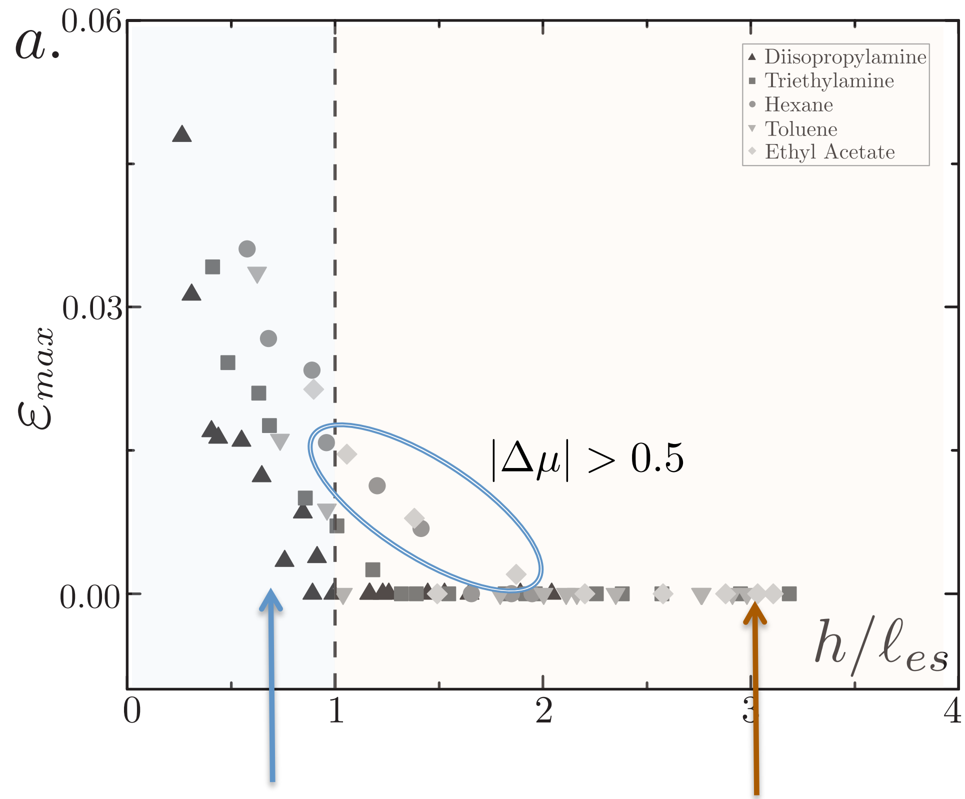
“Elastoswelling” length $$\ell_{es} \sim \left(\varepsilon_{eq}^{2} V_f\right)^{1/3}$$

Deformation Transition

Material	δ_s $(cal^{1/2}cm^{-3/2})$	μ (D)	ε_{eq}
PDMS	7.3	0.6-0.9	–
Diisopropylamine	7.3	1.2	**1.13**
Triethylamine	7.5	0.7	**0.58**
Hexanes	7.3	0.0	**0.35**
Toluene	8.9	0.4	**0.31**
Ethyl acetate	9.0	1.8	**0.18**

$$\ell_{es} \sim \left(\varepsilon_{eq}^2 V_f\right)^{1/3}$$

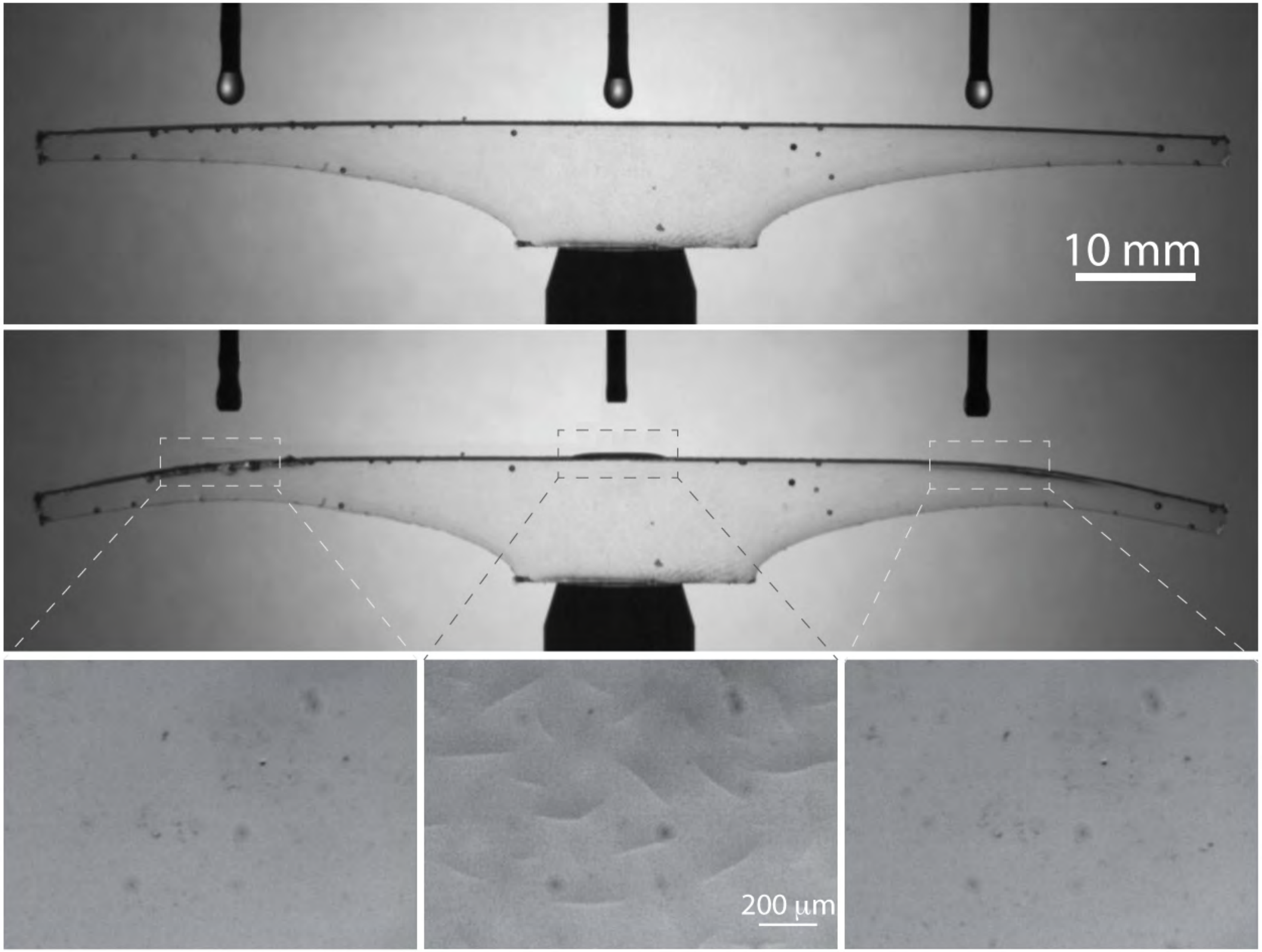




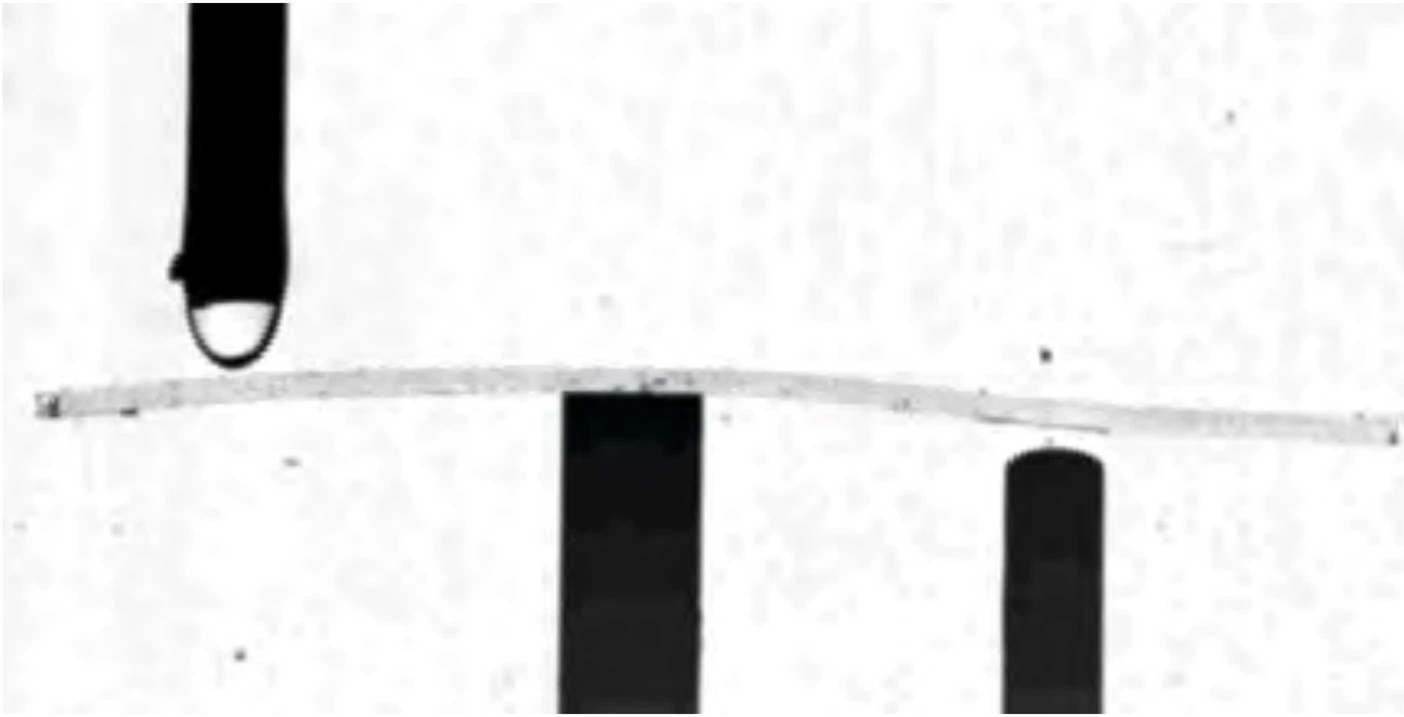
Structural Deformation (Bending/Buckling)

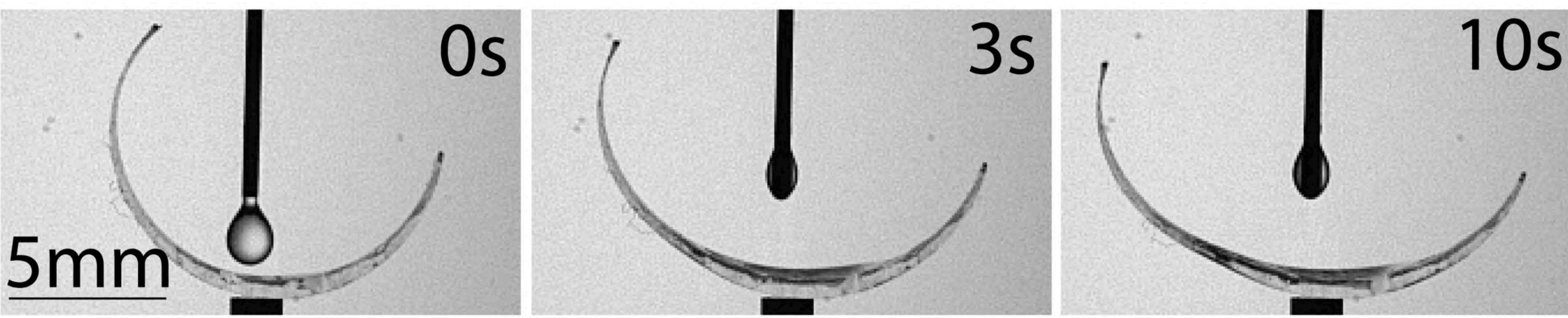
Surface Deformation (Creasing)

A. Pandey and D.P. Holmes. "Swelling-Induced Deformations: A Materials-Defined Transition from Structural Instability to Surface Instability," Soft Matter, **9**, 5524, (2013).


10 mm
200 μm

Controlling Shape

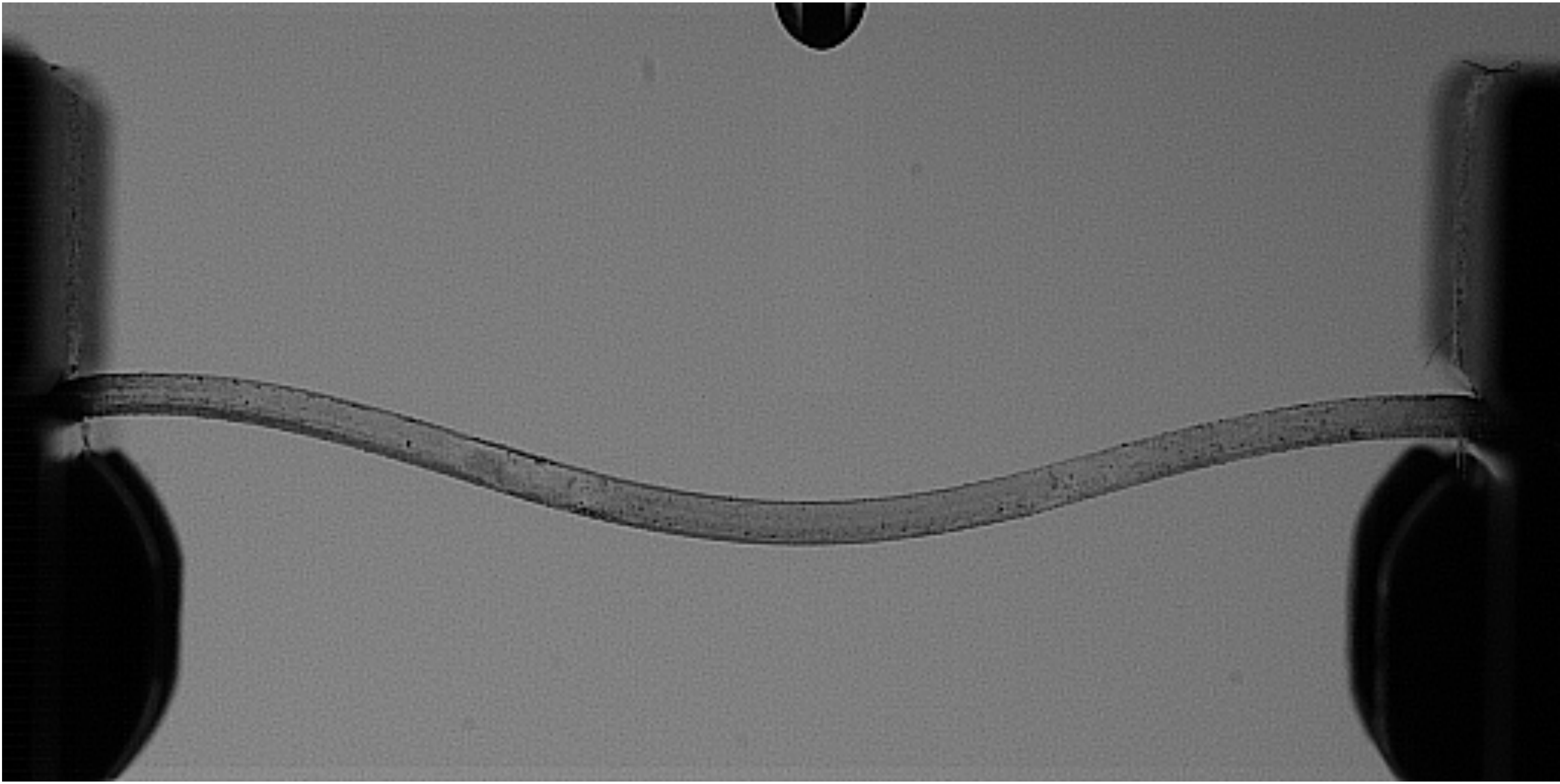




D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. "Bending and Twisting of Soft Materials by Non-homogenous Swelling" *Soft Matter*, **7**, 5188, 2011.

Microfluidic Swelling

Controlling Shape



10 mm

What about wetting?

Fluid Behavior

Viscous Forces

Coiling honey

Interfacial Forces

Wetting of water on a textured surface

Capillary Number: viscous/interfacial

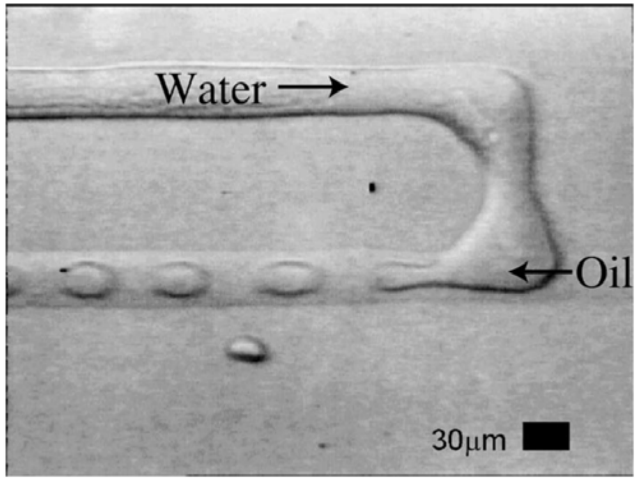
- Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg
- Droplet: http://www.rycobel.be/en/technical-info/articles/1337/measuring-dynamic-absorption-and-wetting

Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial



Monodisperse droplet generation

- Droplet emulsions in immiscible fluids
- Injection of water into stream of oil

Interfacial tension prevents the fluids from flowing alongside each other.

Surface tension acts to reduce the interfacial area.

$$\sigma_c \sim \gamma / R$$

Viscous stresses act to extend and drag the interface downstream.

$$\sigma_v \sim \mu U_0 / h$$

Characteristic droplet size:

$$R \sim \frac{\gamma}{\mu U_0} h = \frac{h}{\mathscr{C}}$$

Capillary number:

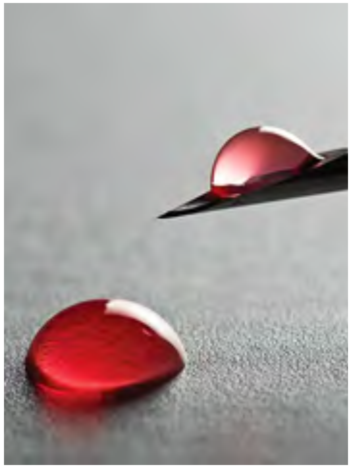
$$\mathscr{C} \equiv \frac{\mu U_0}{\gamma}$$

- Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
- Thorsen, Todd, et al. "Dynamic pattern formation in a vesicle-generating microfluidic device." Physical review letters 86.18 (2001): 4163.
- Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg
- Droplet: http://www.rycobel.be/en/technical-info/articles/1337/measuring-dynamic-absorption-and-wetting

Fluid Behavior

Viscous Forces

Interfacial Forces



Capillary Number: viscous/interfacial

Large **surface-to-volume** ratios in microfluidic devices

- Makes surface effects increasingly important.
- Important when free fluid surfaces are present.

Surface tensions can exert significant stress

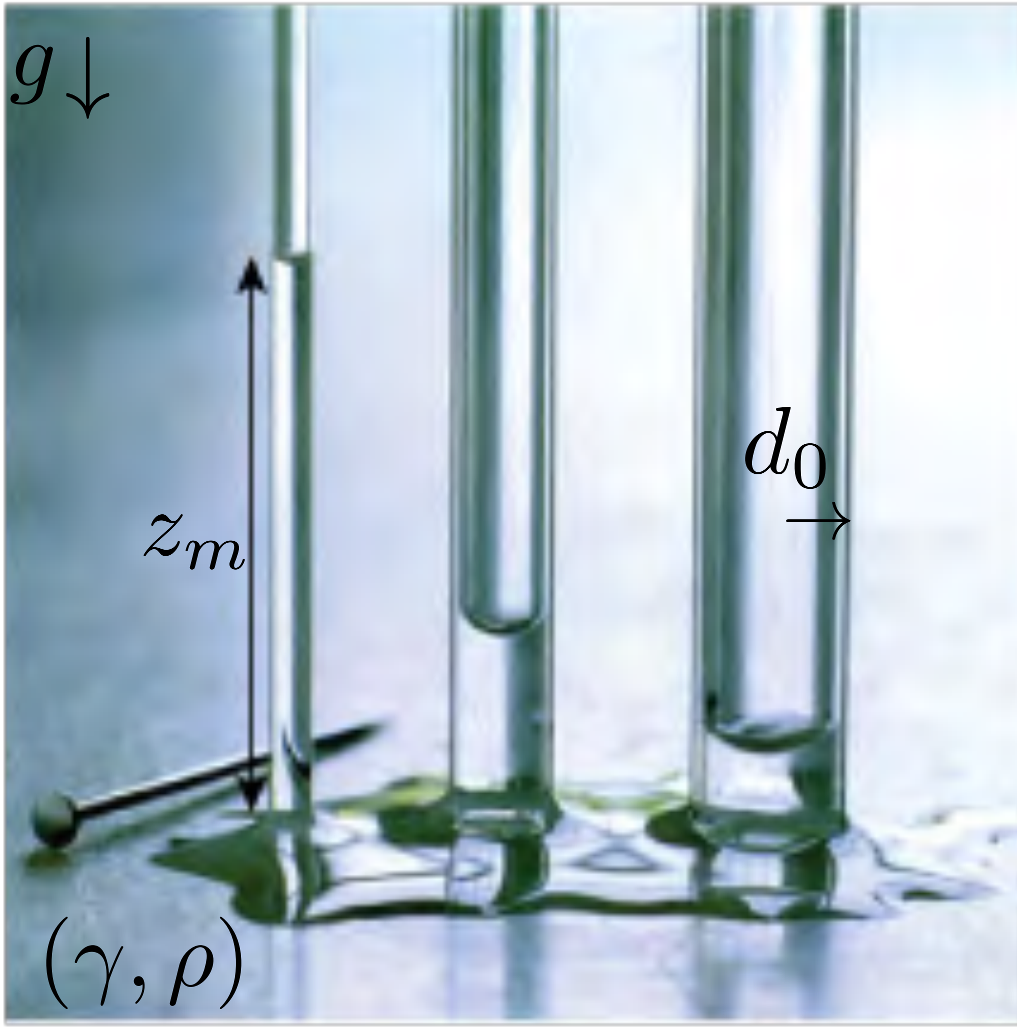
- Result in free surface deformations.
- Can drive fluid motion.

Capillary forces tend to draw fluid into wetting microchannels

- Occurs when **solid-liquid** interfacial **energy** is **lower** than the **solid-gas** interfacial **energy**.

- Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
- Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg
- Droplet: http://www.rycobel.be/en/technical-info/articles/1337/measuring-dynamic-absorption-and-wetting

Capillary Rise



Classical Problem:

- Noted as early as 15th by Leonardo da Vinci.
- Attributed to circulation in plants in 17th century by Montanari.

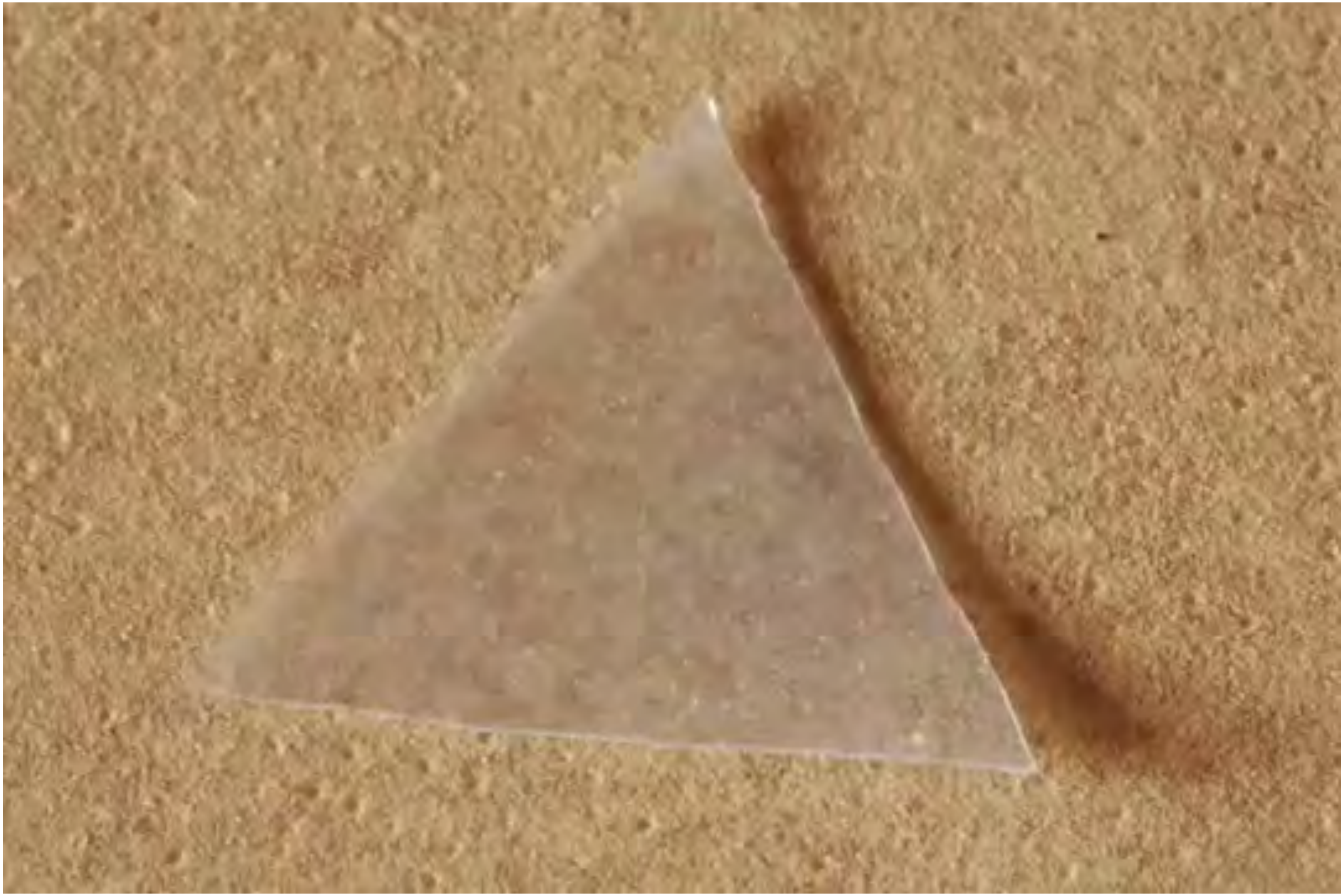
$$U = \underbrace{2\pi\gamma\cos\theta_e r z_m}_{surface\ energy} + \underbrace{\frac{1}{2}\rho g \pi r^2 z_m^2}_{gravitational\ P.E.}$$

$$\frac{dU}{dz_m} = 0$$

$$\ell_{cg} \sim \frac{\gamma\cos\theta_e}{\rho g d_0}$$

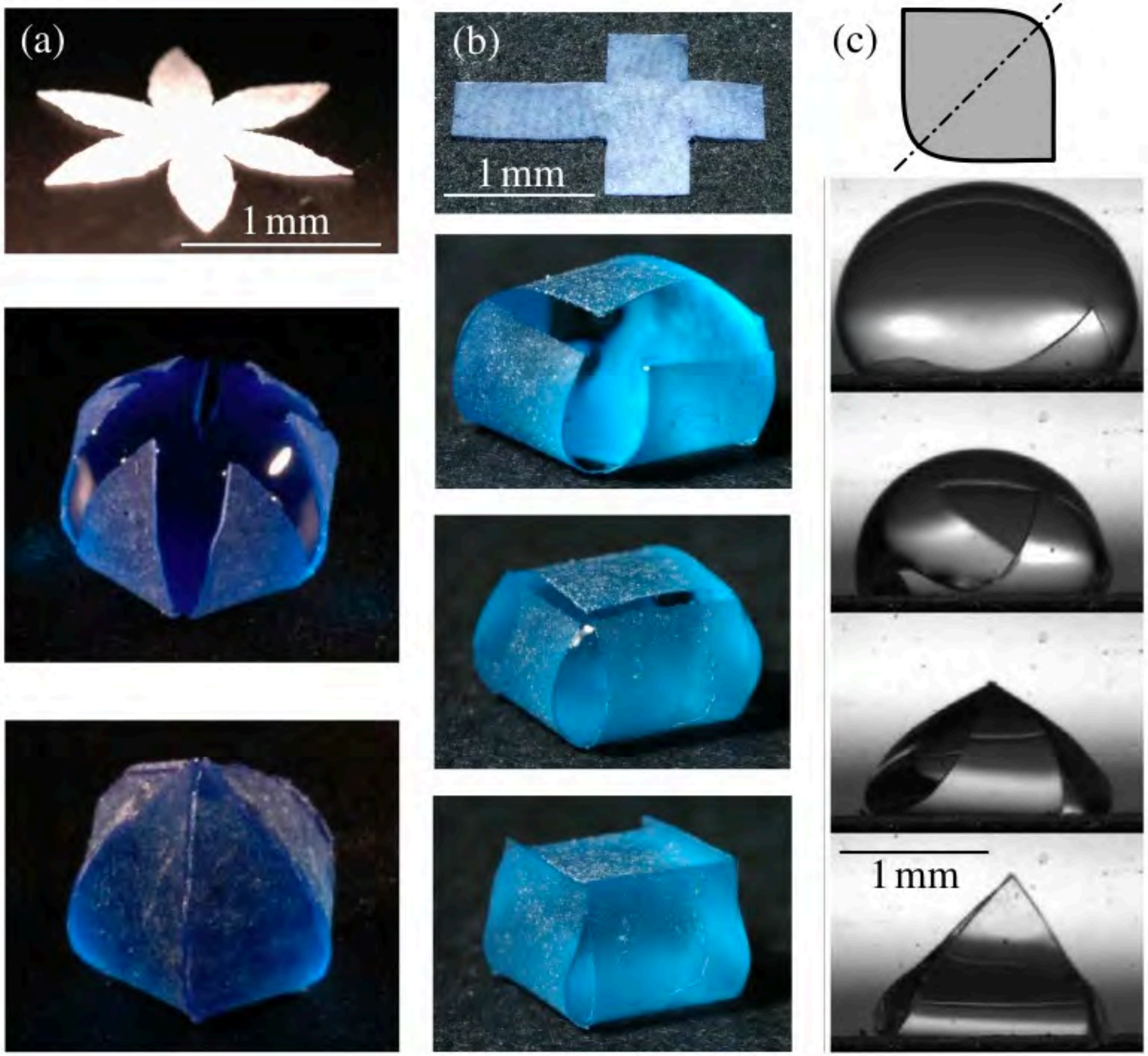
Balance: Surface Tension & Gravity

J.M. Bell and F.K. Cameron, "The flow of liquids through capillary spaces," J. Phys. Chem. **10**, 658-674, (1906).



Py, Charlotte, Paul Reverdy, Lionel Doppler, José Bico, Benoit Roman, and Charles N. Baroud. "Capillary origami: spontaneous wrapping of a droplet with an elastic sheet." Physical Review Letters 98, no. 15 (2007): 156103.

Elastocapillarity

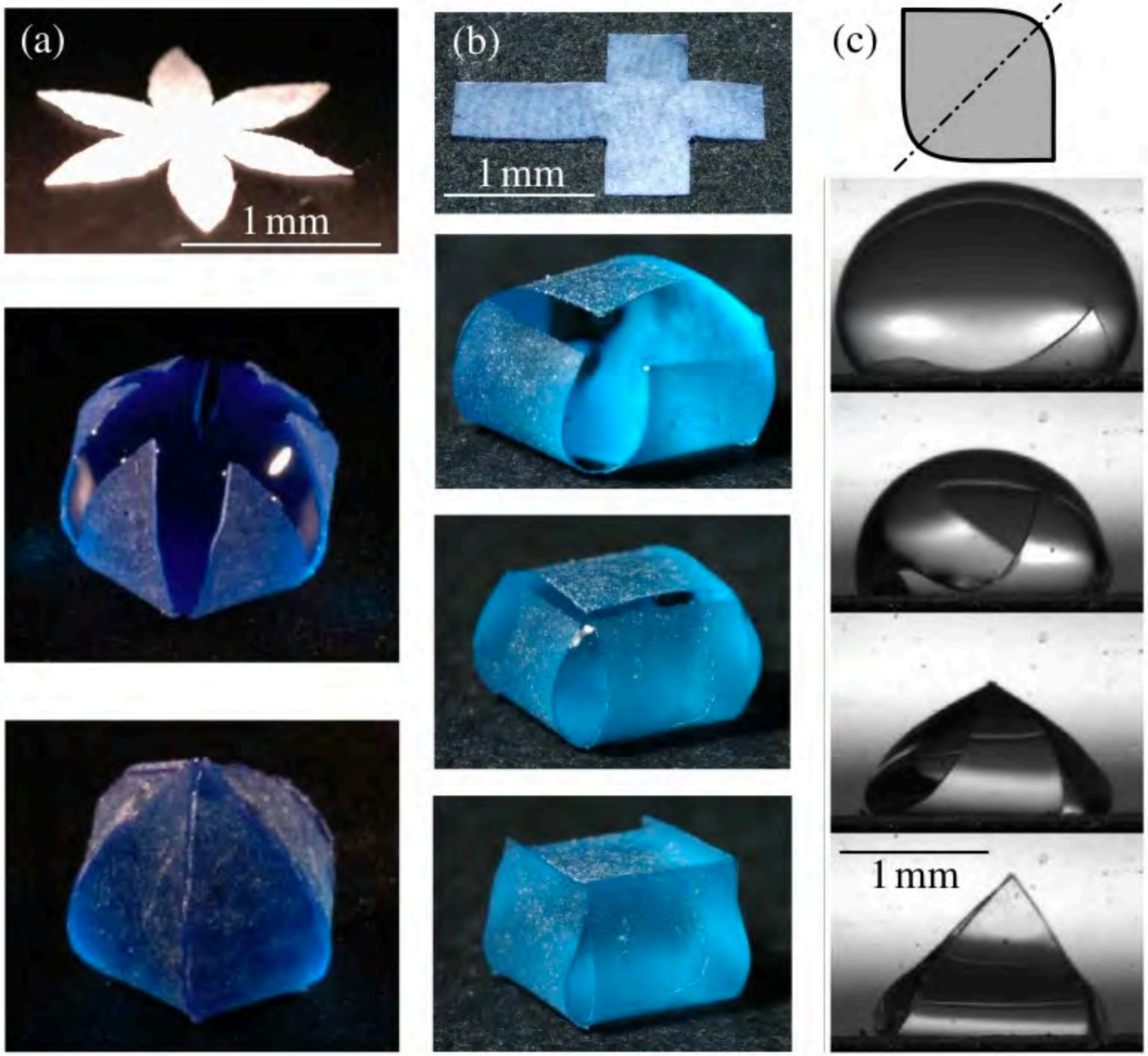


Fluid-structure interaction:

- Droplet bends and folds the sheet.
- Droplet is minimizing the amount of its surface in contact with air.
- Liquid-air surface area is minimized at the expense of bending the sheet.

- Py, Charlotte, Paul Reverdy, Lionel Doppler, José Bico, Benoit Roman, and Charles N. Baroud. "Capillary origami: spontaneous wrapping of a droplet with an elastic sheet." Physical Review Letters 98, no. 15 (2007): 156103.

Elastocapillarity



Fluid-structure interaction:

Elastic energy of a plate – bending:

$$\mathcal{U}_e = \frac{1}{2}\iint_P \mathrm{d}x\mathrm{d}y \int_{-h/2}^{h/2} \mathrm{d}z \left(\sigma_{\alpha\beta}\varepsilon_{\alpha\beta}\right)$$

Relation between in-plane strain to out-of-plane bending:

$$\varepsilon_{\alpha\beta}(x) = z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{z}{R}$$

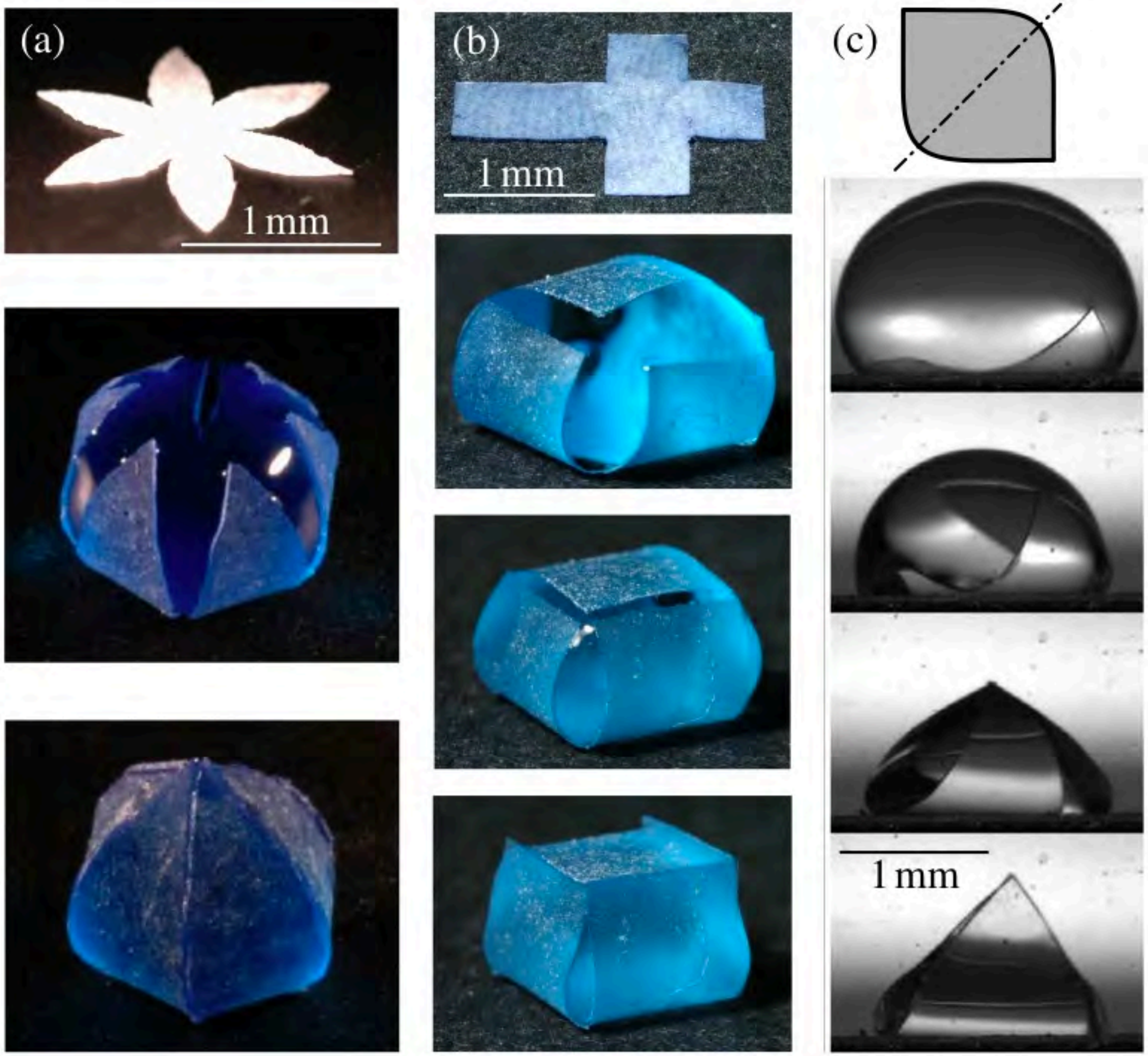
Bending energy:

$$\mathcal{U}_b = \frac{1}{2}\iint_P \mathrm{d}x\mathrm{d}y\, \frac{Eh^3}{12}\left(\frac{1}{R}\right)^2$$

$$\mathcal{U}_b \sim Eh^3$$

• Py, Charlotte, Paul Reverdy, Lionel Doppler, José Bico, Benoit Roman, and Charles N. Baroud. "Capillary origami: spontaneous wrapping of a droplet with an elastic sheet." Physical Review Letters 98, no. 15 (2007): 156103.

Elastocapillarity



Fluid-structure interaction:

Bending energy:

$$\mathcal{U}_b \sim Eh^3$$

Surface energy:

$$\mathcal{U}_\gamma \sim \gamma L^2$$

Elastocapillary length:

$$\ell_{ec} \sim \sqrt{\frac{Eh^3}{\gamma}} \sim \sqrt{\frac{B}{\gamma}}$$

Elastocapillary bending of sheet:

$$7\ell_{ec} \le L \le 12\ell_{ec}$$

- Py, Charlotte, Paul Reverdy, Lionel Doppler, José Bico, Benoit Roman, and Charles N. Baroud. "Capillary origami: spontaneous wrapping of a droplet with an elastic sheet." Physical Review Letters 98, no. 15 (2007): 156103.

Elastocapillarity

Capillary rise between flexible fibers.

$$\underbrace{B\frac{\partial^4 d}{\partial x^4}}_{\text{Beam bending}} = \underbrace{\gamma\kappa_f}_{\text{Laplace pressure}}$$

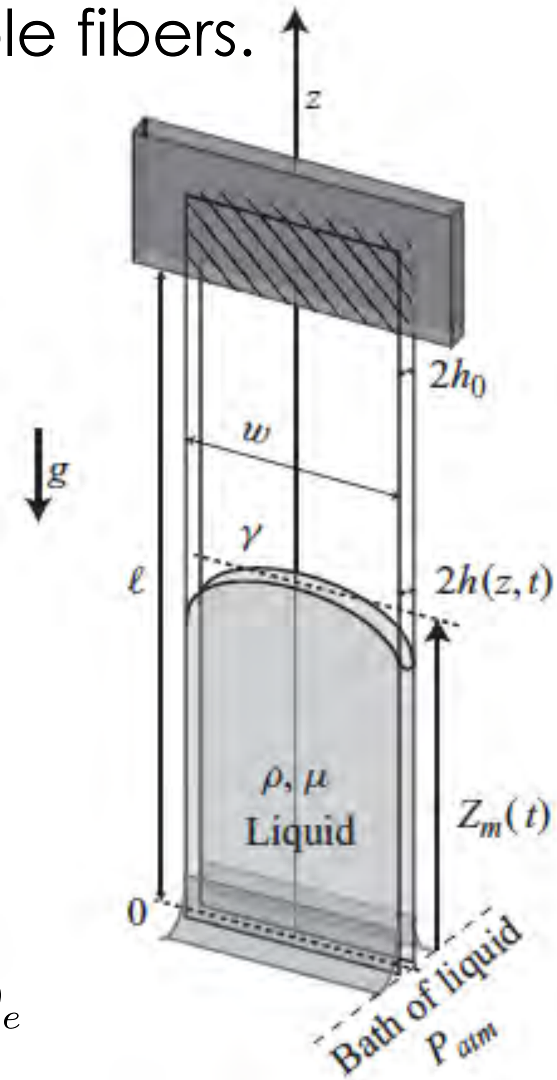
Curvature of meniscus

$$\kappa_f = \frac{1}{d(z_m)}\cos\theta_e + \frac{2}{b}\cos\theta_e$$

Assume gap is much smaller than width: $d_0 \ll b$

Initial gap: $d(z_m, t) = d_0$

Approximation of meniscus curvature: $\kappa_f \approx \dfrac{1}{d_0}\cos\theta_e$



H.-Y. Kim and L. Mahadevan, "Capillary rise between elastic sheets, " J. Fluid Mech. **548**, 141-150, (2006).
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C. Duprat, J.M. Aristoff, and H.A. Stone, "Dynamics of elastocapillary rise," J. Fluid Mech. **679**, 641-654, (2011).

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Elastocapillarity

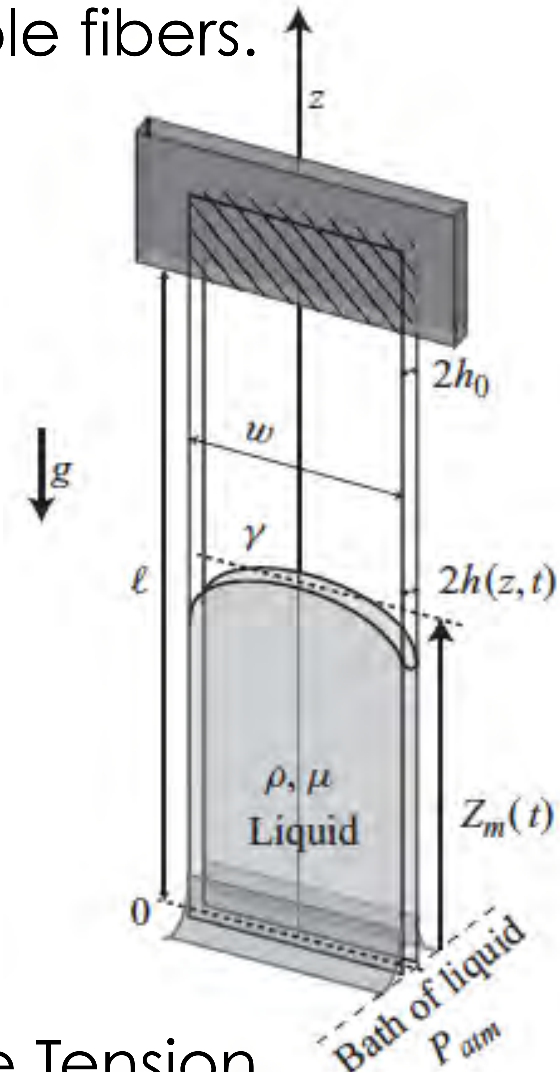
Capillary rise between flexible fibers.

$$\underbrace{B\frac{\partial^4 d}{\partial x^4}}_{\text{Beam bending}} = \underbrace{\gamma\kappa_f}_{\text{Laplace pressure}}$$

Approximation: $\kappa_f \approx \dfrac{1}{d_0}\cos\theta_e$

Scaling: $B\dfrac{d_0}{\ell^4} \sim \dfrac{\gamma}{d_0}\cos\theta_e$

$$\ell_{ec} \equiv \left(\frac{Bd_0^2}{\gamma\cos\theta_e}\right)^{1/4}$$



Balance: Bending & Surface Tension

H.-Y. Kim and L. Mahadevan, "Capillary rise between elastic sheets, " J. Fluid Mech. **548**, 141-150, (2006).
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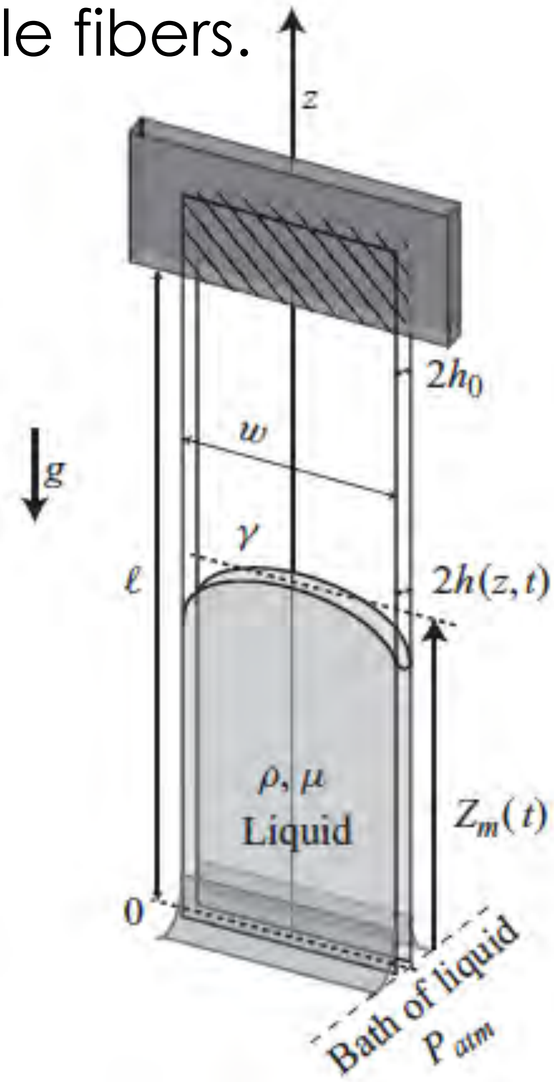
Elastocapillarity

Capillary rise between flexible fibers.



Potential energies:

Elastic energy

$$\mathcal{U}_e \sim \frac{Bwh_0^2}{\ell^3}$$

Gravitational potential energy

$$\mathcal{U}_g \sim \rho g w h_0 \ell^2$$

Surface energy

$$\mathcal{U}_c \sim w\ell\gamma$$

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Elastocapillarity

Capillary rise between flexible fibers.

Characteristic length scales:

Elastocapillary length

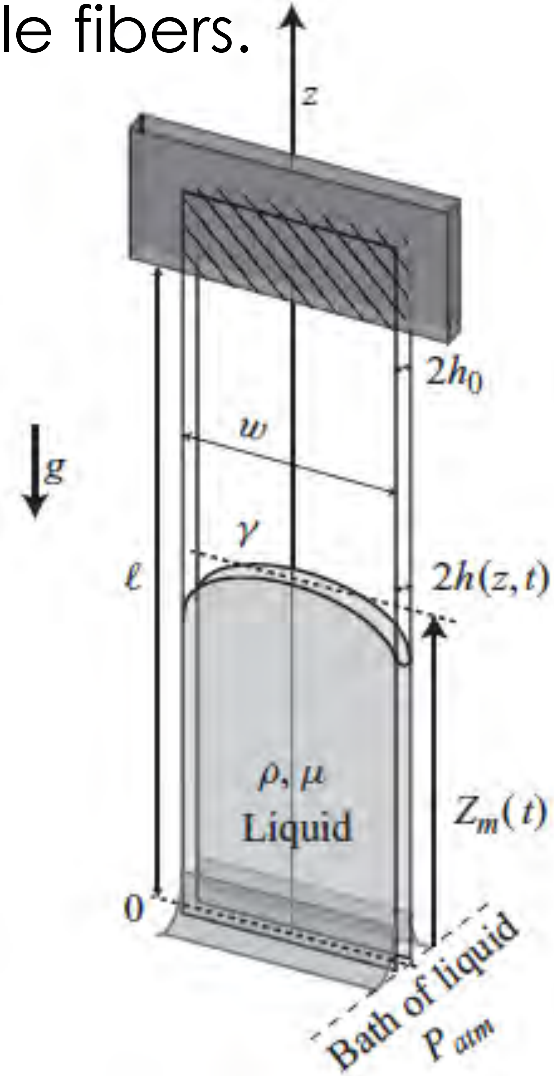
$$\ell_{ec} = \frac{\mathcal{U}_e}{\mathcal{U}_c} = \left(\frac{Bh_0^2}{\gamma}\right)^{1/4}$$

Capillary gravity length

$$\ell_{cg} = \frac{\mathcal{U}_c}{\mathcal{U}_g} = \frac{\gamma}{\rho g h_0}$$

Elastogravity length

$$\ell_{eg} = \frac{\mathcal{U}_e}{\mathcal{U}_g} = \left(\frac{Bh_0}{\rho g}\right)^{1/5}$$



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Elastocapillarity

Capillary rise between flexible fibers.

Dimensionless parameters

Bond number:

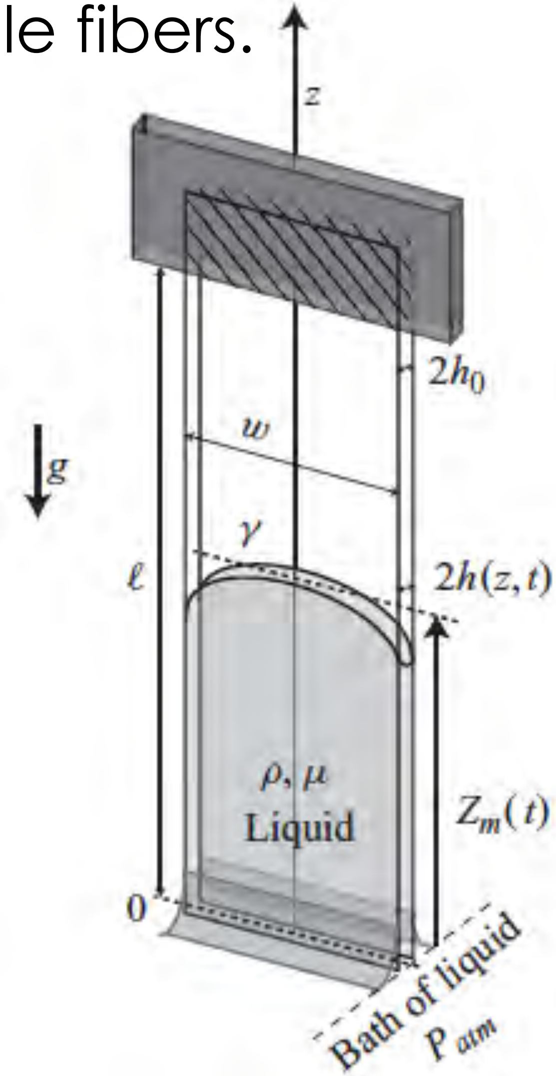
$$\mathscr{B} = \frac{\ell}{\ell_{cg}}$$

Elastocapillary number

$$\mathscr{E} = \left(\frac{\ell}{\ell_{ec}}\right)^4$$

Elastogravity number

$$\mathscr{G} = \mathscr{B}\mathscr{E} = \left(\frac{\ell}{\ell_{eg}}\right)^5$$



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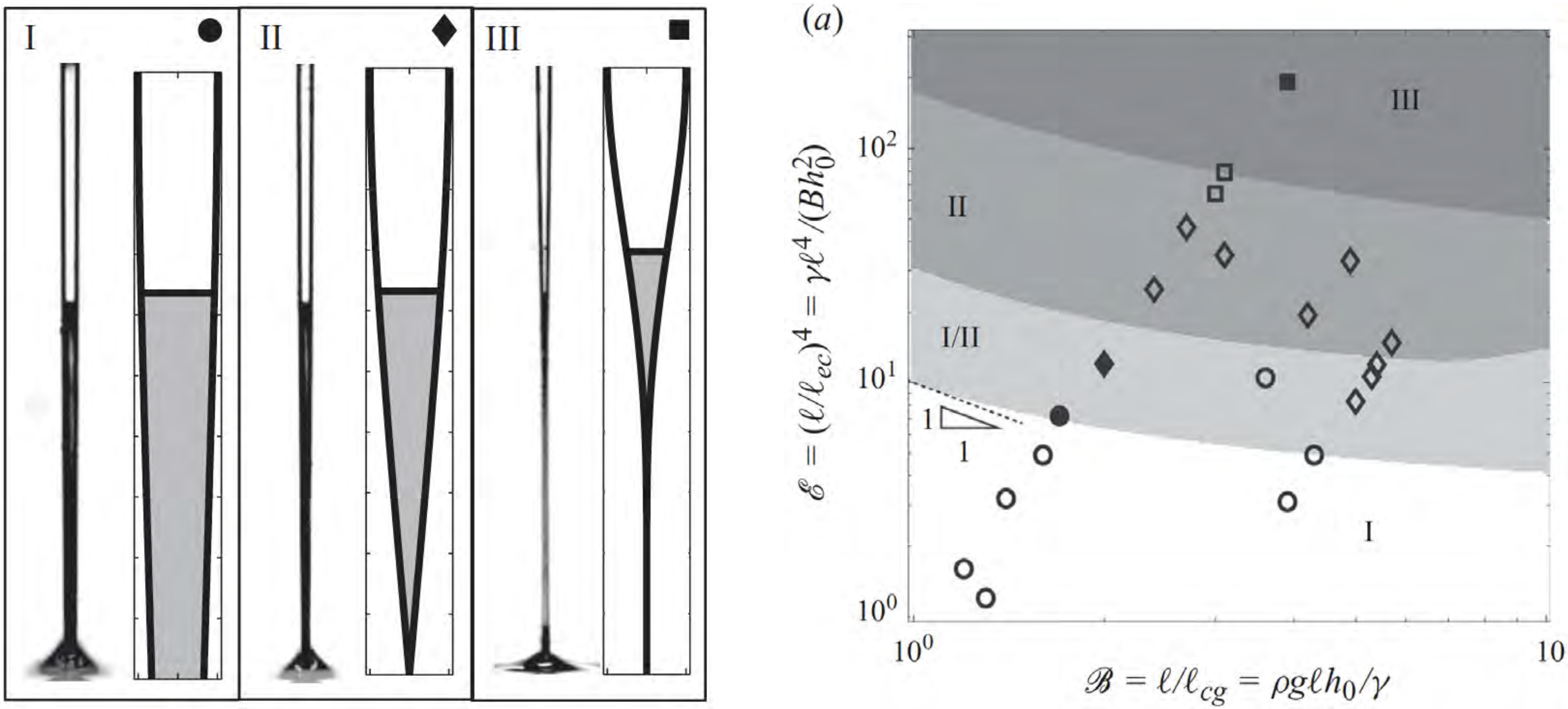
Elastocapillarity



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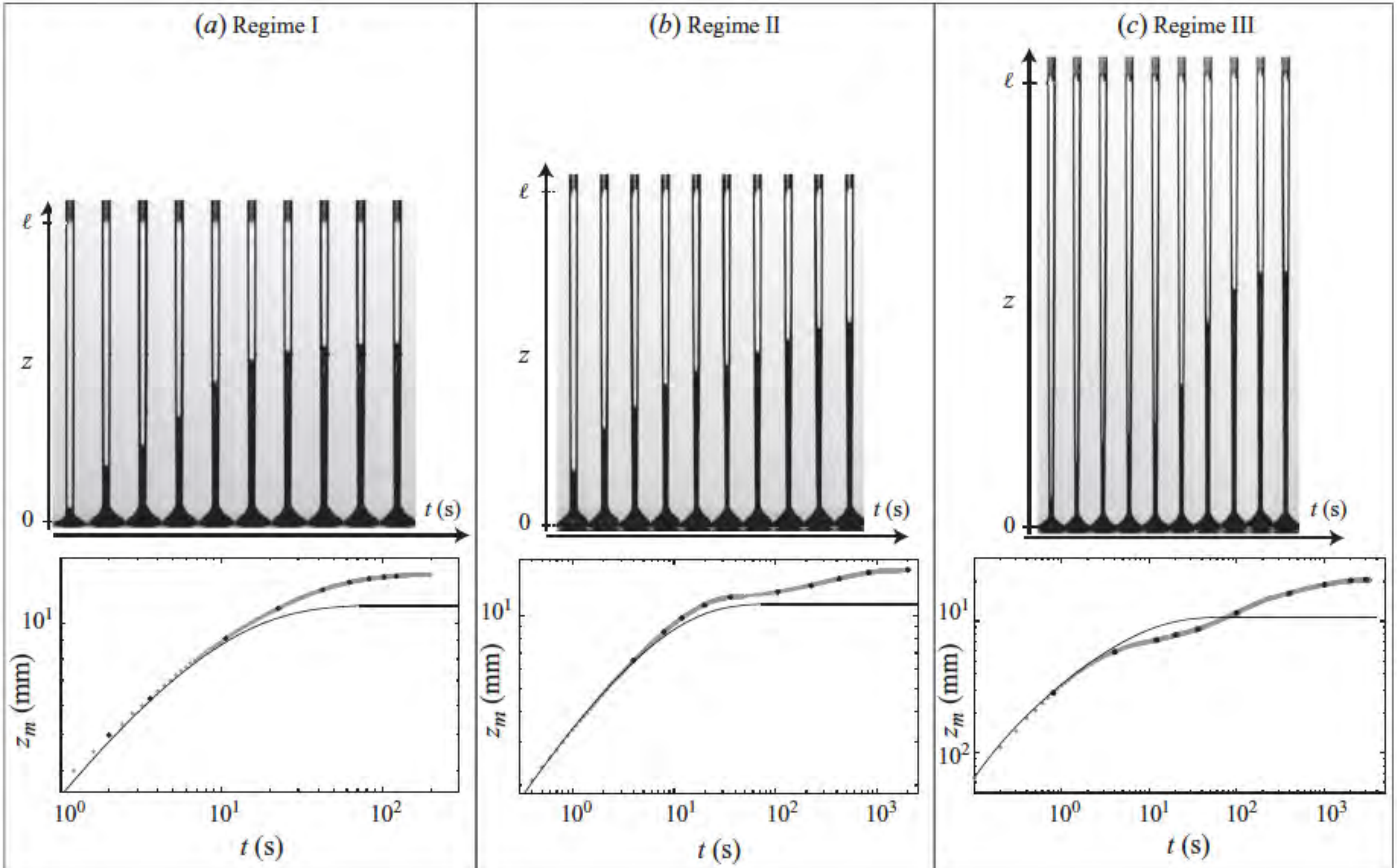
Elastocapillarity



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Capillarity & Swelling

Solid: Polyvinylsiloxane

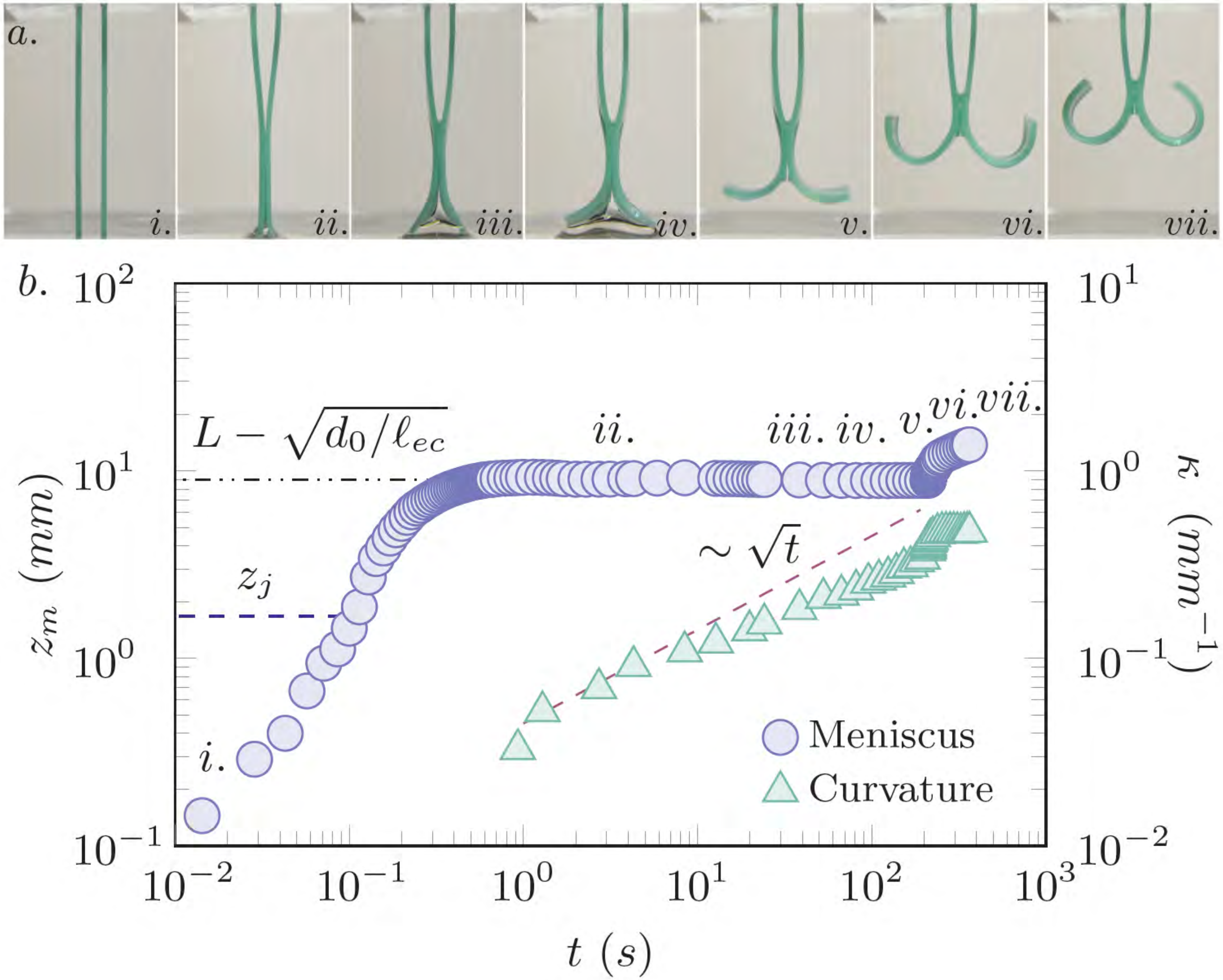
Fluid: Silicone Oil (5 cSt)

20x faster than real time

$E \approx 1\, MPa\, (PVS)$
$L = 20\; mm$
$d \approx 2\; mm$
$h \approx 0.5\; mm$

D.P. Holmes, A. Pandey, P.-T. Brun, and S. Protière, *In Preparation*, (2015).

Capillarity & Swelling



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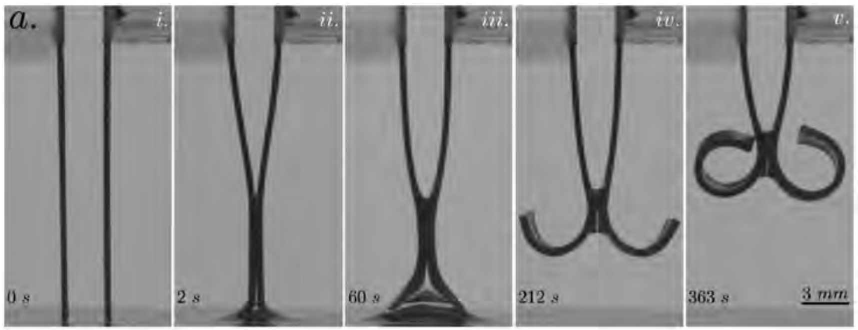
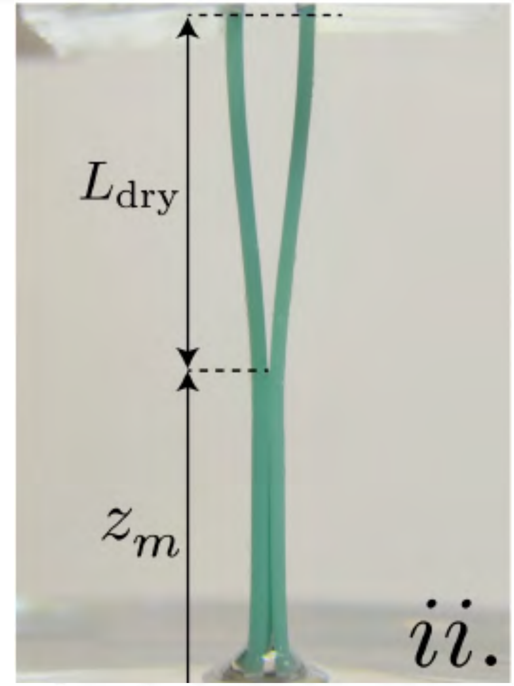
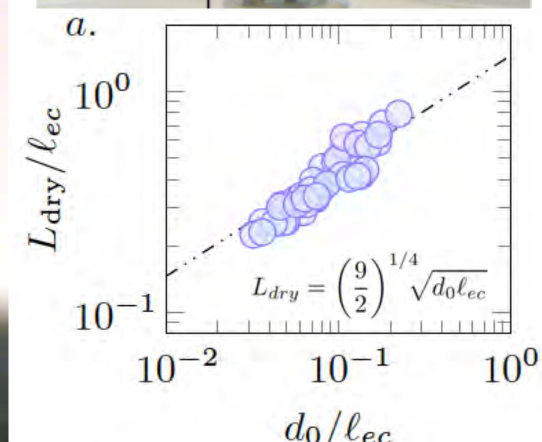
Capillarity & Swelling







1. Elastocapillary rise between flexible fibers.

   At short times, elastocapillary rise dominates the deformation.

2. Swelling-induced bending.

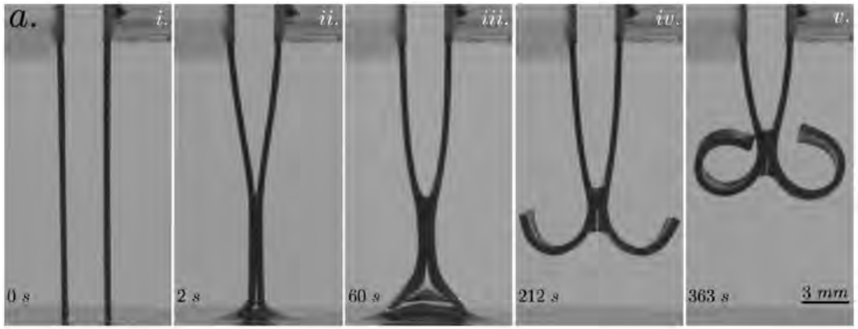
   Bending is constrained by surface tension, as the beam bends with a lower curvature than a free swelling beam.

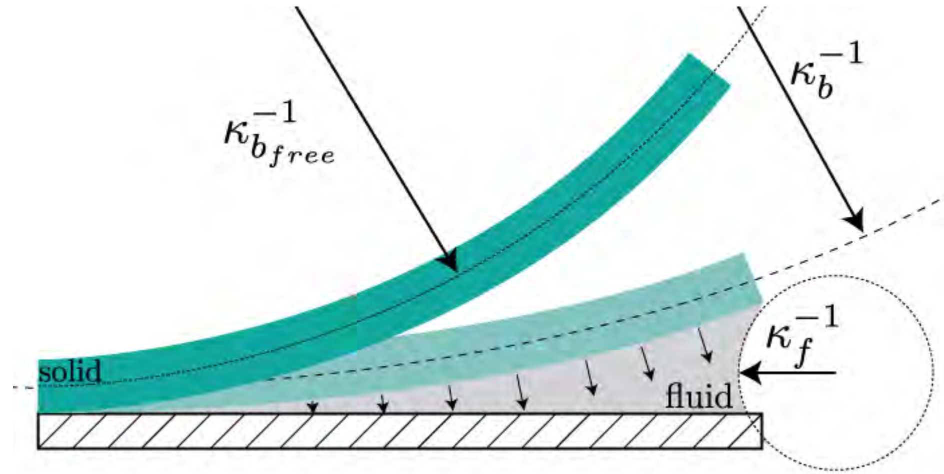
3. Bending dominates surface tension.

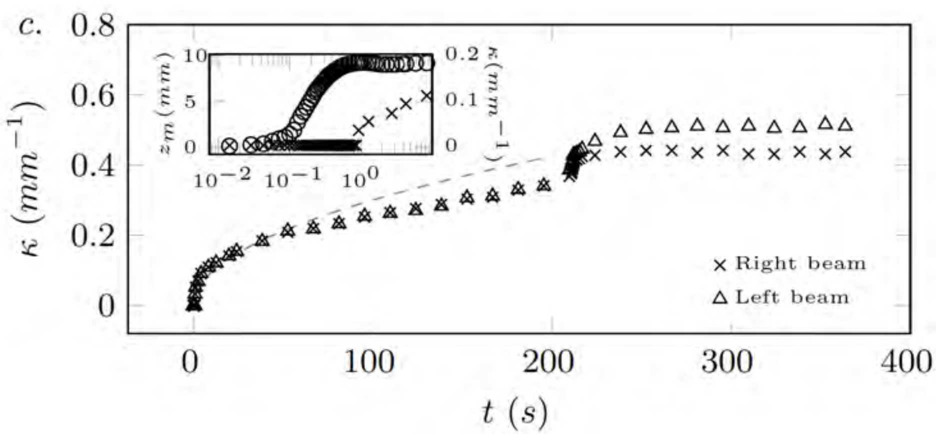
   Separation occurs as the “natural” curvature of the beam exceeds the fluids ability to confine it.

Bico, José, et al. "Adhesion: elastocapillary coalescence in wet hair." Nature 432.7018 (2004): 690-690.
D.P. Holmes, A. Pandey, P.-T. Brun, and S. Protière, *In Preparation*, (2015).

Capillarity & Swelling







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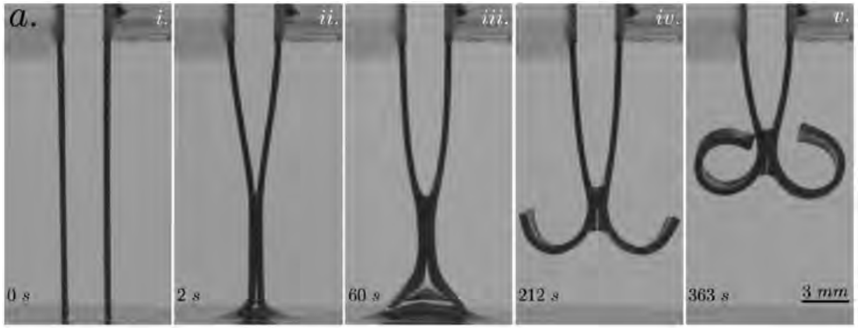
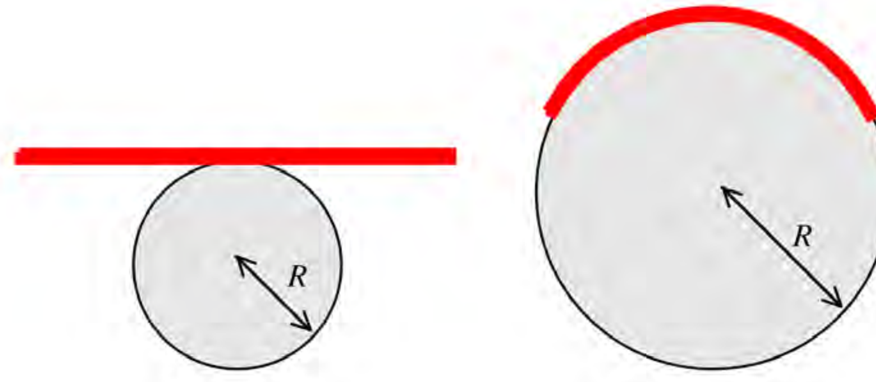
Capillarity & Swelling





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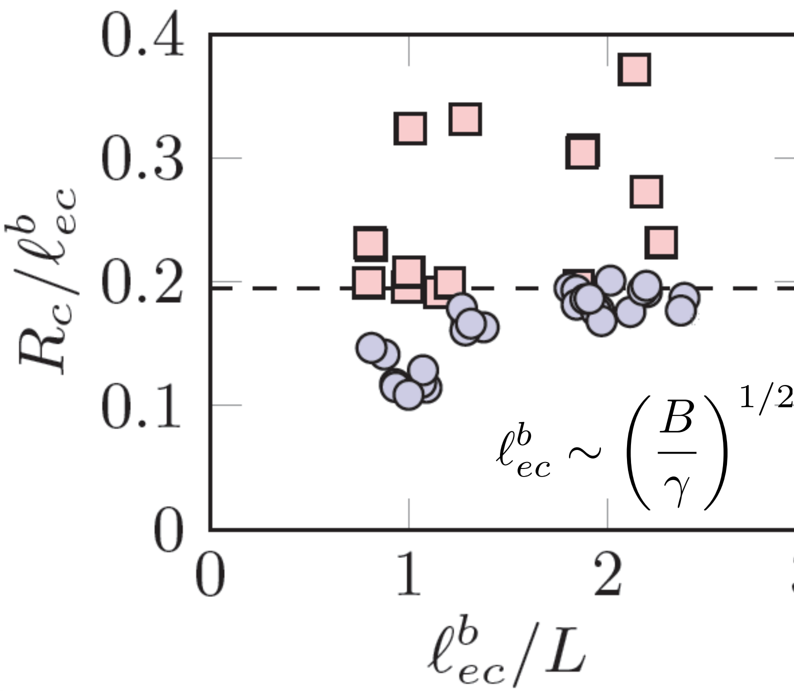
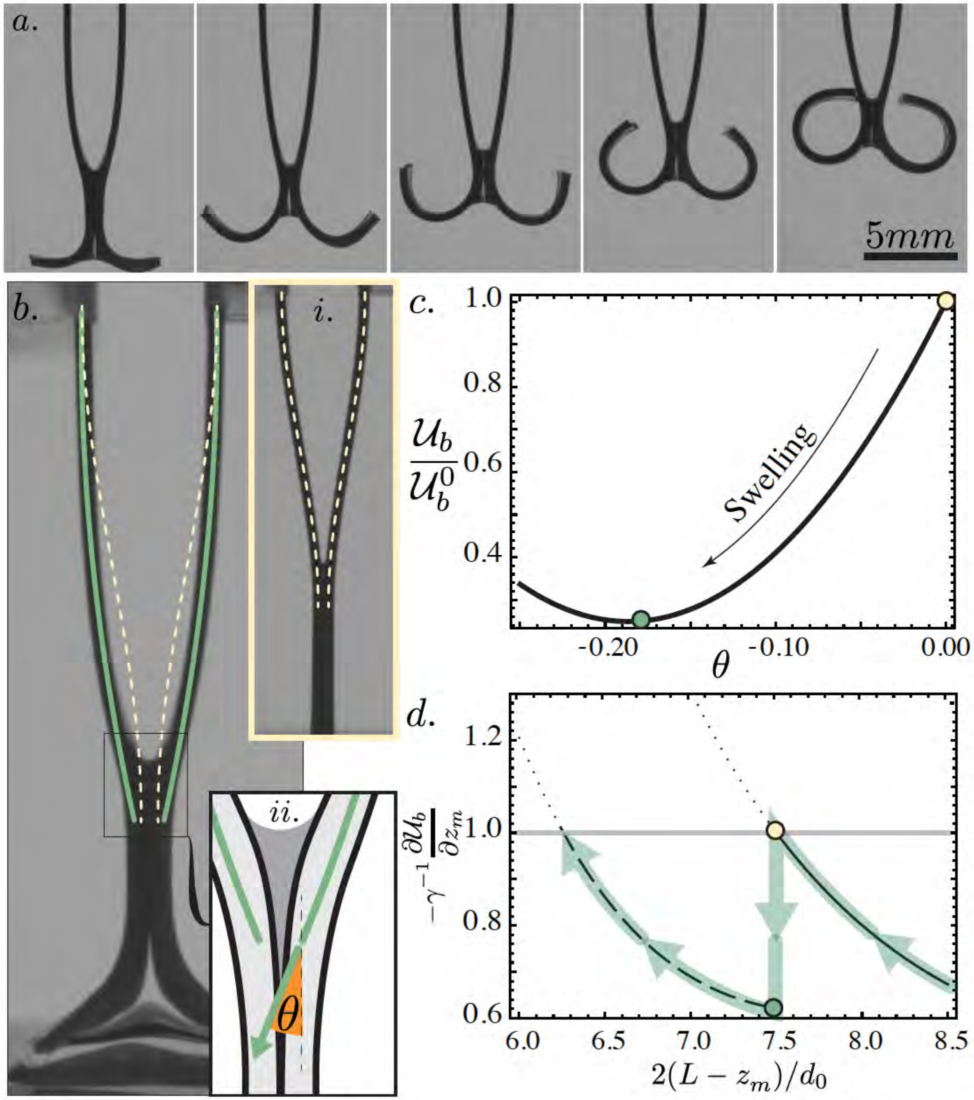


Roman, Benoit, and José Bico. "Elasto-capillarity: deforming an elastic structure with a liquid droplet." Journal of Physics: Condensed Matter 22.49 (2010): 493101.
D.P. Holmes, A. Pandey, P.-T. Brun, and S. Protière, *In Preparation*, (2015).

Swelling & Peeling



D.P. Holmes, A. Pandey, P.-T. Brun, and S. Protière, *In Preparation*, (2015).

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Baobab Flowering

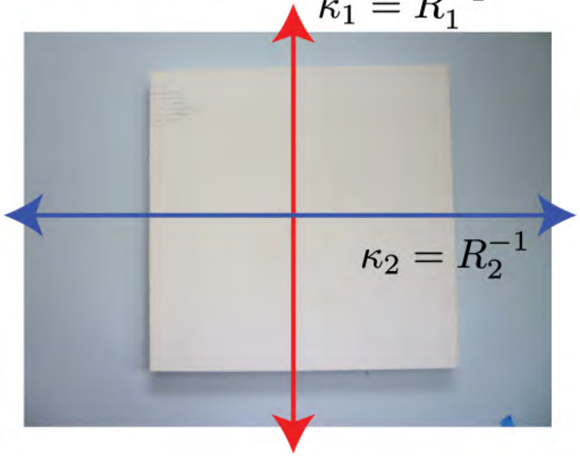
BBC - Planet Earth: Seasonal Forests

What about geometry?

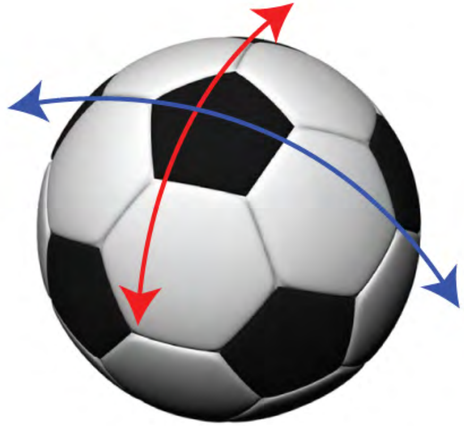
Mechanics of Thin

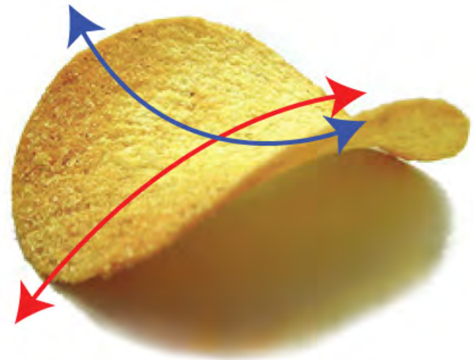
Gauss Curvature

$\kappa_1 = R_1^{-1}$



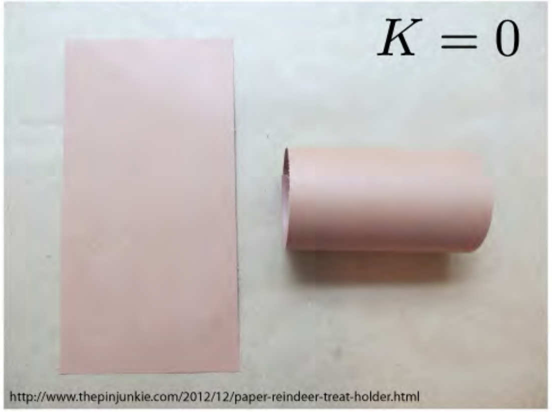
$\kappa_2 = R_2^{-1}$

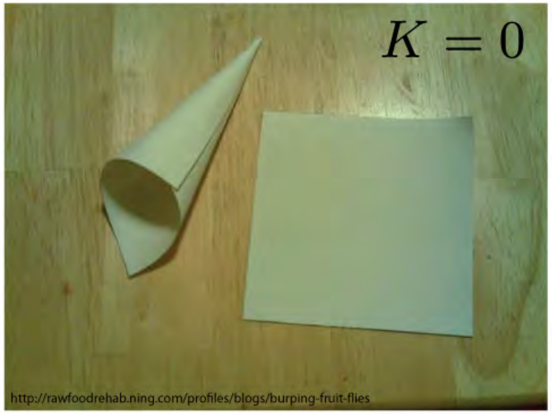




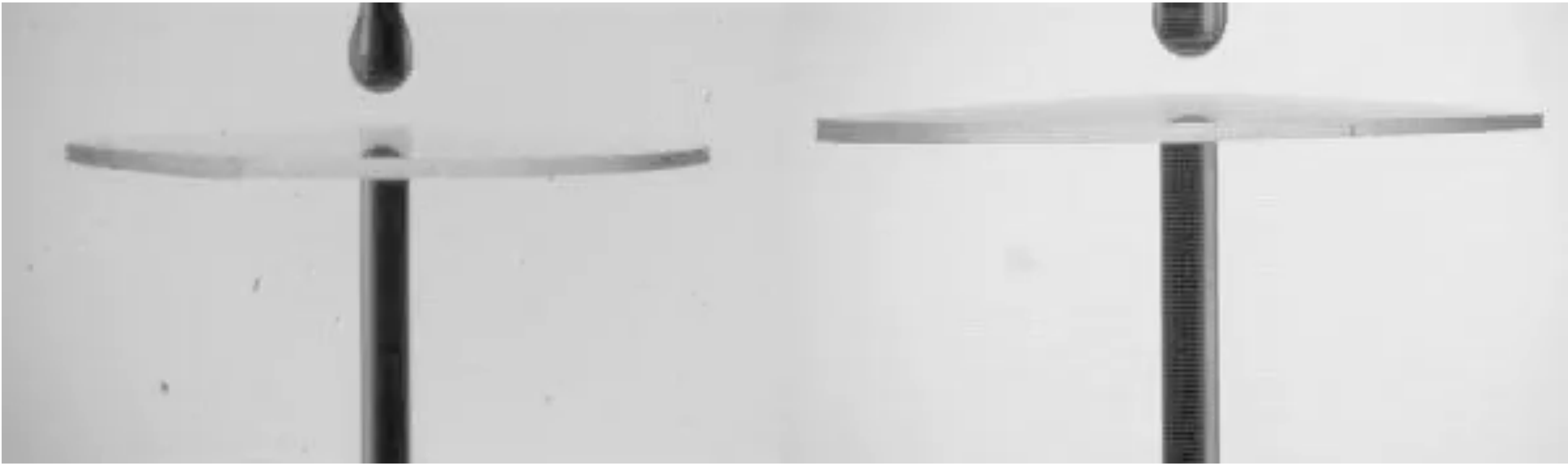
$K = \kappa_1\kappa_2 = \Diamond^4[w, w,] = 0$ $\quad$ $K = \kappa_1\kappa_2 = \Diamond^4[w, w,] > 0$ $\quad$ $K = \kappa_1\kappa_2 = \Diamond^4[w, w,] < 0$

Developable





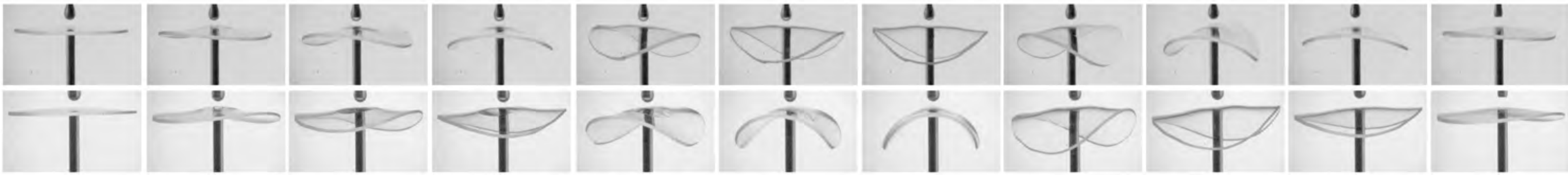
Dancing Disks



Axisymmetric Disks

D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. “Bending and Twisting of Soft Materials by Non-homogenous Swelling” *Soft Matter*, **7**, 5188, 2011.

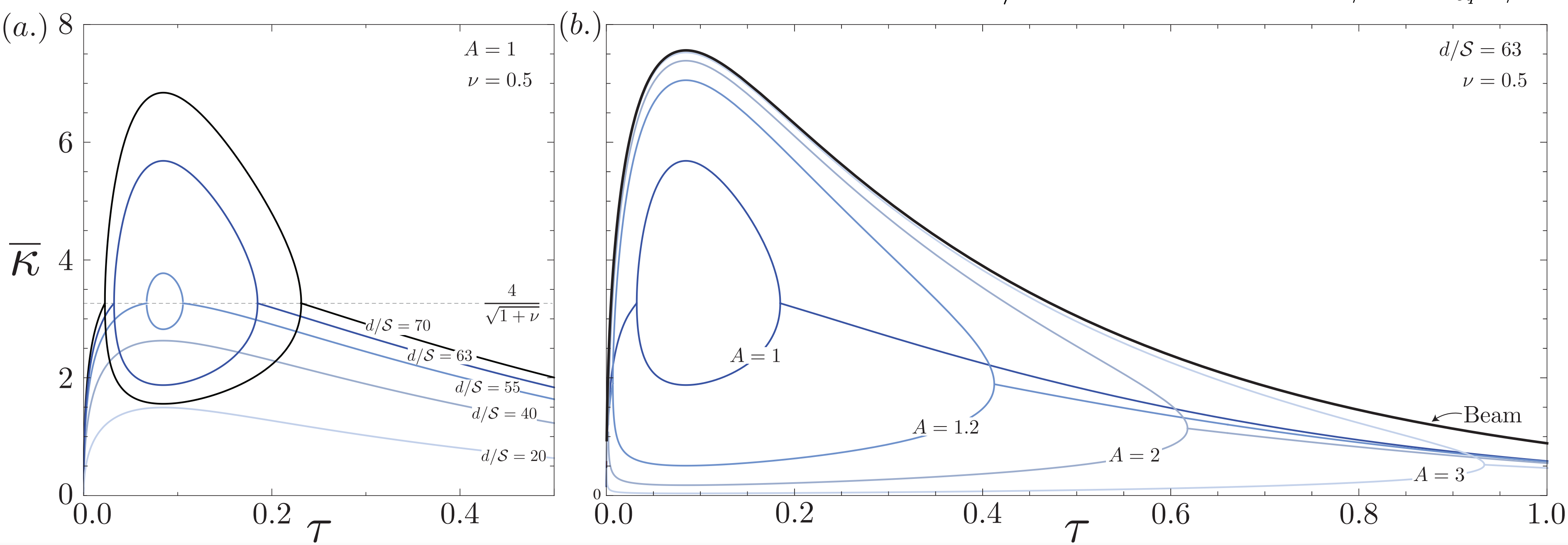
Dynamic Plate Shape



Diffusive dynamics: $$\psi(\overline{x}_3,\tau) = 1 + \sum_{n=0}^{\infty} \mathcal{A}(n)\sin[\lambda(n)\overline{x}_3]\, e^{-\lambda(n)^2\tau}$$

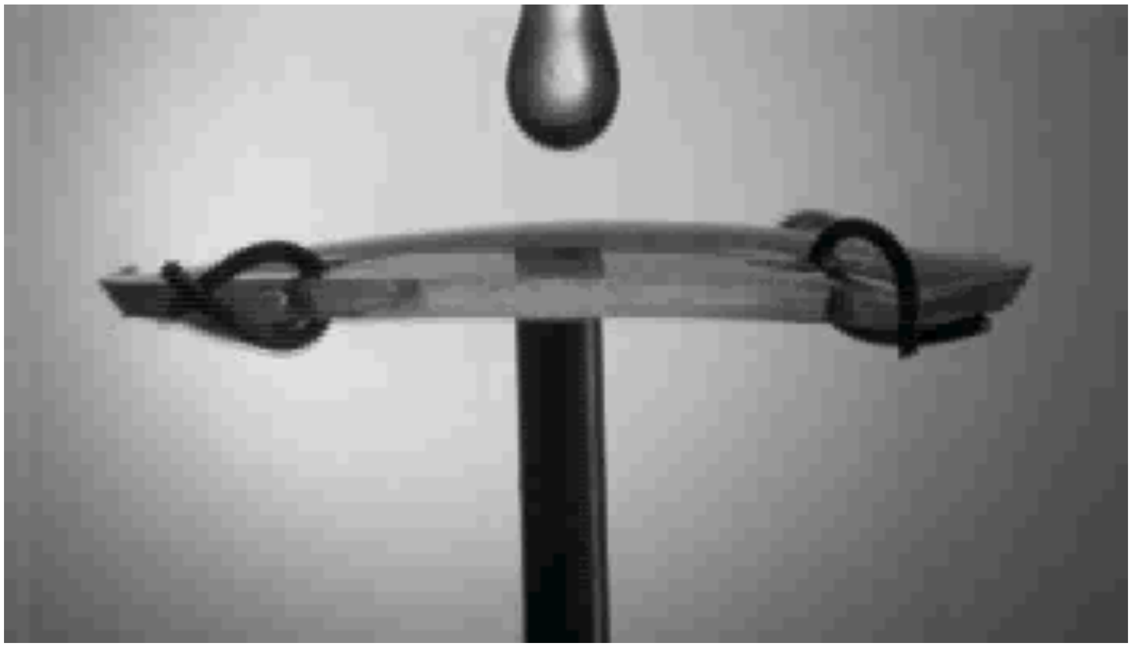
Bending dynamics: $$\overline{\kappa}_\varepsilon(\tau) = \frac{d}{\mathcal{S}}\int_{-1/2}^{1/2}\psi(\overline{x}_3,\tau)\overline{x}_3\ \mathrm{d}\overline{x}_3$$

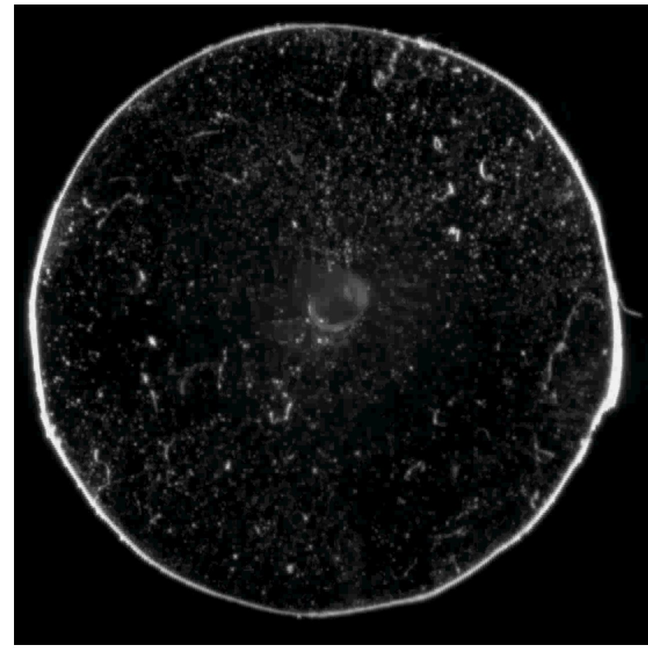
$d/\mathcal{S} = 12\varepsilon_{eq}L^2/h^2$



D.P. Holmes, A. Pandey, M. Pezzulla, and P. Nardinocchi,. *In Preparation* (2014).

Dynamics: Twisting

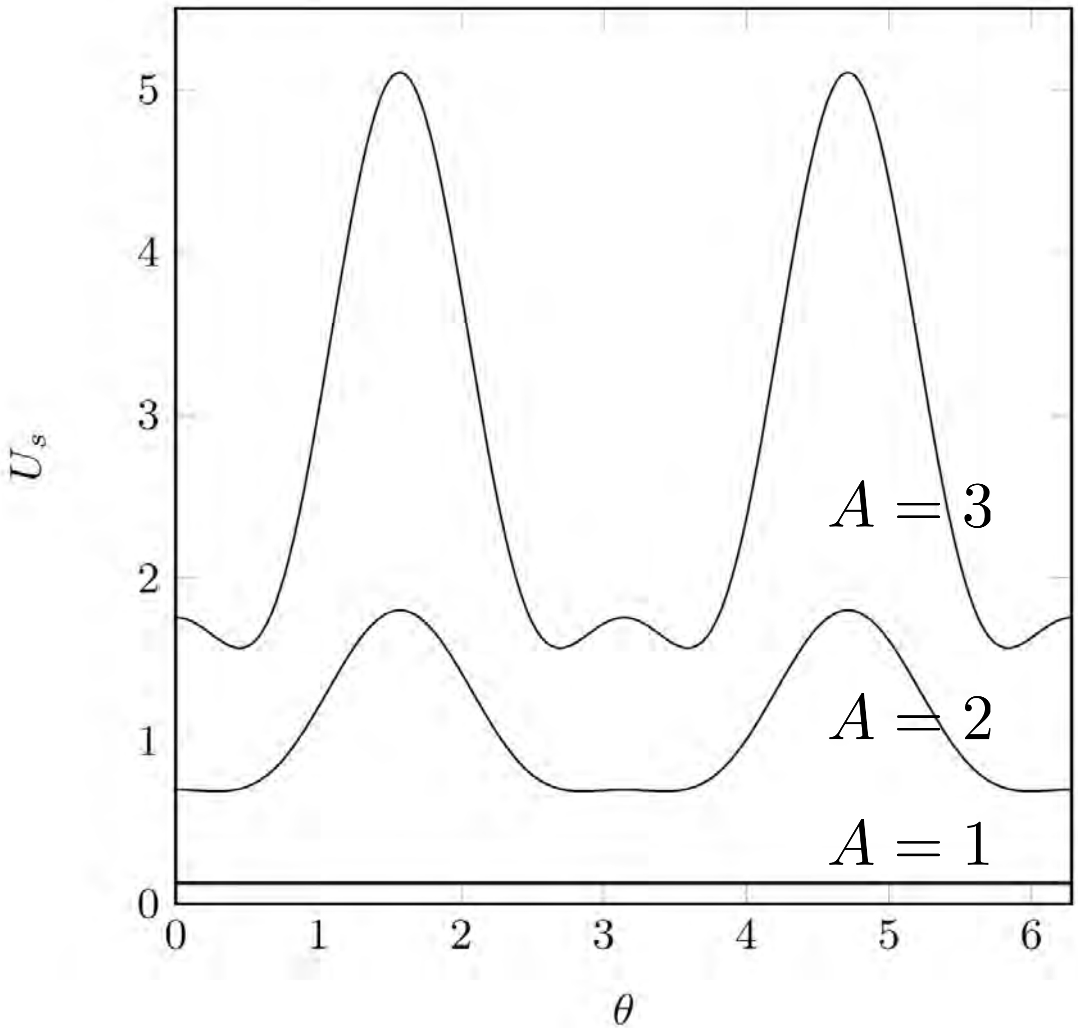




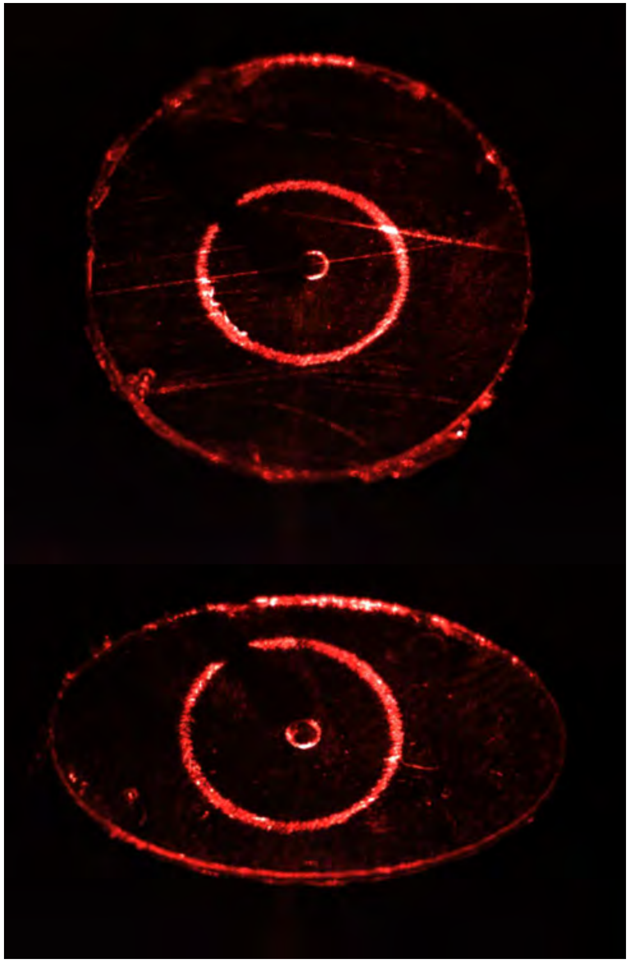
D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. “Bending and Twisting of Soft Materials by Non-homogenous Swelling” *Soft Matter*, **7**, 5188, 2011.

mechanics of slender structures

Dynamics: Twisting



$a/b = 1$



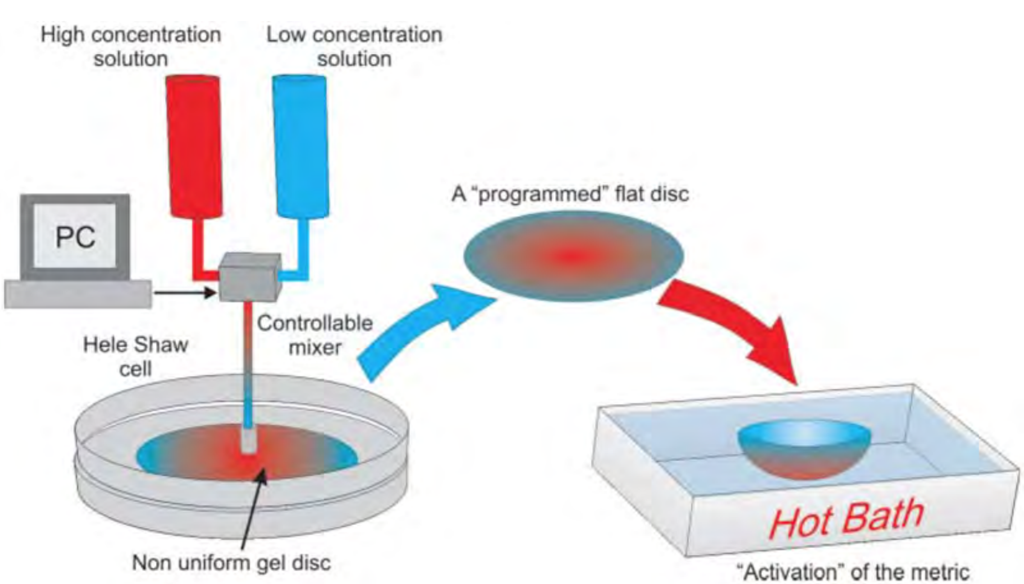
$a/b = 1.4$

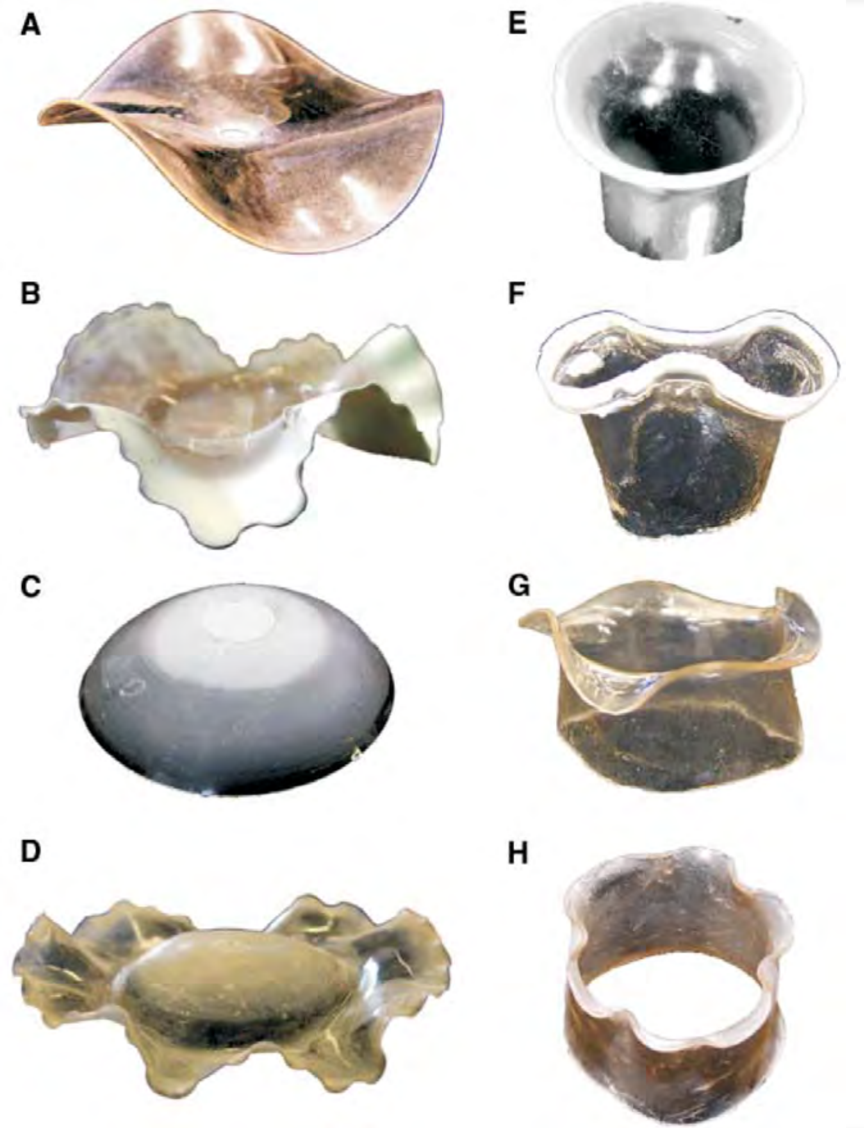
How do thin structures grow?

Permanent shape change and mass increase.

- Radial growth
- Through-thickness growth

Shaping Sheets



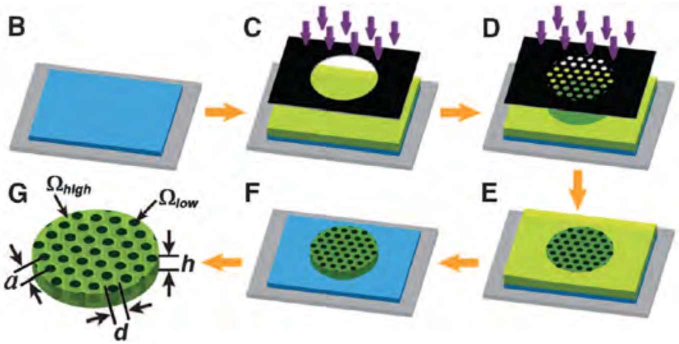


Shaping elastic sheets by prescribing non-Euclidean metrics

- Prepare gels that undergo nonuniform shrinkage.
- Buckling thin films based on chosen metrics.

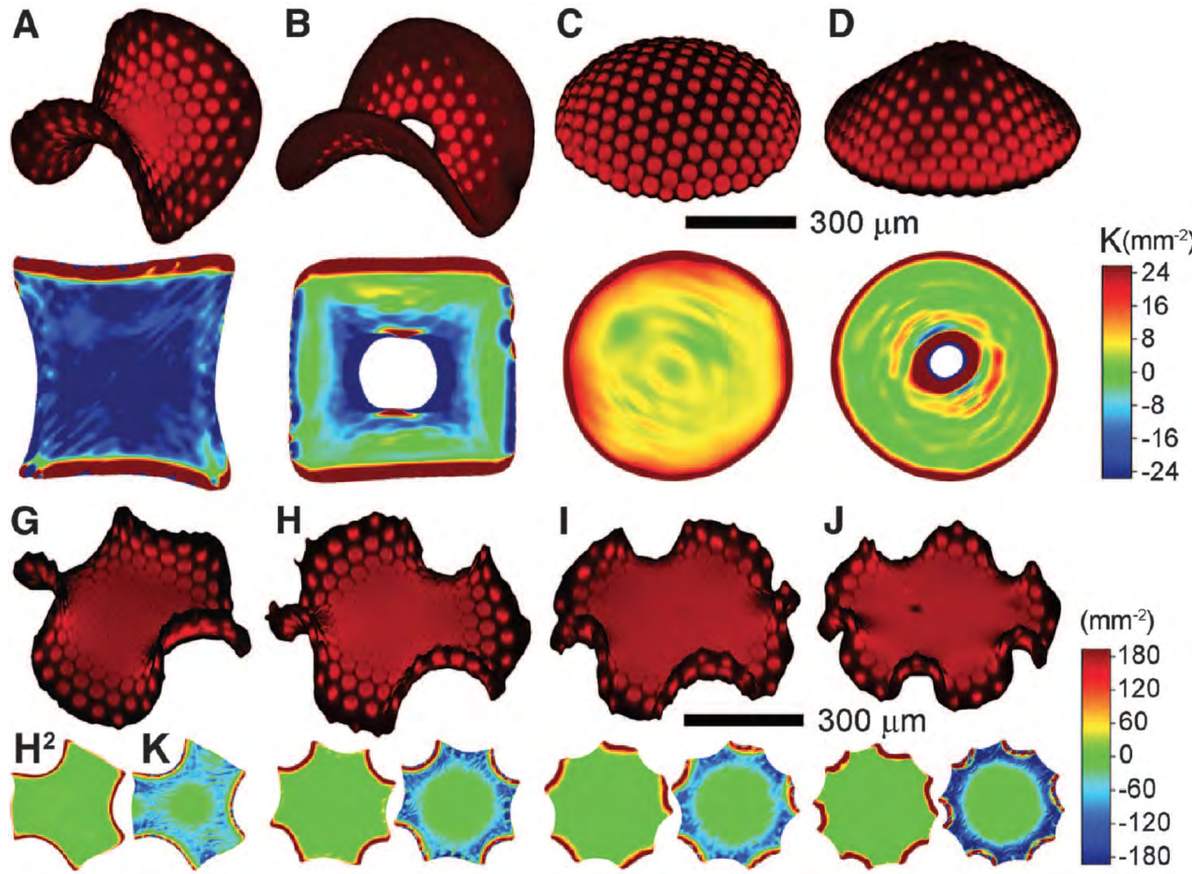
Y. Klein, E. Efrati, and E. Sharon, "Shaping of Elastic Sheets by Prescription of Non-Euclidean Metrics" *Science*, **315**, 1116, 2007.

Shaping Sheets



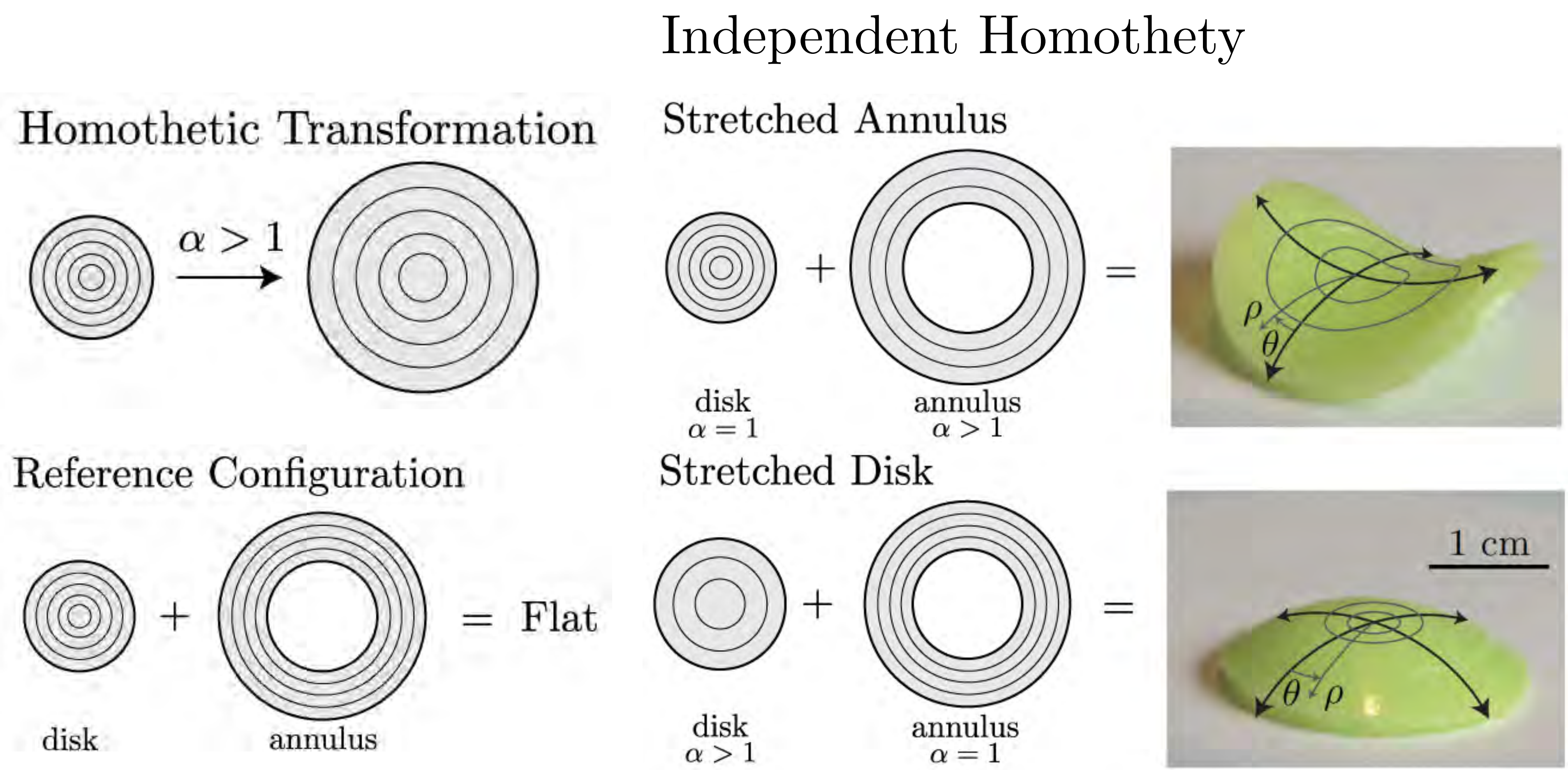
Shaping elastic sheets by halftone gel lithography

- Photopattern thin films.
- Thermal-actuated shape change.
- Swell to embedding based on prescribed metric.



J Kim, J.A. Hanna, M. Byun, C.D. Santangelo, and R.C. Hayward, "Designing Responsive Buckled Surfaces by Halftone Gel Lithography" *Science*, **335**, 1201, 2012.

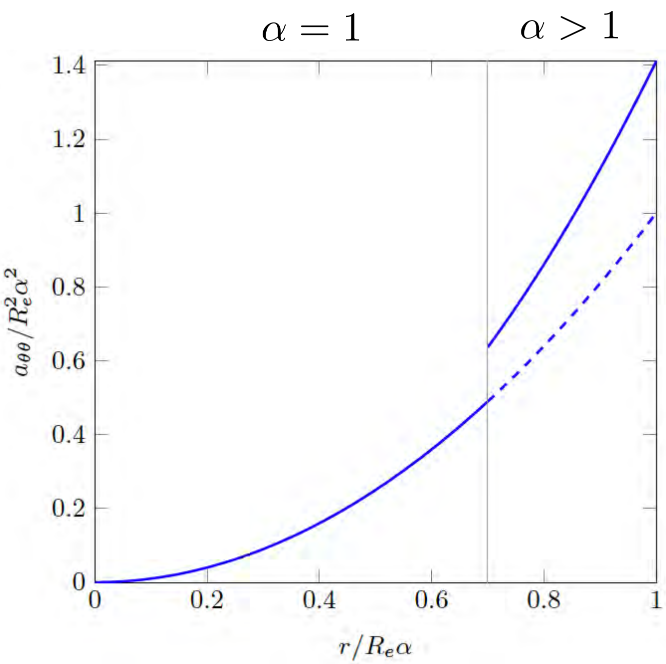
Geometric Composite

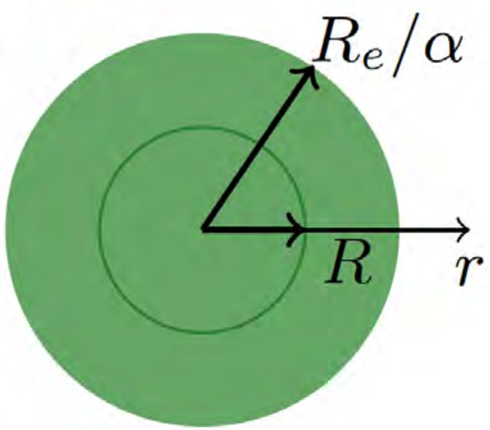


Goal: Use swelling to predictably & permanently morph plates into shells

M. Pezzulla, S. Shillig, P. Nardinocchi, and D.P. Holmes. "*Morphing of Geometric Composites via Residual Swelling*," *Soft Matter*, **11**, 5812-5820, (2015).

Geometric Composite





Stretching Dominated

Will bend as much as possible while minimizing stretching.

Stretching Energy of the Plate (incompressible)

$$\mathcal{U}_s \simeq h \int_A E[\mathrm{tr}^2(a - \overline{a}) + \mathrm{tr}(a - \overline{a})^2]\sqrt{|\overline{a}|}\mathrm{d}A$$

Realized metric (a) — Target metric ($\overline{a}$)

Stretching Energy (Assume all strains zero, except: $a_{\theta\theta} - \overline{a}_{\theta\theta}$)

$$\mathcal{U}_s \simeq Eh \int_0^R \frac{(a_{\theta\theta} - r^2)^2}{r^3}\,\mathrm{d}r + Eh \int_R^{R_e/\alpha} \frac{(a_{\theta\theta} - \alpha^2 r^2)^2}{\alpha^2 r^3}\,\mathrm{d}r$$

First Fundamental Form

$$ds^2 = d\rho^2 + a_{\theta\theta}(\rho)d\theta^2$$

Gaussian curvature

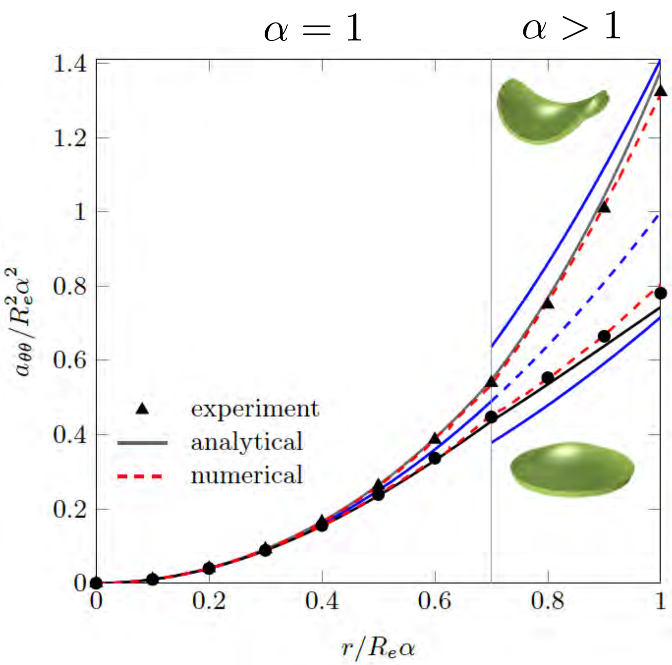
$$-\partial_{\rho\rho}\sqrt{a_{\theta\theta}}/\sqrt{a_{\theta\theta}}$$

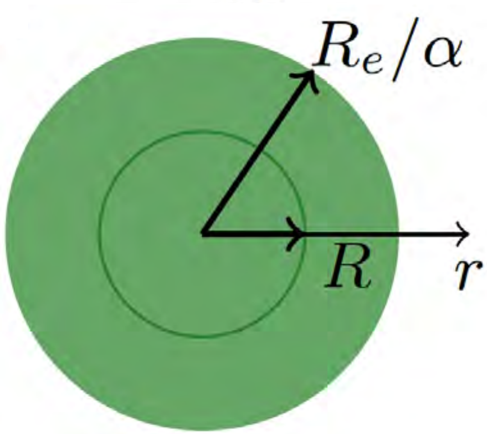
Minimize Stretching Energy (Constant *K* metric)

$$a_{\theta\theta}(\rho) = (\sin(\sqrt{K}\rho)/\sqrt{K})^2$$

M. Pezzulla, S. Shillig, P. Nardinocchi, and D.P. Holmes. "*Morphing of Geometric Composites via Residual Swelling*," *Soft Matter*, **11**, 5812-5820, (2015).

Geometric Composite





Minimize Stretching Energy (Constant *K* metric)

$$a_{\theta\theta}(\rho) = (\sin(\sqrt{K}\rho)/\sqrt{K})^2$$

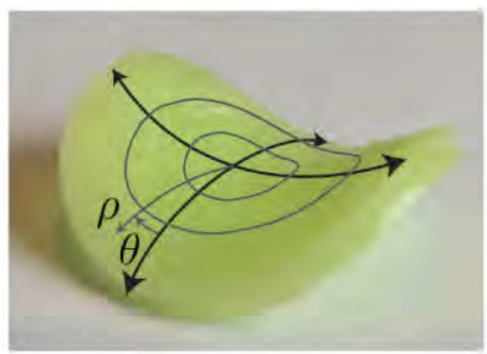
Taylor Expand $a_{\theta\theta}(\rho)$ (Assume: $|K| < \alpha^2/R_e^2$)

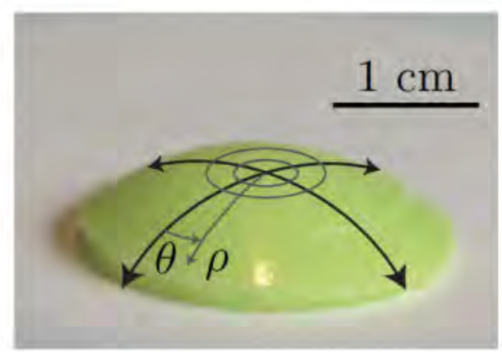
$$a_{\theta\theta}(\rho) = \rho^2 - \frac{K}{3}\rho^4 + \mathcal{O}(\rho^5)$$

Flat metric

Kind of non-Euclidean metric

Experiments: Mechanical Strain





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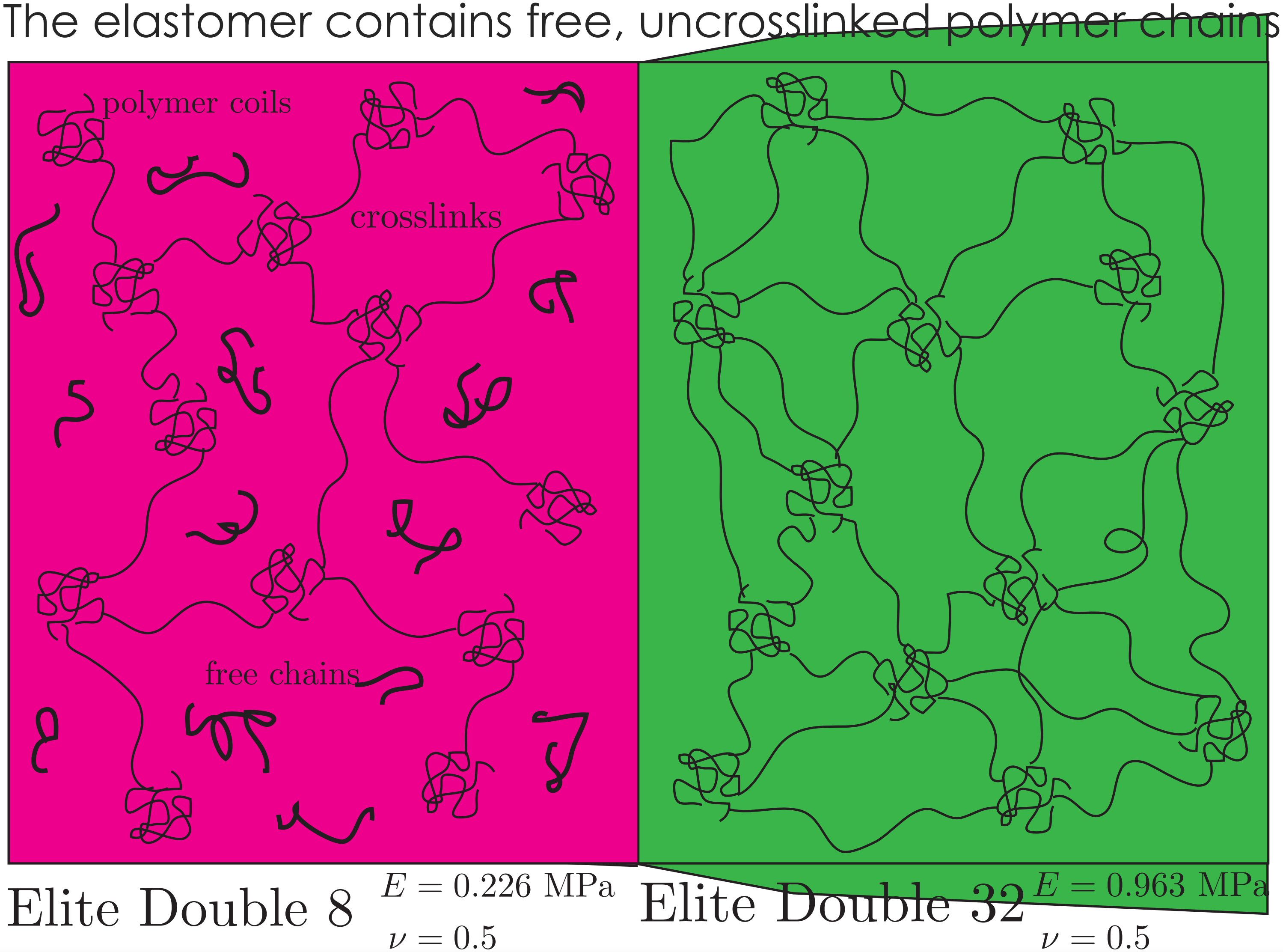
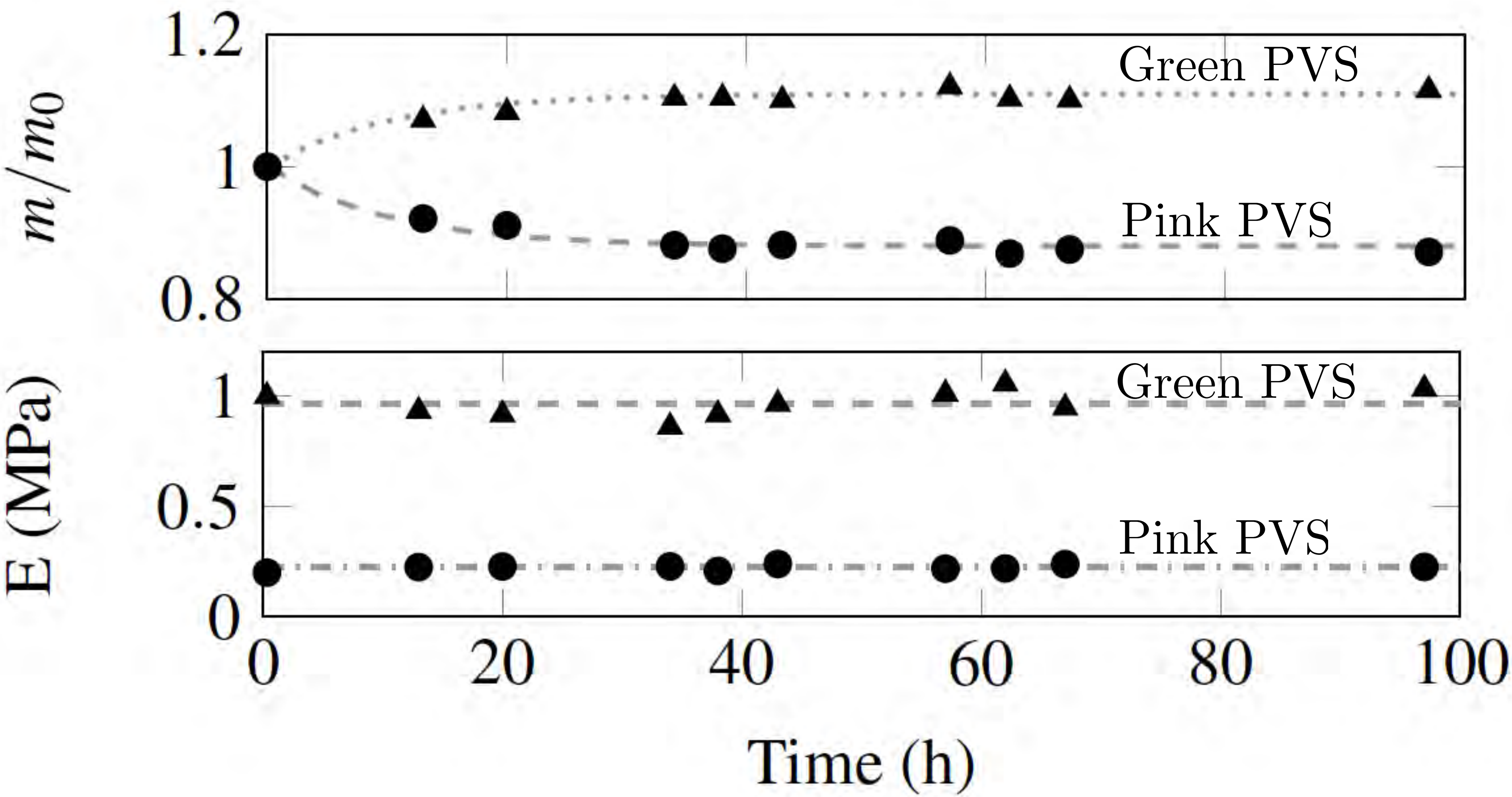
Residual Swelling

The elastomer contains free, uncrosslinked polymer chains.



M. Pezzulla, S. Shillig, P. Nardinocchi, and D.P. Holmes. "*Morphing of Geometric Composites via Residual Swelling*," *Soft Matter*, **11**, 5812-5820, (2015).

Residual Swelling



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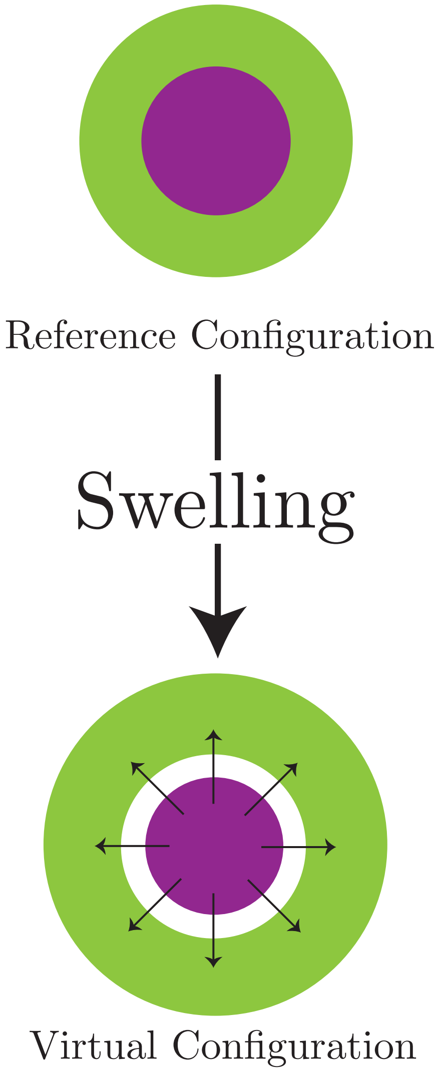
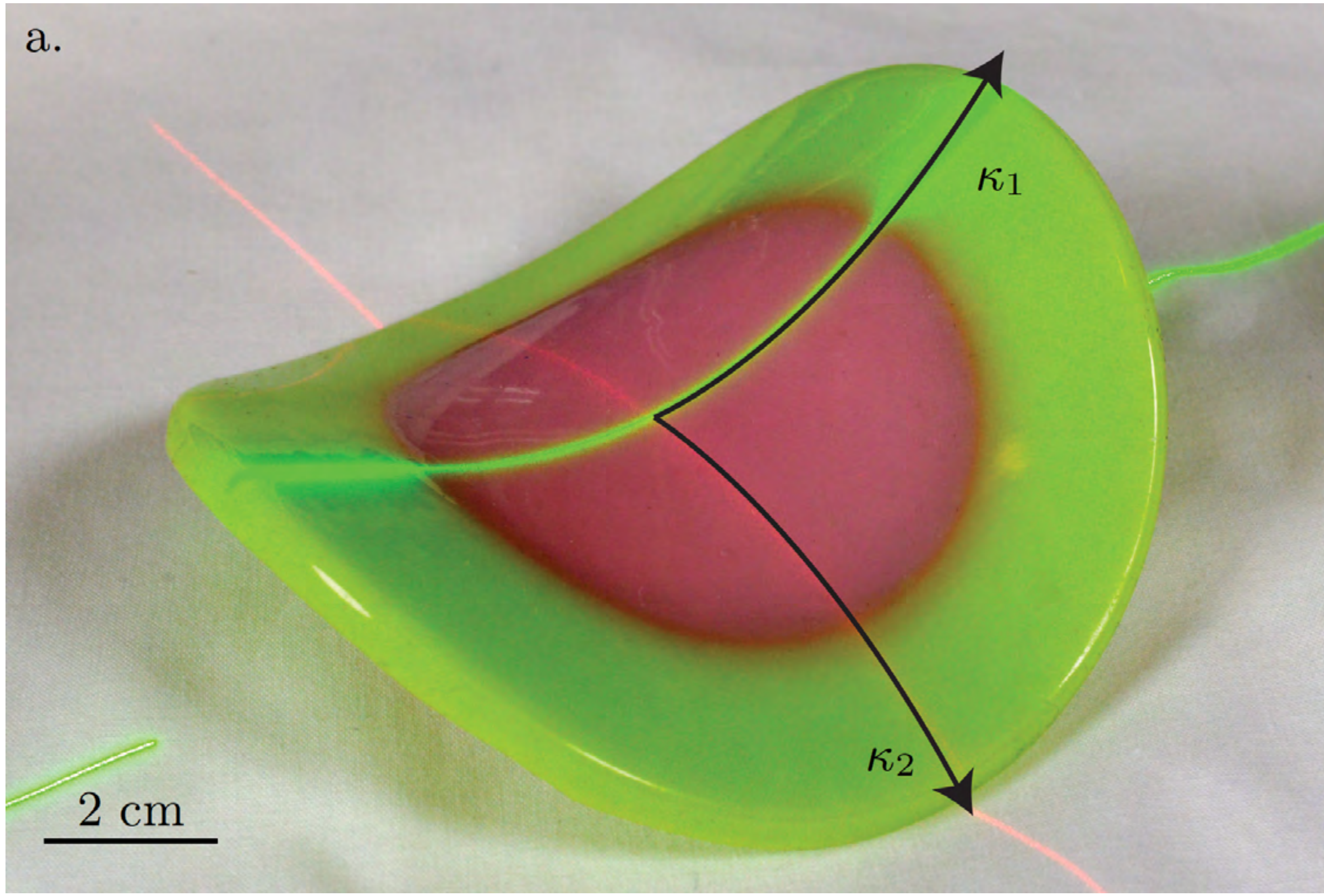
Residual Swelling





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Residual Swelling

Stretching Energy

$$\mathcal{U}_s \simeq \int_0^R \frac{(a_{\theta\theta} - \alpha^{-2}r^2)^2}{\alpha^{-2}r^3}\,\mathrm{d}r + \frac{E_a}{E_d}\int_R^{R_e} \frac{(a_{\theta\theta} - \alpha^2 r^2)^2}{\alpha^2 r^3}\,\mathrm{d}r$$

Modulus difference

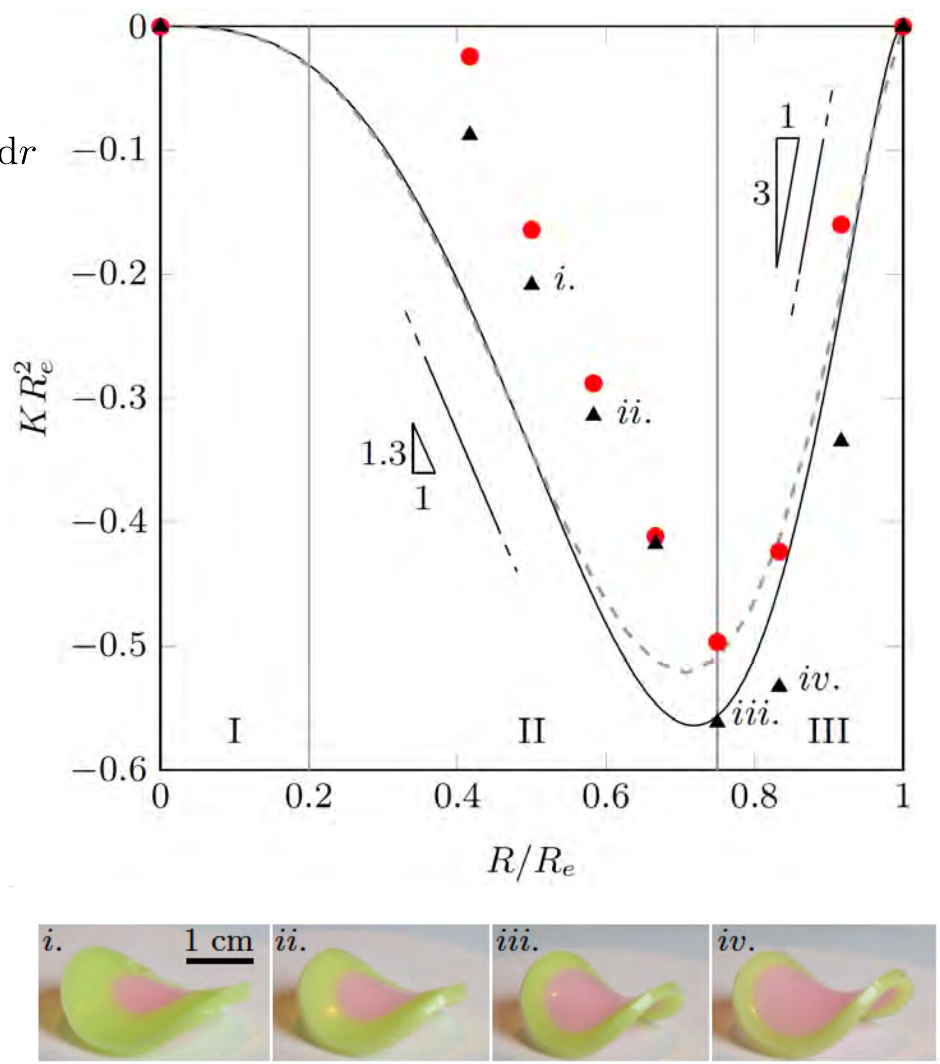
Stretching ratio

- Depends on chemical and material properties.
- Should vary with R/R_e
- Will depend on concentration gradient of free chains.

ansantz

- from conservation of mass & proportional to mass uptake in annulus.

$$\alpha = 1 + \eta(c_d - c_a)\left(\frac{R}{R_e}\right)^2\left[1 - \left(\frac{R}{R_e}\right)^2\right]$$



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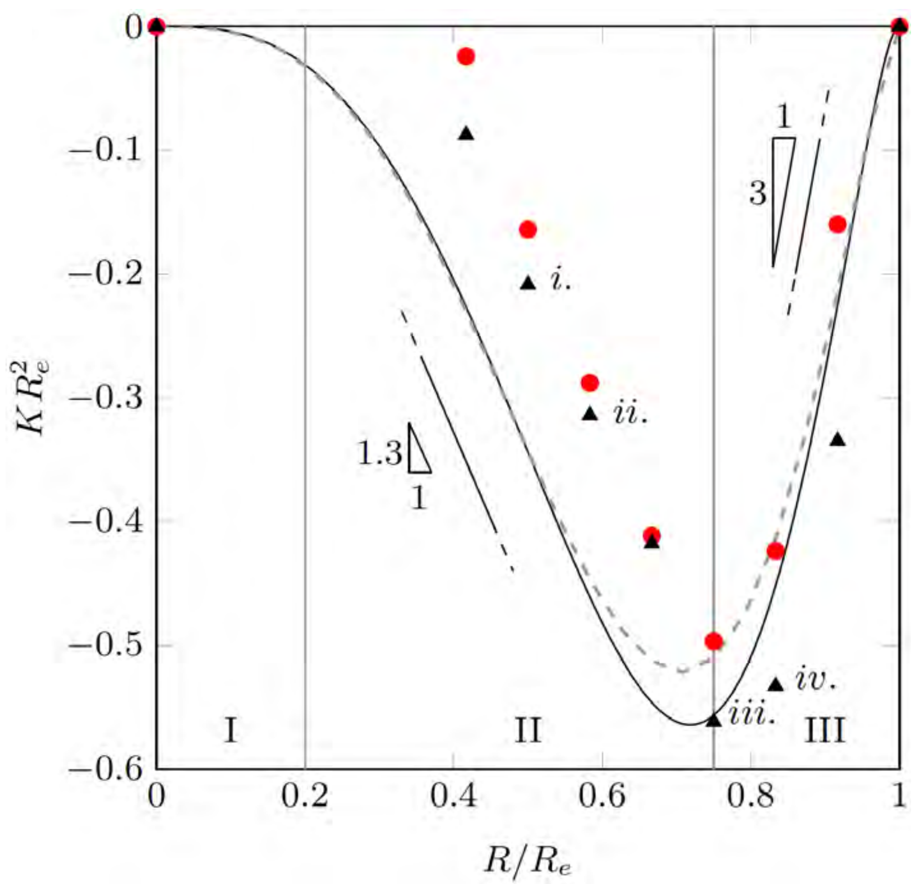
Residual Swelling

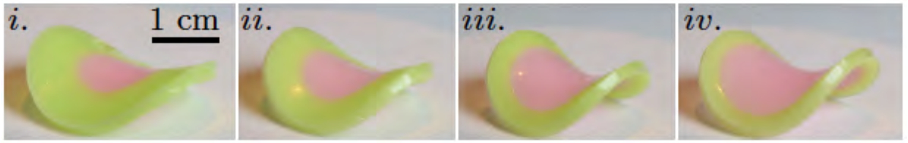
Approximate Analytical Solution

- Taylor expand about $(\alpha - 1)$
- $\overline{E} = E_a / E_d$ and $\overline{R} = R/R_e$

$$KR_e^2 \simeq 96(1-\alpha_{\max})\bar{E}\bar{R}^3 \frac{\left(1-\bar{R}^2\right)\left(1-\bar{R}^3\right)}{\bar{R}^6\left(1-\bar{E}\right)+\bar{E}}$$

2D analog to Timoshenko's model for thermal beam bending of **bimetallic strips**.





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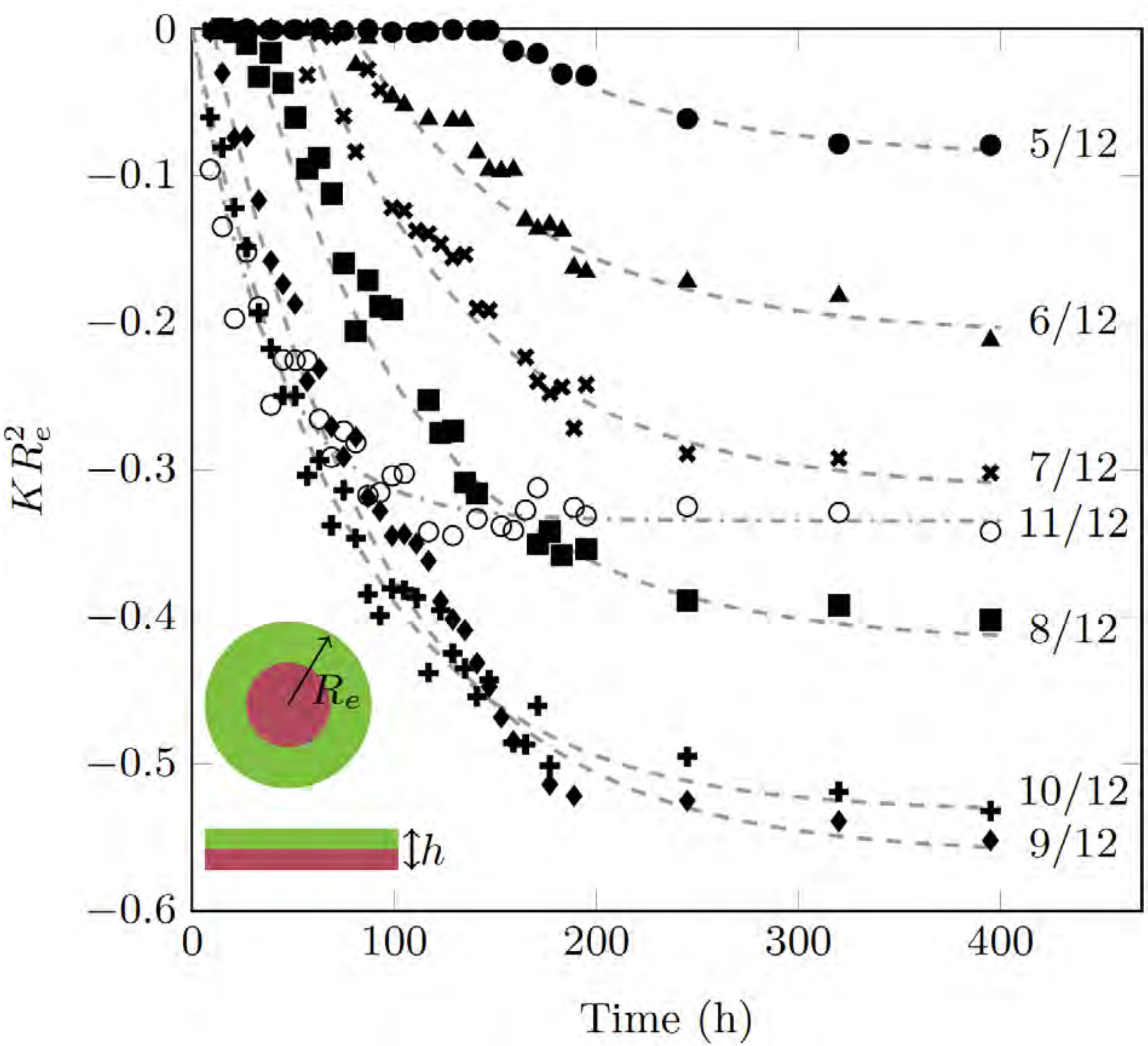
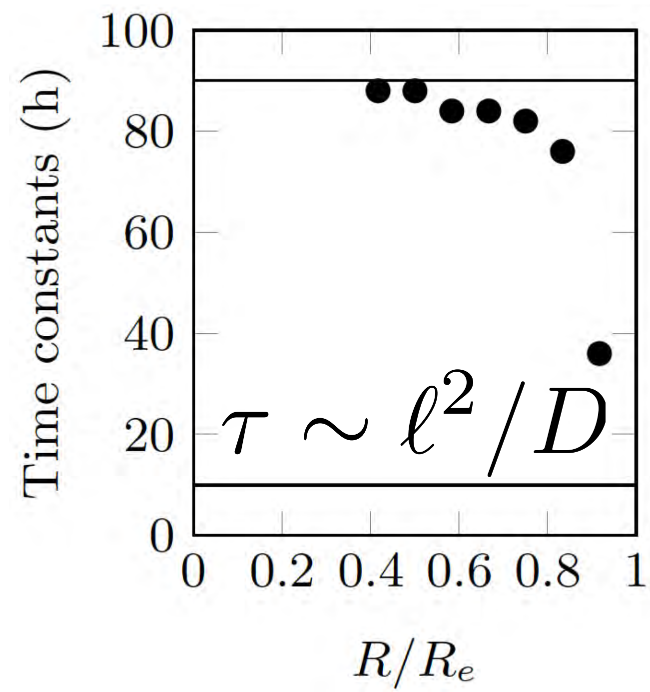
Swelling Dynamics





Diffusion of free chains

Characteristic length:

$$R/R_e < 1 \longrightarrow \ell \sim R$$

$$R/R_e \approx 1 \longrightarrow \ell \sim h$$

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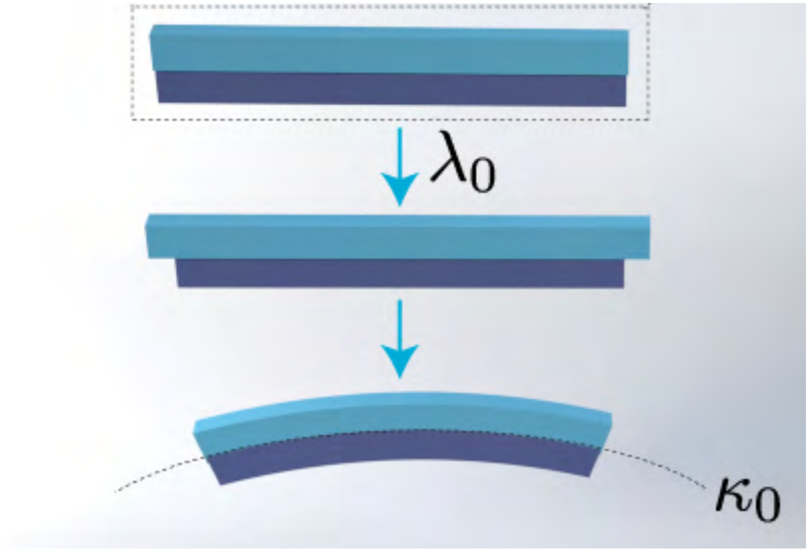
Growing Sheets

Consider a thin structure with a growing top layer.

- Caused by **swelling**, **growth**, **heating**, etc.

Elastic energy density depends on material properties & metric tensors

$$\overline{\mathcal{U}} = \underbrace{\int [(1-\nu)|\mathbf{a}-\bar{\mathbf{a}}|^2 + \nu \mathrm{tr}^2(\mathbf{a}-\bar{\mathbf{a}})]\sqrt{|\bar{\mathbf{a}}|}\,\mathrm{d}A}_{\text{Stretching Energy}} + \underbrace{\frac{h^2}{3}\int [(1-\nu)|\mathbf{b}-\bar{\mathbf{b}}|^2 + \nu \mathrm{tr}^2(\mathbf{b}-\bar{\mathbf{b}})]\sqrt{|\bar{\mathbf{a}}|}\,\mathrm{d}A}_{\text{Bending Energy}}$$



Lateral distances (Λ_0^2) and curvatures (κ_0) that make the sheet stress free.

Metric tensor / Curvature tensor

$$\bar{\mathbf{a}} = \Lambda_o^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{\mathbf{b}} = \kappa_o \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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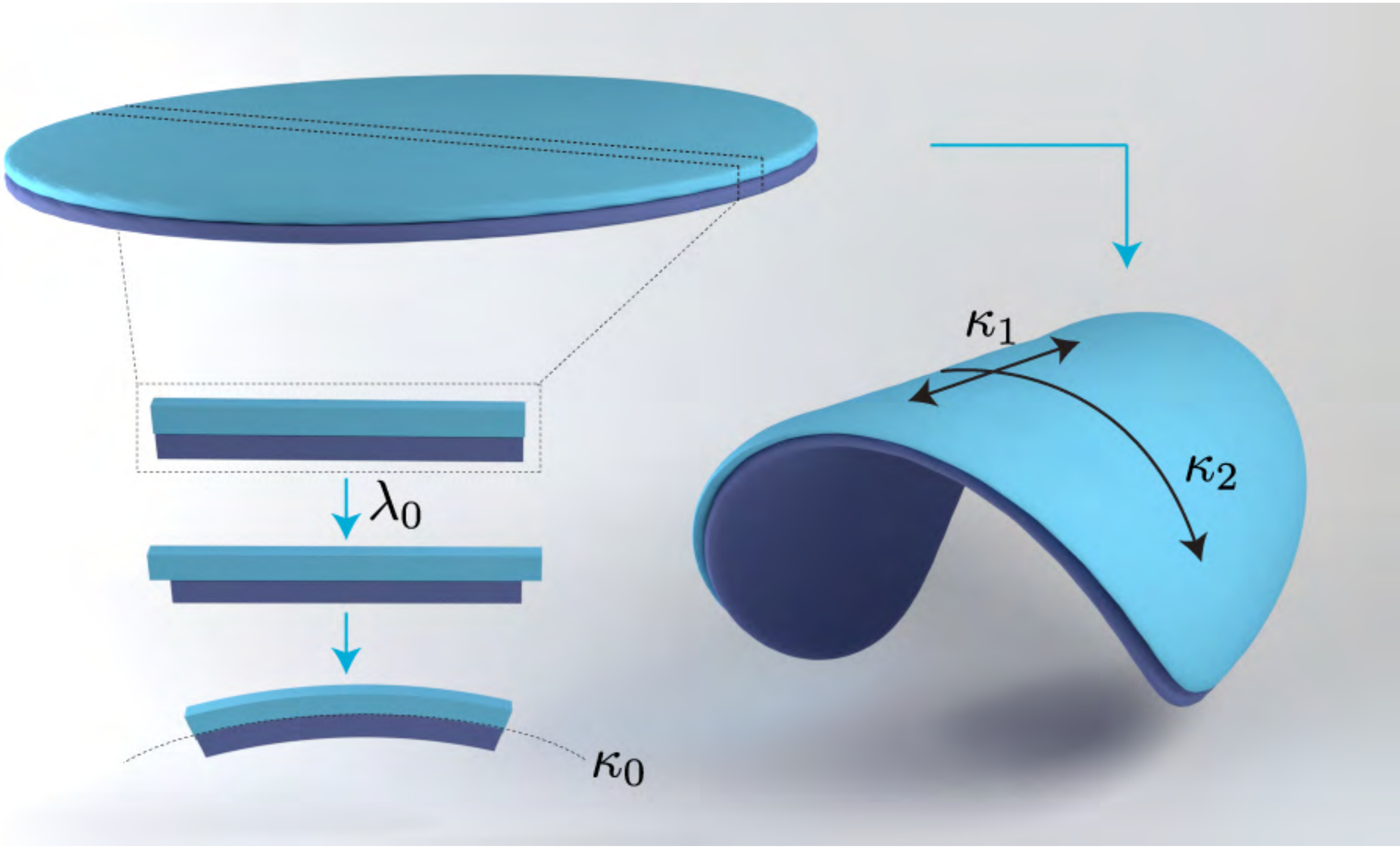
Isometric Limit



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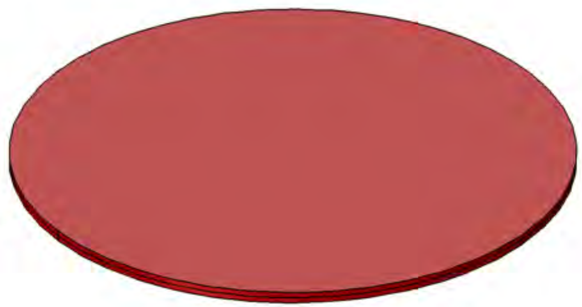
Isometric Limit



Numerical Simulations using COMSOL Multiphysics

- Finite, incompatible tridimensional elasticity with a Neo-Hookean incompressible material.
- Distortions used to simulate prestretch.
- Top layer subjected to distortion field:

$$\mathbf{F}_o = \lambda(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \mathbf{e}_3 \otimes \mathbf{e}_3$$

Residual Swelling Experiments

- PVS Bilayer (Total thickness ~ 600um)
- Axisymmetric, circular plate.
- Pink (top) source of swelling for green (bot).
- Real time = 75 minutes (Video: 640x RT)

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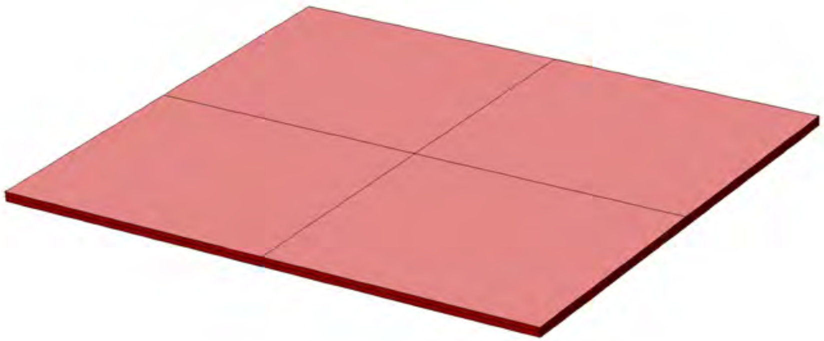
Isometric Limit

Ellipse

Rectangle



Cross

Leaf

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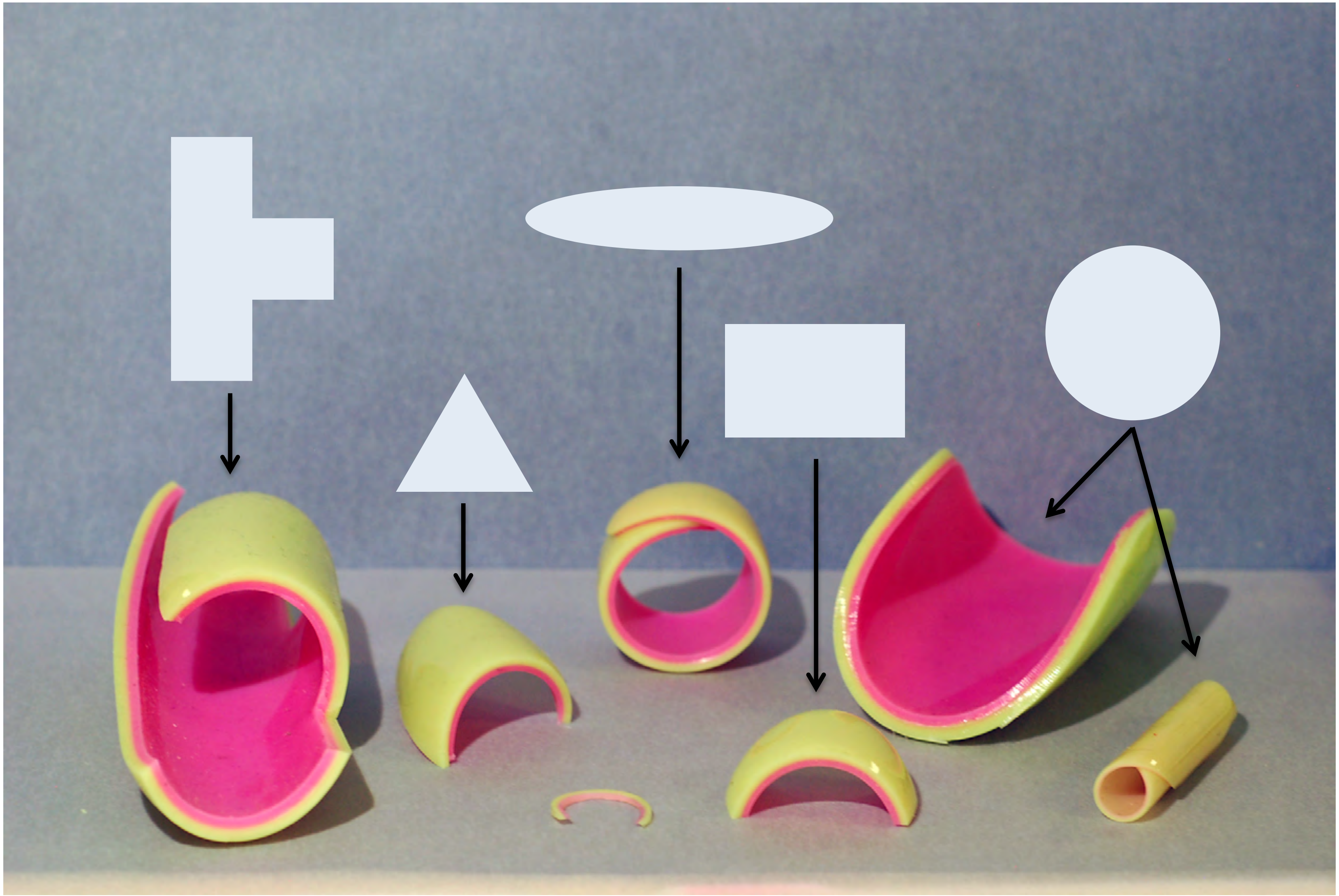
Isometric Limit



Isometric Limit

In the **isometric limit**, the stretching energy is zero.

- i.e. $\mathbf{a} = \bar{\mathbf{a}}$

Curvature tensor (Cartesian)

- Second fundamental form:

$$L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2$$

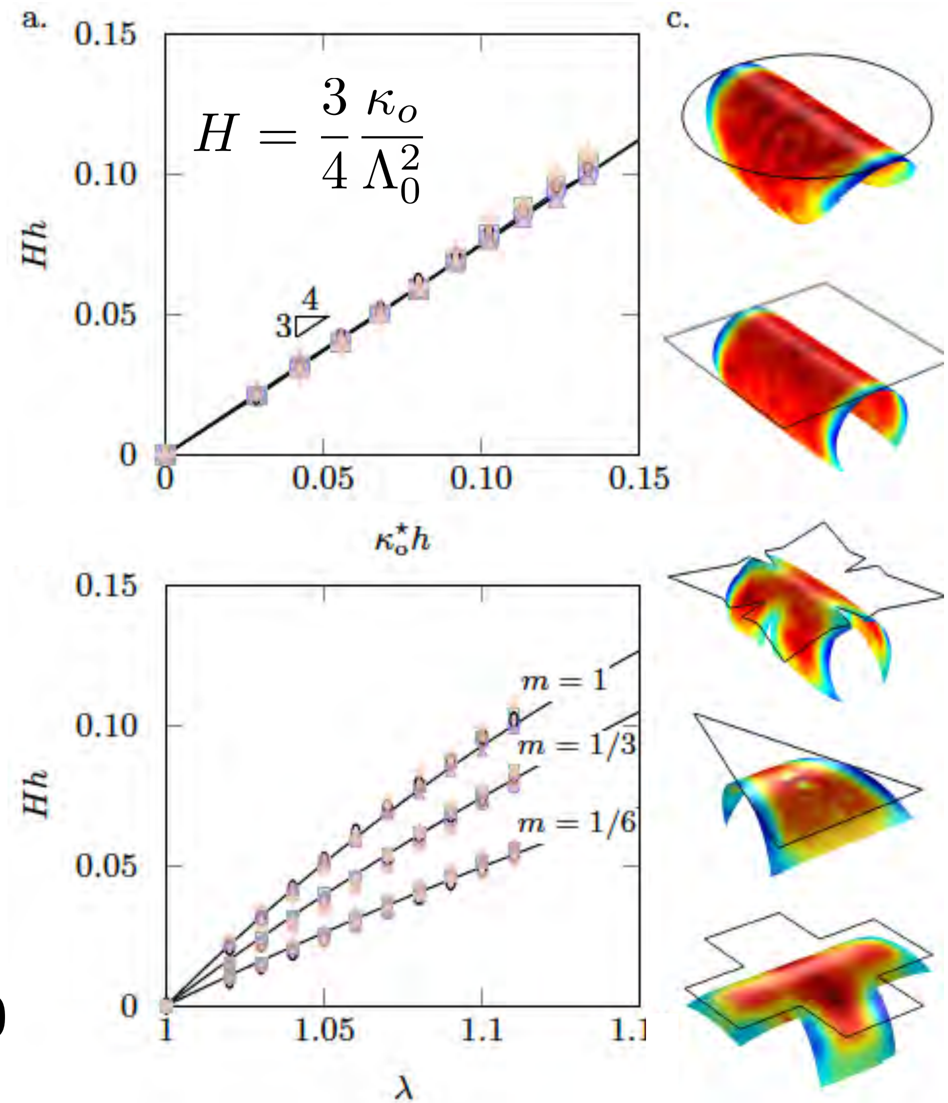
Minimize bending energy

- Constrain the mid-surface to be flat
- Impose Lagrange multiplier enforcing

$$\underbrace{\Lambda_0^{-4}\left(LN - M^2\right)}_{\text{Gaussian Curvature}} = 0$$

Minimization yields:

$$L + N = \kappa_0(1+\nu) \quad \text{and} \quad K = 0$$



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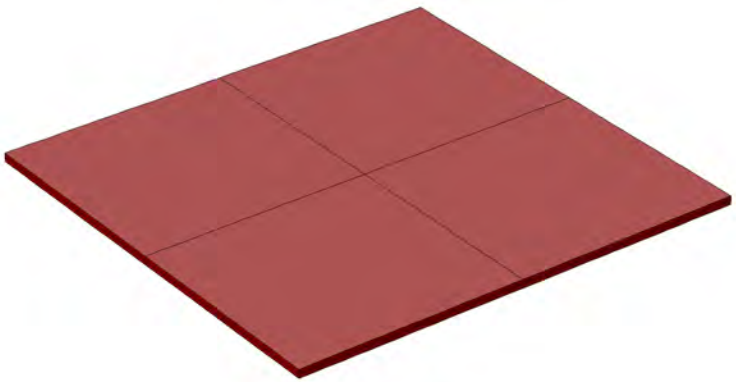
Bifurcation

Rectangular Sheet



Stretching Energy Density

In the limit of **large stretching**, the sheet adopts an **isometry**.

For **small stretching**, the sheet is initially spherical curved.

- Bifurcation from spherical to cylindrical.

Classical problem (limited to circular and elliptical disks)

- Stoney formula relating stress to curvature.
- Strain mismatch work (Hyer, Freund, Seffen, etc.)

Bifurcation

Stretching Energy

Assuming a metric with constant K (Gauss normal coords)

$$\overline{\mathcal{U}}_s = \frac{1}{9}\lambda_0^6 K^2 \int_A r^4 \,\mathrm{d}A$$

Bending Energy

In the spherical shape: $L = N$

$$\overline{\mathcal{U}}_b = h^2 \lambda_0^{-2} A (L - \kappa_0)^2$$

Shape factor:

- Assume metric is axisymmetric.
- Assume K is homogenous.

$$a_{rr} = \overline{a}_{rr} \qquad a_{r\theta} = 0$$

$$a_{\theta\theta}(\rho) = \rho^2 - \frac{K}{3}\rho^4 + \mathcal{O}(\rho^5)$$

Shape factor:

$$\mathcal{S} \equiv \left(\frac{2}{9} \frac{1}{A} \int_A r^4 \,\mathrm{d}A \right)^{1/4}$$

Structural Slenderness:

$$\gamma = h/\mathcal{S}$$

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Bifurcation

Energetic cost to continue bending into a **spherical cap**:

$$\overline{\mathcal{U}}_{bb} = (1/9)L^4\Lambda_o^{-2}\int r^4 dA + h^2 A\Lambda_o^{-2}(L-\kappa_o)^2$$

Energy to bend as a **cylinder**:

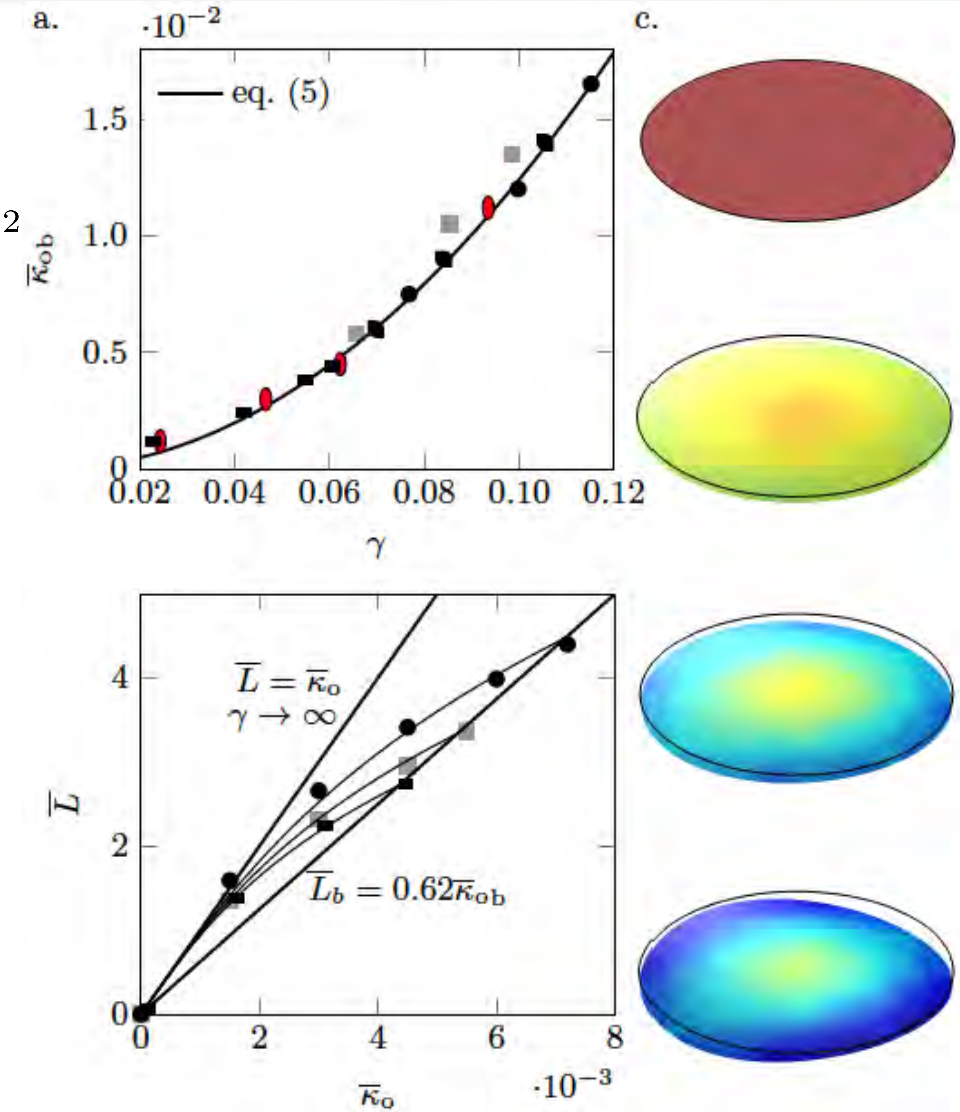
$$\overline{\mathcal{U}}_{ab} = (1/4)h^2 A\Lambda_o^{-2}\kappa_o^2$$

Energy balance:

$$\frac{1}{2}\overline{L}_{\mathrm{b}}^4 + \gamma^4\left(\overline{L}_{\mathrm{b}}^2 - 2\overline{L}_{\mathrm{b}}\overline{\kappa}_{o\mathrm{b}} + \frac{3}{4}\overline{\kappa}_{o\mathrm{b}}^2\right) = 0$$

Bifurcation curvature:

$$\overline{\kappa}_{ob} = \frac{2\sqrt{2}}{3^{3/4}}\gamma^2$$



Shell Growth

…with M. Trejo, J. Bico, and B. Roman.

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Swelling Structures

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1. A. Pandey and D.P. Holmes. "Swelling Induced Deformations: A Materials-Defined Transition from Structural Instability to Surface Instability." *Soft Matter*, **9**, 7049, (2013.)
2. D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. "Bending and Twisting of Soft Materials by Non-homogenous Swelling" *Soft Matter*, **7**, 5188, (2011).
3. M. Pezzulla, S. Shillig, P. Nardinocchi, and D.P. Holmes. "*Morphing of Geometric* Composites via Residual Swelling," *Soft Matter*, **11**, 5812-5820, (2015).