Drag on a Sphere

- depends on: Radius (R), velocity (v), density (ρ), viscosity (μ)

$$D_f = f(R, v, \rho, \mu) \longrightarrow [D_f] = [R^a v^b \rho^c \mu^d]$$

$$MLT^{-2} = L^a \left(\frac{L}{T}\right)^b \left(\frac{M}{L^3}\right)^c \left(\frac{M}{LT}\right)^d = L^{a+b-3c-d}\, T^{-b-d}\, M^{c+d}$$

$$\left.\begin{aligned} L&: a+b-3c-d = 1 \\ T&: -b-d = -2 \\ M&: c+d = 1 \end{aligned}\right\} \begin{aligned} b &= 2-d \\ c &= 1-d \\ a &= 2-d \end{aligned} \longrightarrow D_f \sim R^{2-d} v^{2-d} \rho^{1-d} \mu^d \sim R^2 v^2 \rho \underbrace{\left(\frac{\mu}{Rv\rho}\right)^d}_{\Pi^d}$$

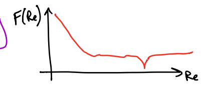
$Re \equiv \frac{Rv\rho}{\mu}$

dimensionless product/group

$\boxed{\text{physical Similarity}}$

model

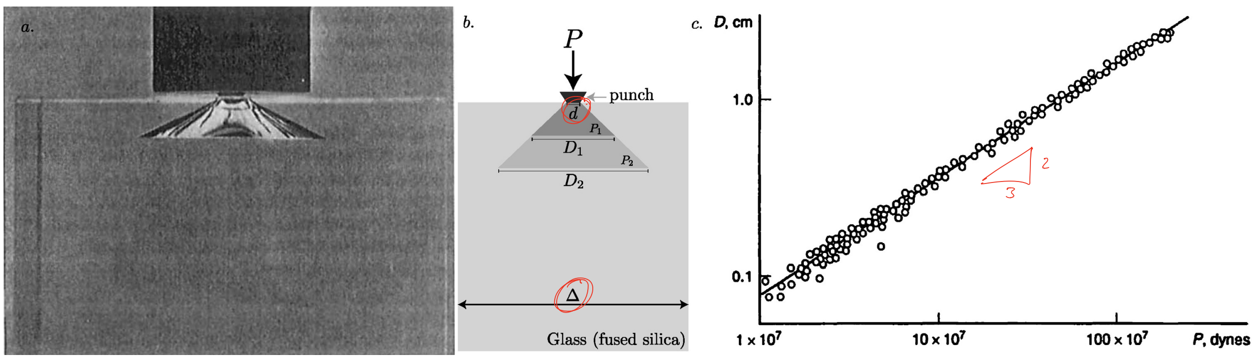
$$\frac{\mu_m}{R_m V_m \rho_m} = \frac{\mu}{Rv\rho} \longrightarrow \boxed{V_m = \frac{\mu_m R \rho}{\mu R_m \rho_m} V}$$



$$\boxed{D_f = \rho R^2 v^2 F(\Pi)}$$

$$\left\{\begin{aligned} D_f &= \alpha R^2 v^2 \rho \Pi^d \\ D_f &= \alpha_1 R^2 v^2 \rho \Pi^{d_1} + \alpha_2 R^2 v^2 \rho \Pi^{d_1} \\ D_f &= \rho R^2 v^2 \underbrace{(\alpha_1 \Pi^{d_1} + \alpha_2 \Pi^{d_2} + \dots)}_{F(\Pi)} \end{aligned}\right.$$

Brittle Fracture

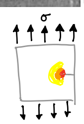


Strain Energy Release Rate

↳ related to "Stress Intensity Factor"

$$K \sim \sigma \sqrt{s}$$

avg. stress — distance from crack



$$D = f(P, K, \nu) \quad \frac{F}{A}\sqrt{L} \sim MLT^{-2} L^{1/2} L^{-2} \sim \frac{MLT^{-2}}{L^{3/2}}$$

$\sigma \quad \sqrt{s}$

$[P] = F = MLT^{-2}$

$$\boxed{D \sim \left(\frac{P}{K}\right)^{2/3} \Phi(\nu)}$$

valid: $\frac{d}{(P/K)^{2/3}} \ll 1$ AND $\frac{\Delta}{(P/K)^{2/3}} \gg 1$

intermediate asymptotics

Similarity Variables

Dimensional Analysis: Derivatives/Integrals

$A(x) = x^2 \longrightarrow [A] = L^2$

$\frac{dA}{dx} = 2x \longrightarrow \left[\frac{dA}{dx}\right] = \frac{1}{L}[A] = L$

$\frac{d^2A}{dx^2} = 2 \longrightarrow \left[\frac{d^2A}{dx^2}\right] = \frac{1}{L^2}[A] = 1$

integrals

Derivatives

$$\boxed{D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}} \qquad \underbrace{u|_{x=0} = u_0, \quad u|_{x=\infty} = 0}_{\text{B.C.}} \qquad \underbrace{u|_{t=0} = 0}_{\text{I.C.}}$$

$$[D\, u_{,xx}] = [u_{,t}]$$

$[u] = ML^{-3}$ $\quad \therefore [D] = L^2 T^{-1}$

$$u = f(x, t, D, u_0)$$

$$[u] = [x^a t^b D^c u_0^d] \longrightarrow \left.\begin{aligned} ML^{-3} &= L^a T^b (L^2 T^{-1})^c (ML^{-3})^d \\ ML^{-3} &= L^{a+2c-3d}\, T^{b-c}\, M^d \end{aligned}\right\}$$

$$\begin{aligned} L&: a+2c-3d = -3 \\ T&: b-c = 0 \\ M&: d = 1 \end{aligned} \qquad \therefore \begin{aligned} d &= 1 \\ b &= c = -\tfrac{a}{2} \end{aligned}$$

$$\longrightarrow \boxed{u \sim u_0 \left(\frac{x}{\sqrt{Dt}}\right)^a} \longrightarrow \boxed{u = u_0 F(\eta)}$$

$\eta \equiv \frac{x}{\sqrt{Dt}}$ similarity variable

Chain Rule

$$\frac{\partial}{\partial t}[F(\eta)] = \frac{\partial F(\eta)}{\partial \eta}\frac{\partial \eta}{\partial t} \longrightarrow D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$\to u_0 F'(\eta) \frac{\partial \eta}{\partial t}$

$\to \frac{\partial\left(\frac{x}{\sqrt{Dt}}\right)}{\partial t} = -\frac{x}{2D^{1/2} t^{3/2}}$

$\to -u_0 F'(\eta) \frac{\eta}{2t}$

$\to \frac{\partial^2 u}{\partial x^2} = u_0 F''(\eta) \frac{1}{Dt}$

$$\cancel{D}\left[\cancel{u_0} F''(\eta) \frac{1}{\cancel{D}t}\right] = -\cancel{u_0} F'(\eta) \frac{\eta}{2\cancel{t}} \longrightarrow \boxed{F''(\eta) = -\tfrac{1}{2}\eta F'(\eta)} \quad \boxed{F(0) = 1 \quad F(\infty) = 0}$$

let: $G = F'$

$\therefore G' = -\frac{1}{2}\eta G \longrightarrow G = \alpha \exp\left[-\eta^2/4\right]$

$\therefore F(\eta) = \beta + \alpha \int_0^\eta \exp\left[-s^2/4\right] ds$

$F(0) = 1 \quad \therefore \beta = 1$

$F(\infty) = 0 \quad \therefore 1 + \alpha \underbrace{\int_0^\infty \exp\left[-s^2/4\right] ds}_{\sqrt{\pi}} = 0$

$$\left.\begin{aligned} F(\eta) = \underbrace{1 - \frac{1}{\sqrt{\pi}} \int_0^\eta \exp\left[-s^2/4\right] ds}_{1 - \operatorname{erf}(\eta) = \operatorname{erfc}(\eta)} \end{aligned}\right\} \boxed{u(x,t) = u_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}\right)}$$