Projectiles

- $\underline{\text{scale}}$ the variables using $\underline{\text{characteristic}}$ values

x_c & t_c ($\underline{\text{constants}}$)

$$\ddot{x} = -\frac{gR^2}{(R+x)^2} \qquad x(0)=0 \qquad \dot{x}(0)=v_0$$

let: $x = x_c u$, $t = t_c s$ } u & s are dimensionless

$$\frac{1}{t_c^2}\frac{d^2}{ds^2}(x_c u) = -\frac{gR^2}{(R+x_c u)^2}$$

$$\boxed{R^2\left(1+x_c\frac{u}{R}\right)^2} \quad \text{or} \quad x_c^2\left(\frac{R}{x_c}+u\right)^2$$

$$\boxed{\underbrace{\frac{x_c}{gt_c^2}}_{\Pi_1}\frac{d^2u}{ds^2} = -\frac{1}{\left(1+\underbrace{x_c/R}_{\Pi_2}\,u\right)^2}}$$

$$\boxed{\text{I.C.'s:} \quad u(0)=0 \quad \frac{du(0)}{ds} = \underbrace{\frac{t_c}{x_c}v_0}_{\Pi_3}}$$

Recall the chain rule:

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = \frac{1}{t_c}\frac{d}{ds} \qquad s=\frac{t}{t_c} \quad \therefore \frac{ds}{dt}=\frac{1}{t_c}$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{1}{t_c^2}\frac{d^2}{ds^2}$$

$\underline{\text{Dimensionless Groups:}}$

1. Do not involve $\underline{\text{variables}}$ (u & s) only the $\underline{\text{parameters}}$ (x_c, t_c, g, R, v_0)
2. $\underline{\text{Dimensionless}}$: accomplished by rearranging our equations
3. They are $\underline{\text{independent}}$: not possible to write Π_i in terms of Π_j, Π_k

Characteristic Values

Rule 1: Set Π_i in I.C./B.C. equal to 1 $\longrightarrow$ $\Pi_3 = \frac{t_c}{x_c}v_0 = 1$ $\quad \therefore x_c = v_0 t_c \rightarrow \boxed{t_c = \frac{v_0}{g}}$

Rule 2: Set Π_j that appear in the $\underline{\text{reduced}}$ problem equal to 1 $\longrightarrow$ $\Pi_1 = \frac{x_c}{gt_c^2} = 1$ $\quad \therefore \boxed{x_c = \frac{v_0^2}{g}}$

$$\Pi_1\frac{d^2u}{ds^2} = -\frac{1}{(1+\Pi_2 u)^2} = -1$$

$\Pi_2 \rightarrow 0$: reduced problem

Dimensionless Equation

$$\frac{d^2u}{ds^2} = -\frac{1}{\left(1+\left(v_0^2/gR\right)u\right)^2} \qquad u(0)=0 \quad \frac{du(0)}{ds}=1 \quad \Bigg] \rightarrow \boxed{\frac{d^2u}{ds^2} = -\frac{1}{(1+\epsilon u)^2}}$$

$\epsilon \equiv \Pi_2$ is a small parameter

- is ϵ small?
  $R = 6.4\times10^6$ m
  $g = 9.81$ m/s^2 } $\boxed{\epsilon \approx 1.6\times10^{-8}v_0^2}$

Reaction-Diffusion (KPP)

$$D\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t} - \lambda\underbrace{(\gamma-c)}_{\text{nonlinear}}c$$

$[\gamma]=[c]$

$[c]=[c_0]=\frac{M}{L^3}$

$[D]=\frac{L^2}{T}$

$[\lambda(\gamma-c)c] = \left[\frac{\partial c}{\partial t}\right] \quad \therefore [\lambda] = \frac{L^3}{MT}$

Change of Variables

$x = x_c u$

$t = t_c s$

$c = c_c v$

B.C.
$c(0,t) = c(\ell,t) = 0$

I.C
$c(x,0) = c_0\sin\left(5\pi\frac{x}{\ell}\right)$

$$\frac{Dc_c}{x_c^2}\frac{\partial^2 v}{\partial u^2} = \frac{c_c}{t_c}\frac{\partial v}{\partial s} - \lambda c_c v(\gamma - c_c v)$$

$$\underbrace{\frac{Dt_c}{x_c^2}}_{\Pi_1}\frac{\partial^2 v}{\partial u^2} = \frac{\partial v}{\partial s} - \underbrace{\lambda t_c c_c}_{\Pi_2}v\left(\underbrace{\frac{\gamma}{c_c}}_{\Pi_3} - v\right)$$

$$v(0,s) = v\left(\underbrace{\frac{\ell}{x_c}}_{\Pi_4}, s\right) = 0$$

$$v(u,0) = \underbrace{\frac{c_0}{c_c}}_{\Pi_5}\sin\left(5\pi\frac{x_c u}{\ell}\right)$$

Rule 1: $\Pi_4 = \frac{\ell}{x_c} = 1 \quad \therefore x_c = \ell$

$\Pi_5 = \frac{c_0}{c_c} = 1 \quad \therefore c_c = c_0$

$$\boxed{\Pi_1}\frac{\partial^2 v}{\partial u^2} = \frac{\partial v}{\partial s} - \boxed{\Pi_2}v(\Pi_3 - v)$$

$\Pi_1 \rightarrow 0$: weak diffusion

$\Pi_2 = \lambda t_c c_c = 1$
$\therefore \boxed{t_c = \frac{1}{c_0\lambda}}$

$$\boxed{\epsilon\frac{\partial^2 v}{\partial u^2} = \frac{\partial v}{\partial s} - \left(\frac{\gamma}{c_0} - v\right)v}$$

$\Pi_2 \rightarrow 0$: weak nonlinearity

$\Pi_1 = \frac{Dt_c}{x_c^2} = 1 \longrightarrow \boxed{t_c = \frac{\ell^2}{D}}$

$$\boxed{\begin{aligned}&\frac{\partial^2 v}{\partial u^2} = \frac{\partial v}{\partial s} - \epsilon\left(\frac{\gamma}{c_0} - v\right)v\\ &v(0,s) = v(1,s) = 0\\ &v(u,0) = \sin(5\pi u)\end{aligned}}$$