Introduction to modeling tsunamis with GeoClaw

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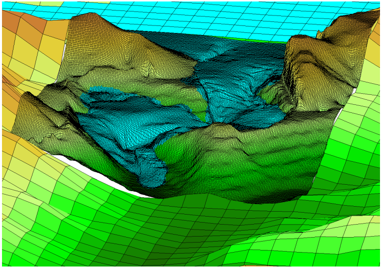

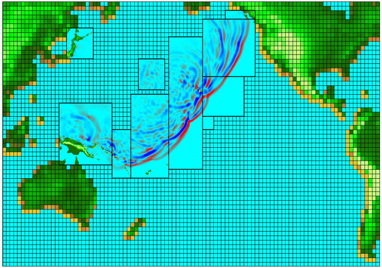

Overview

1. Preview of GeoClaw (shallow free-surface flows)
2. Depth-averaged models
3. Tsunamis and the shallow water equations
4. Hyperbolic systems
5. Finite volume methods and adaptive mesh refinement

Mt. Meager debris flow: B.C. August 2010

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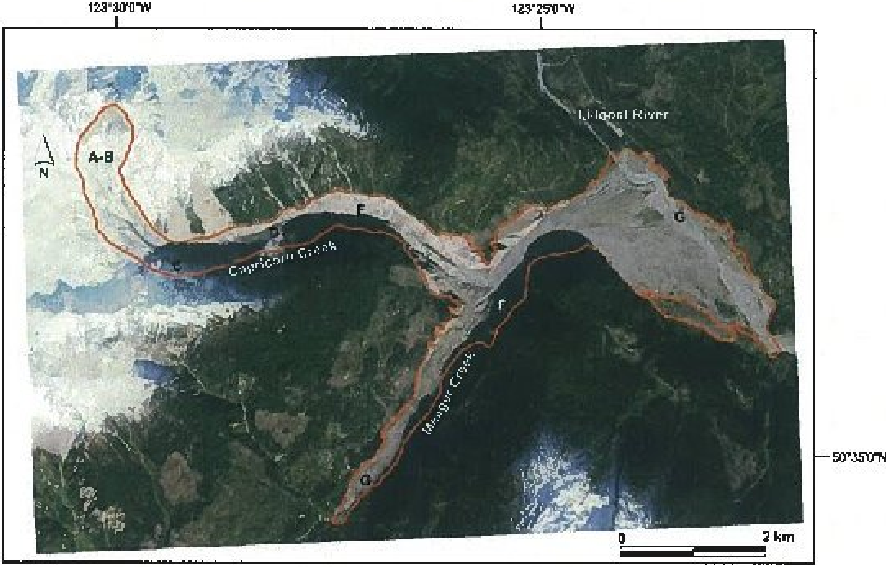

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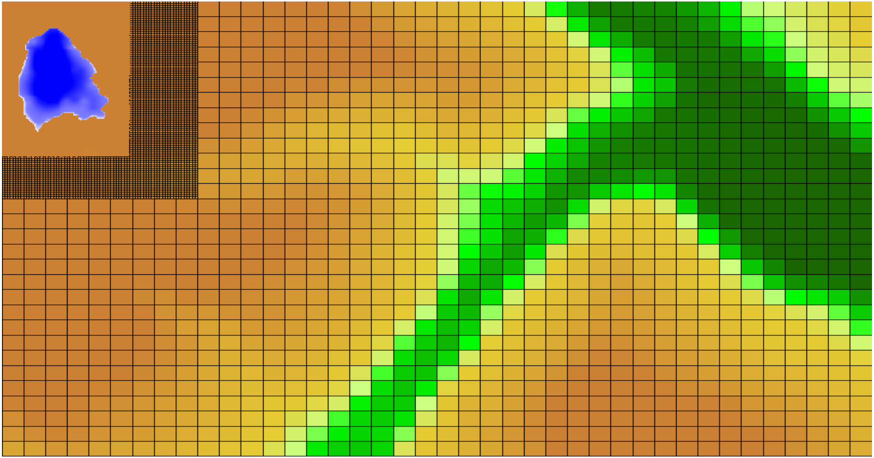

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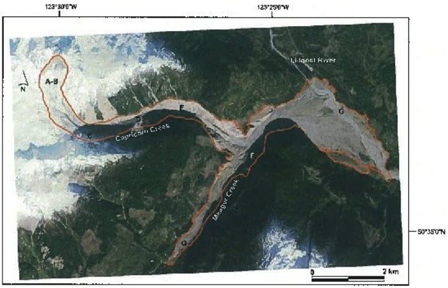

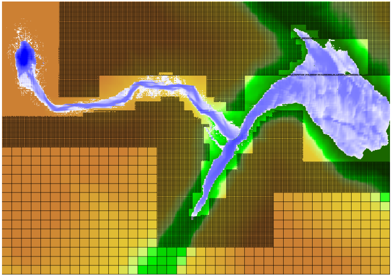

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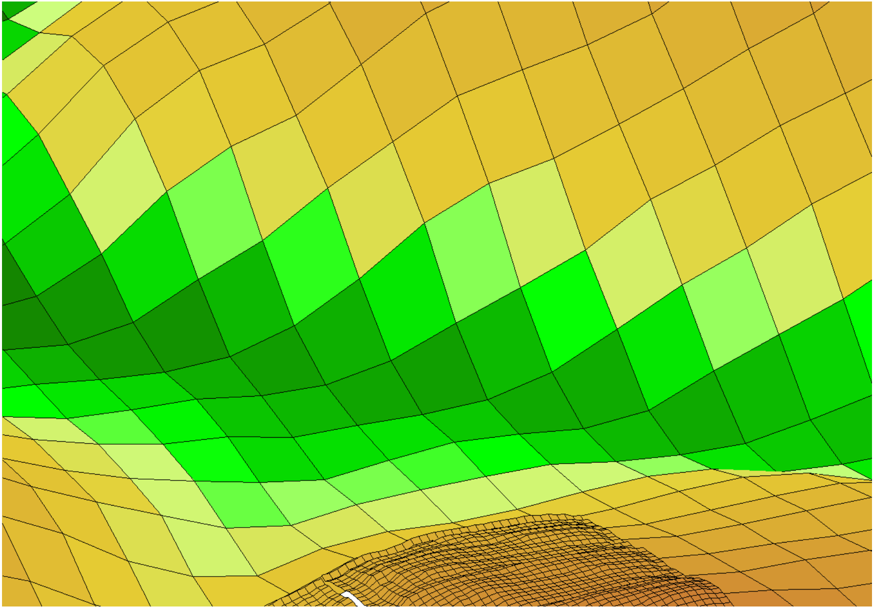
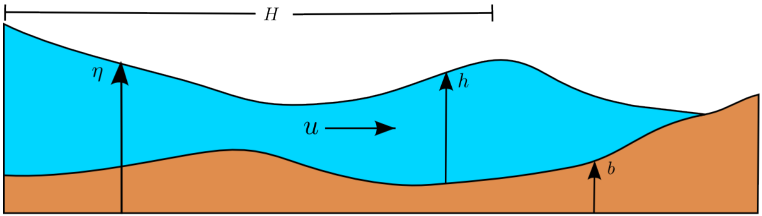


Depth-averaged equations

Flow between a fixed bottom $b(x, y)$ and free surface $\eta(x, y, t)$:



1. Start with full 3D equations.
2. Make assumptions about stress model and vertical flow profile.
3. Integrate from $z = b$ to $z = \eta$ applying boundary conditions.
4. Yields a system in 2D for $q(x, y, t) = (h, hu, hv, \ldots, \ldots)^{\mathrm{T}}$.

Depth-averaged equations

e.g., incompressible, inviscid fluid $\mathbf{u} = (u, v, w)^{\mathrm{T}}$:

$$\begin{aligned}\nabla \cdot \mathbf{u} &= 0,\\ \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho &= 0,\\ \rho \frac{\partial \mathbf{u}}{\partial t} + \rho \nabla \cdot (\mathbf{u}\mathbf{u}^{\mathrm{T}}) + \nabla \cdot p\mathbf{I} - \rho \mathbf{g} &= 0,\end{aligned}$$

with boundary conditions at $z = b(x, y, t)$ and $z = \eta(x, y, t)$:

$$\begin{aligned}p|_{\eta} &= 0,\\ w|_{\eta} &= \frac{\partial \eta}{\partial t} + u|_{\eta} \frac{\partial \eta}{\partial x} + v|_{\eta} \frac{\partial \eta}{\partial y},\\ w|_{b} &= \frac{\partial b}{\partial t} + u|_{b} \frac{\partial b}{\partial x} + v|_{b} \frac{\partial b}{\partial y},\end{aligned}$$

Depth-averaged equations

e.g., incompressible, inviscid fluid (neglecting y)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uw)}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial w}{\partial t} + \frac{\partial (uw)}{\partial x} + \frac{\partial (w^2)}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0.$$

with boundary conditions at $z = b(x,y,t)$ and $z = \eta(x,y,t)$:

$$p|_\eta = 0,$$

$$w|_\eta = \frac{\partial \eta}{\partial t} + u|_\eta \frac{\partial \eta}{\partial x},$$

$$w|_b = \frac{\partial b}{\partial t} + u|_b \frac{\partial b}{\partial x},$$

Depth-averaged equations

integrate through the depth and apply boundary conditions:

$$\int_b^\eta \left[\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right] dz = 0,$$
$$\int_b^\eta \left[\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uw)}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right] dz = 0,$$
$$\int_b^\eta \left[\frac{\partial w}{\partial t} + \frac{\partial (uw)}{\partial x} + \frac{\partial (w^2)}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g\right] dz = 0.$$

$$p|_\eta = 0,$$
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Depth-averaged equations

integrate through the depth and apply boundary conditions: $\rightarrow$

$$\frac{\partial h}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} = 0,$$
$$\frac{\partial (h\bar{u})}{\partial t} + \frac{\partial}{\partial x}(h\overline{u^2} + \frac{1}{\rho}h\bar{p}) = -\frac{1}{\rho}p|_b \frac{\partial b}{\partial x},$$

where,

$$\bar{f} := \frac{1}{h}\int_b^\eta f\,dz.$$

Depth-averaged equations

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Note:

- No simplifying assumptions have been made yet.
- b.c.'s + incompressibility → evolution equation for h.
- averaging vertical dynamics → system is no longer closed.
- $\overline{u^2} = (\bar{u})^2 \Leftrightarrow u(x,z,t) = U(x,t)$

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The shallow water assumptions:

- $\overline{u^2} = (\bar{u})^2$
- $p(x,z,t) = \rho g\,(\eta(x,t) - z)$ (hydrostatic pressure)

The shallow water equations (swe)

$$\frac{\partial h}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} = 0,$$
$$\frac{\partial (h\bar{u})}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}^2 + \frac{1}{2}gh^2) = -gh\frac{\partial b}{\partial x},$$

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$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0,$$
$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}(hu^2 + \frac{1}{2}gh^2) = -gh\frac{\partial b}{\partial x},$$

where $u(x,t) = \bar{u}$.
Some properties:

- swe are a nonlinear hyperbolic system (wave propagation)
- swe lack dispersion
- hyperbolic systems present numerical challenges
- specialized numerical methods have been designed for such systems

In 2D:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0,$$
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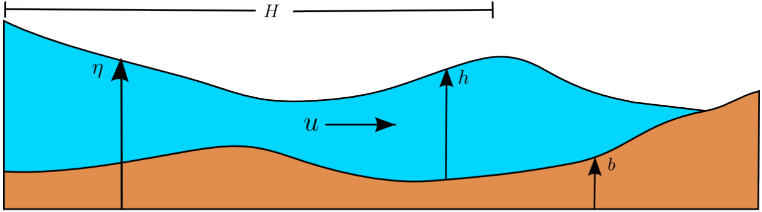
Solution vector: $q(x, y, t) = (h, hu, hv)^{\mathrm{T}}$

The shallow water equations (swe)

How accurate are the shallow water equations?

- "shallowness" assumption: $\epsilon = h/H \ll 1$.
- shallow water equations most accurate when $\epsilon \ll 1$.
- shallow water equations are often used when $\epsilon \not\ll 1$
- reliability of swe solutions depends on problem scales and degree of detail desired



Tsunamis

Tsunami initiation:

- Coseismic tsunamis
  - generated by fault motion and coincident alteration of the seafloor
  - spatial scale of seafloor displacement determines initial wave profile
  - wave energy can travel across entire oceans
- Landslides
  - submarine and terrestrial landslides
  - can produce very large-amplitude tsunamis
  - typically more localized than coseismic tsunamis

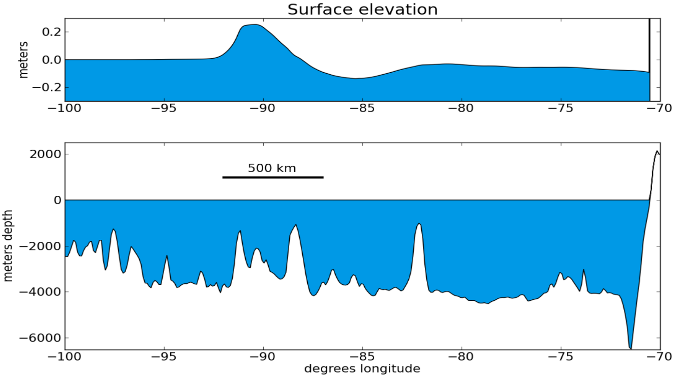
Tsunamis

Tsunami spatial scales:

- characteristic horizontal length-scale ≈ 100 km
- ocean depth < 10 km
- $\rightarrow$ tsunami initiation and propagation primarily a long-wave phenomenon (depth/wavelength $\ll 1$).
- transport of energy over long distances (ocean basins) is reasonably approximated by depth-averaged equations (*e.g.*, swe)

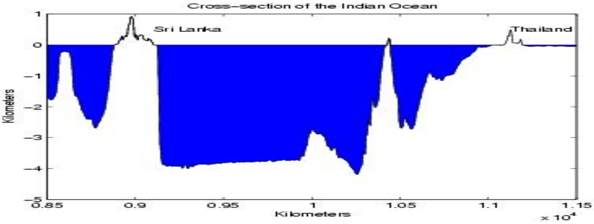
Tsunamis

Tsunami spatial scales:



Tsunami initiation:

- tsunamis begin as long-wavelength perturbation to the motionless ocean at rest
- motionless steady-state: $\eta = h + b \equiv 0$



Tsunamis

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- swe exactly govern motionless water over variable $b(x, y)$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = \frac{\partial \eta}{\partial t} - \frac{\partial b}{\partial t} = 0,$$
$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}(hu^2 + \frac{1}{2}gh^2) + \frac{\partial (huv)}{\partial y} = -gh\frac{\partial b}{\partial x},$$
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Tsunamis

Tsunami initiation:

- seafloor displacement alters $\eta(x,y,t)$: $(\eta_t = b_t)$
- creates potential energy in the form of hydraulic head: $\eta(x,y,t) \neq 0$
- steady-state balance is perturbed
- hydraulic head $\rightarrow$ horizontal momentum

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Hyperbolic systems

The swe belong to a larger class of PDEs with common properties

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written compactly:

$$q_t + f(q)_x + g(q)_y = \psi(q, x, y)$$

Hyperbolic systems

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$$
\begin{aligned}
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\end{aligned}
$$

written compactly:

$$q_t + f(q)_x + g(q)_y = \psi(q, x, y)$$

Hyperbolic systems

- Hyperbolic conservation law (in 2D):

$$q_t + f(q)_x + g(q)_y = \psi(q, x, y),$$

where $\quad q(x,y,t) \in \mathbb{R}^m, \quad f(q), g(q) \in \mathbb{R}^m, \quad \psi \in \mathbb{R}^m.$

- General hyperbolic system

$$q_t + A(q)q_x + B(q)q_y = \psi(q, x, y),$$

where $\quad A(q), B(q) \in \mathbb{R}^{m \times m}$

  - eg: hyperbolic conservation law:

$$f(q)_x = f'(q)q_x = A(q)q_x$$

Hyperbolic systems

Conservation law (PDE) is derived from a weaker law over region Ω:

$$\frac{d}{dt}\int_{\Omega} q + \int_{\partial\Omega} [f(q), g(q)] \cdot \vec{n} = \int_{\Omega} \psi$$

- Example: Rectangle $\Omega = \mathcal{C} = [x_1, x_2] \times [y_1, y_2]$ with area $\Delta x \Delta y$:

$$\frac{d}{dt} Q\, \Delta x \Delta y + [F_2(t) - F_1(t)]\, \Delta y + [G_2(t) - G_1(t)]\, \Delta x = \Psi\, \Delta x \Delta y$$

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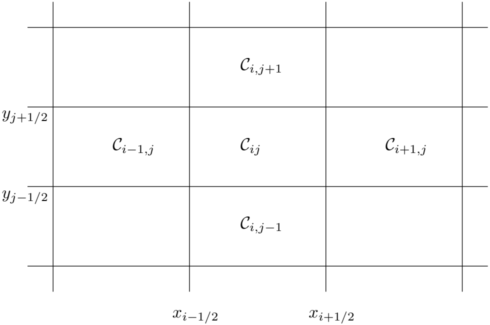
- Example: Rectangle $\Omega = \mathcal{C} = [x_1, x_2] \times [y_1, y_2]$ with area $\Delta x \Delta y$:

$$\frac{d}{dt}Q + \frac{1}{\Delta x}\left[F_2(t) - F_1(t)\right] + \frac{1}{\Delta y}\left[G_2(t) - G_1(t)\right] = \Psi$$

- Logically Rectangular Finite Volume Grid.

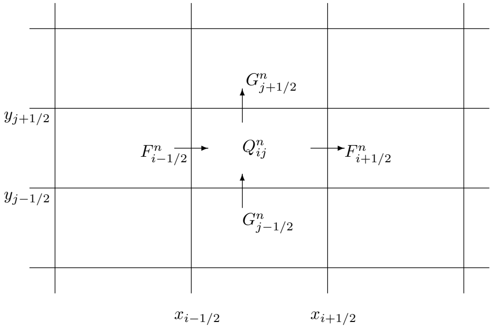


- Solution is piecewise constant:

$$Q_{ij}^n \approx \frac{1}{|\mathcal{C}_{ij}|} \int_{\mathcal{C}_{ij}} q(x, y, t^n)\, dx$$

Finite volume methods

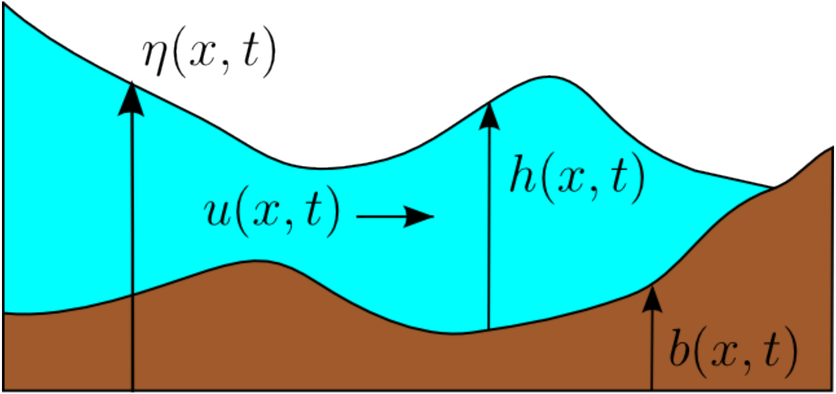
- Logically Rectangular Finite Volume Grid.



- Solution is updated analogous to integral form:

$$Q^n_{ij} \to Q^{n+1}_{ij}$$

Finite volume methods



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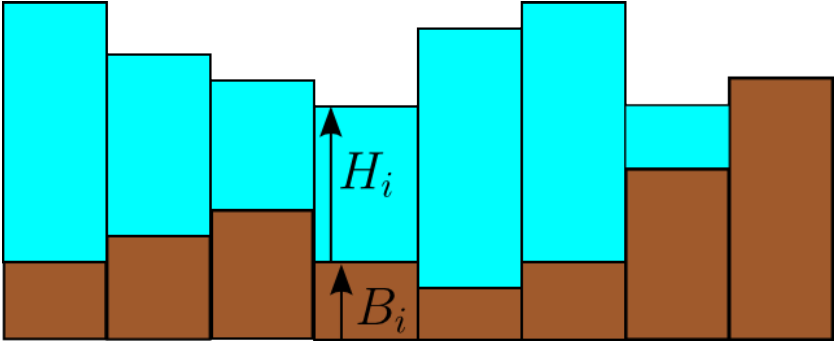

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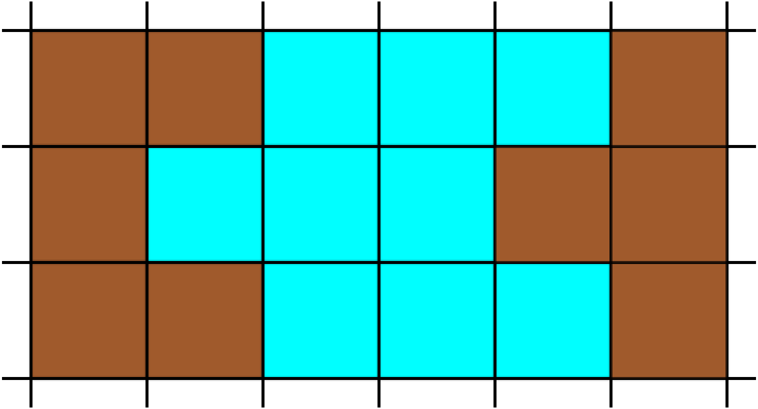


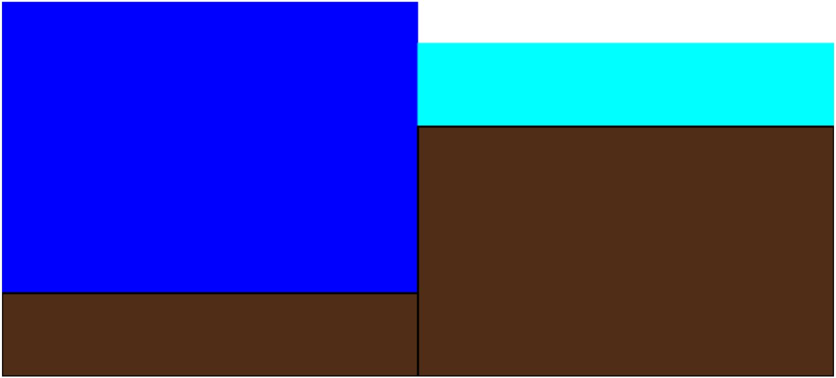
Finite volume methods

- GeoClaw is based on Godunov methods
- numerical update determined by solving Riemann problems
- *e.g.*, between cells $\mathcal{C}_{ij}$ and $\mathcal{C}_{i-1,j}$, solve swe from $t^n \to t^{n+1}$subject to following I.C.:

$$q(x,y,t^n) = \begin{cases} Q^n_{i-1,j} = \quad \text{if} \quad x < x_{i-1/2} \\ Q^n_{i,j} = \quad \text{if} \quad x > x_{i-1/2} \end{cases}$$

- resulting solution at $t = t^{n+1}$, in cell $\mathcal{C}_{i,j}$, is averaged to determine $Q^{n+1}_{i,j}$
- repeat

Finite volume methods



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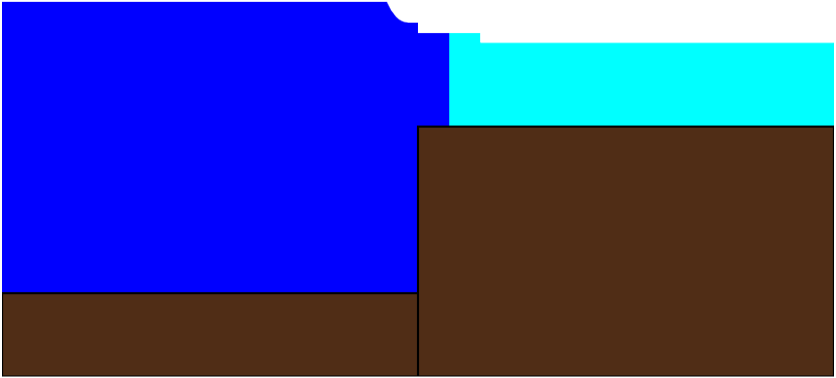

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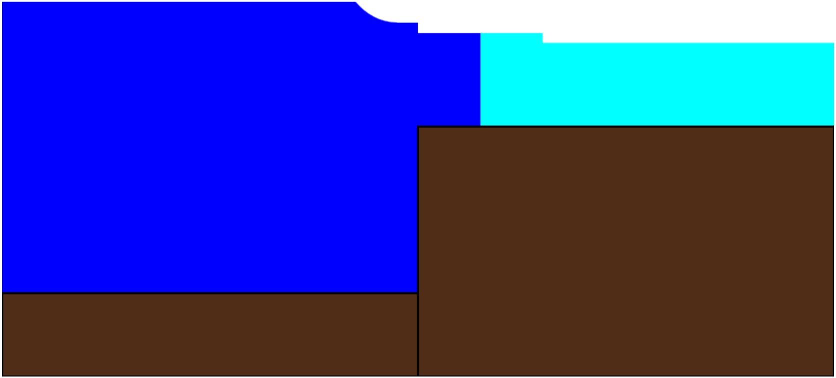

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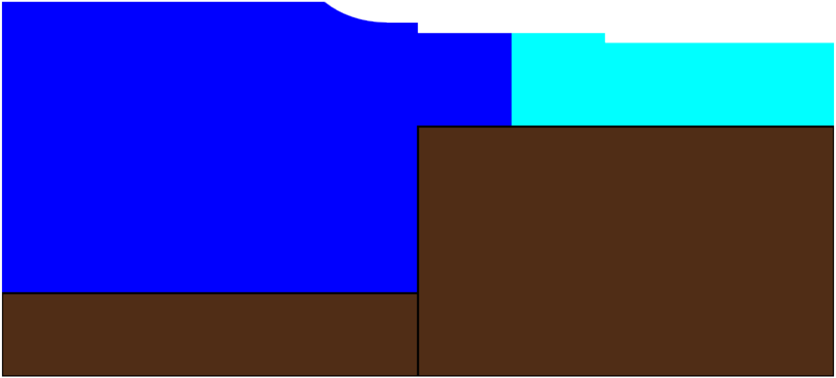

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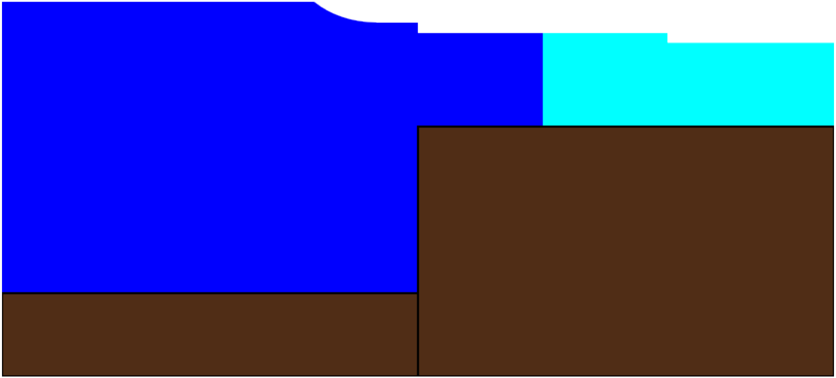

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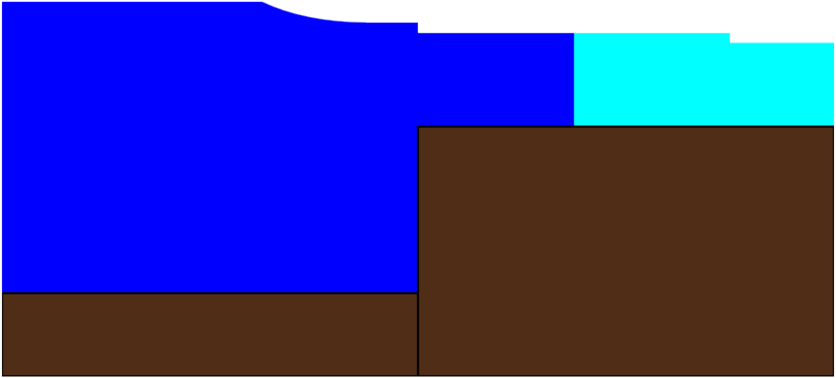

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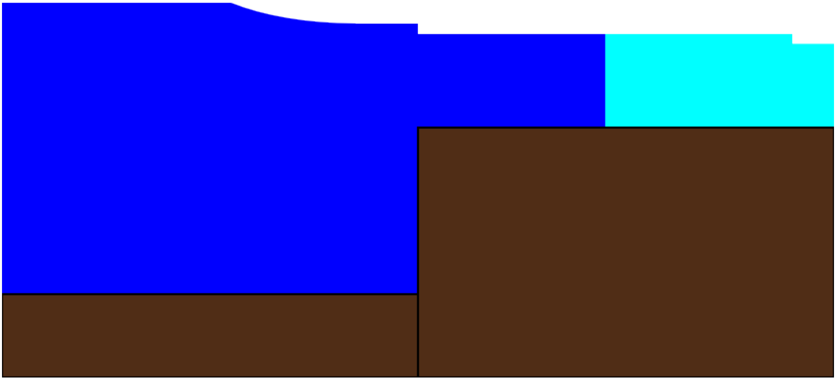

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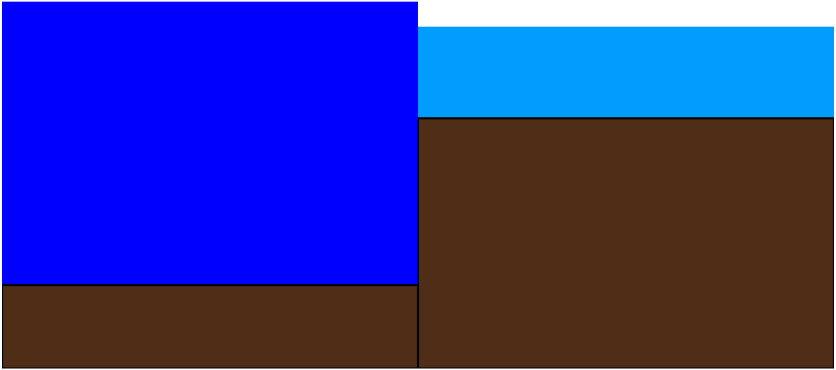


Tsunami spatial scales

swe:

$$q_t + f(q)_x = \psi(q),$$

Jacobian: $f(q)_x = f'(q)q_x$

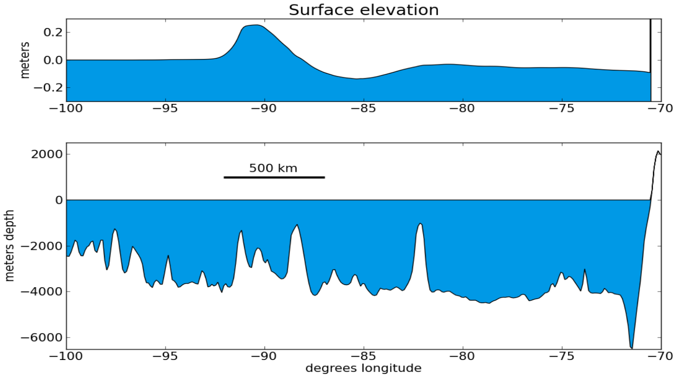
$$q_t + f'(q)q_x = \psi(q),$$

waves travels at the speed of the eigenvalues of $f'(q)$

for swe $\lambda_{1,2,3} = u \pm \sqrt{gh}, u$

- Note: $h = 5$ km $\rightarrow \sqrt{gh} > 200$ m/s
- $h = 10$ m $\rightarrow \sqrt{gh} < 10$ m/s

Tsunami spatial scales



Implications:

- long-waves ≈ 100km compress near shore
- variable-depth (bathymetry) strongly focuses waves
- need higher-resolution ≈ 10 m grids near shore

Tsunami spatial scales

Tsunami modeling requires resolving extreme multiple scales

- Global-scale simulation domain.
- Deep-ocean propagation: wavelength ≈ 100 km.
- waves are localized at any given time.
- waves propagation throughout the domain.
- near-shore wave compression & topographic features $\Rightarrow$ meter-scale resolution needed for inundation modeling.
- grid resolution is highly temporal-spatially dependent!

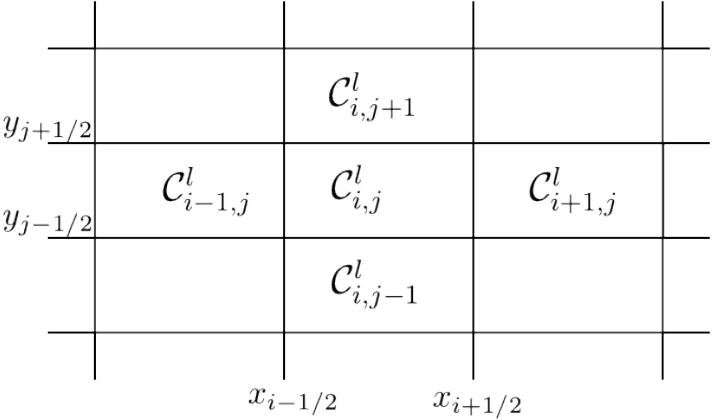
$\Rightarrow$ AMR

R.J. LeVeque, D.L. George and M.J. Berger, 2011: Tsunami modeling with adaptively refined finite volume methods, *Acta Numerica*, 20: 211-289, Arieh Iserles, ed.

Adaptive mesh refinement (AMR)

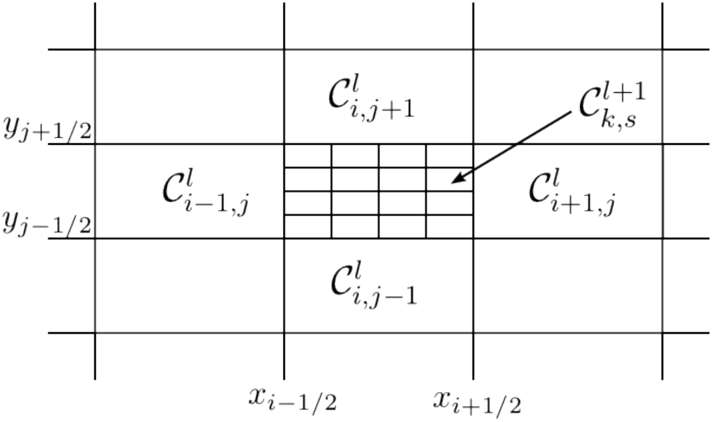
- AMR provides multiple grid resolutions during a computation
- multiple levels $l = 1, \ldots, L$ of nested grids resolve waves
- grid arrangement evolves with the solution
- AMR is patch-based (logically rectangular Cartesian grids)
- goal is to optimally (efficiency and accuracy) accommodate spatially and temporally varying features in the solution

Level l finite volume grid:



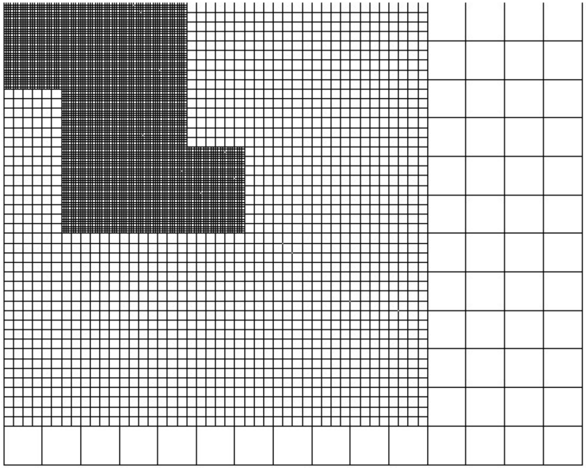
Adaptive mesh refinement (AMR)

level-l cell $\mathcal{C}^l_{i,m}$ is refined to a level-$(l+1)$ grid



Adaptive mesh refinement (AMR)

patches of multiple levels of grids evolve with the solution



Numerical methods and software

Clawpack software:

- open source software package for general hyperbolic systems;
- finite volume methods on logically rectangular grids;
- high-resolution (2nd-order TVD) shock-capturing methods;
- Godunov-type methods utilize Riemann solvers;
- block-structured adaptive mesh refinement (AMR);
- Available @ `www.clawpack.org`:

Numerical methods and software

GeoClaw: subset of Clawpack for problems in geophysics.
tailored to free-surface flows over topoography:

- AMR schemes tailored to free-surface flows, *e.g.*,
  - preserve steady states (*well-balanced*)
  - stable near wet-dry interface
  - prevent introduction of hydraulic head (energy)
- Riemann solvers for inundation and steady-states, *e.g.*,
  - resolve steady-state perturbations (*well-balanced*)
  - capture moving wet-dry interface over topography
  - satisfy physical admissibility requirements
- user tools developed for geophysical data, *e.g.*,
  - dynamic conservative integration of multiple DEMs
  - dynamic topography motion (from *e.g.*, fault params)
- grid mappings to sphere or ellipse for global-scale problems

M.J. Berger, D.L. George, R.J. LeVeque and K.T. Mandli, 2011: The GeoClaw software for depth-averaged flows with adaptive refinement, *Adv. Water. Resources*, 34: 1195-1206.