Basics of surface wave simulation

L. Ridgway Scott

Departments of Computer Science and Mathematics, Computation Institute, and Institute for Biophysical Dynamics,

University of Chicago

KdV equation

The equation balances nonlinear advection with dispersion:

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (1)$$

(Korteweg & de Vries 1895, Boussinesq [Bou77, p. 360]); has a family of solutions

$$u(t,x) = \frac{c}{2}\operatorname{sech}^2\left(\tfrac{1}{2}\sqrt{c}(x - ct)\right)$$

which move at constant speed c without change of shape.

Matches observations of J. Scott Russell (1845).

BBM equation

An equivalent equation that balances nonlinear advection with dispersion is

$$u_t + u_x + 2uu_x - u_{xxt} = 0 \tag{2}$$

(Peregrine 1964, Benjamin, Bona and Mahoney 1972) which has similar solutions [ZWG02]

$$u(t,x) = \frac{3}{2}a\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{a}{a+1}}\left(x-(1+a)t\right)\right)$$

The BBM equation is better behaved numerically.

$$u_t = -\left(1 - \frac{d}{dx^2}\right)^{-1}\frac{d}{dx}\left(u+u^2\right) = B\left(u+u^2\right) \tag{3}$$

Solitary wave exercises

The KdV and BBM equations can be compared by using the underlying advection model $u_t + u_x = 0$.

Thus we can swap time derivatives for (minus) space derivatives: $u_t \approx -u_x$. This suggests the near equivalence of the terms $u_{xxx} \approx -u_{xxt}$.

Derive the solitary wave solution for

$$u_t + u_x + 2uu_x + u_{xxx} = 0 \tag{4}$$

and compare this with the solitary wave for BBM

Show that the two forms converge as the wave amplitude goes to zero.

Tsunami controversy

Terry Tao says ”solitons are large-amplitude (and thus nonlinear) phenomena, whereas tsunami propagation (in deep water, at least) is governed by low-amplitude (and thus essentially linear) equations. Typically, linear waves disperse due to the fact that the group velocity is usually sensitive to the wavelength; but in the tsunami regime, the group velocity is driven by pressure effects that relate to the depth of the ocean rather than the wavelength of the wave, and as such there is essentially no dispersion, thus creating traveling waves that have some superficial resemblance to solitons, but arise through a different mechanism.
It is true, though, that KdV also arises from a shallow water wave approximation. The main distinction seems to be that the shallow water equation comes from assuming that the pressure behaves like the hydrostatic pressure, whereas KdV arises if one assumes instead that the velocity is irrotational (which is definitely not the case for tsunami waves).”

Tsunami analysis

We know that tsunamis must have long wave lengths since their amplitude is small.

Otherwise, no devastating amount of energy (height times width) can be transmitted.

The time scale of tsunami impact is minutes, not hours as occurs in hurricane storm surge.

So the wave needs to be long and fast.

KDV/BBM provide such a mechanism.

Key question: what causes such a long wave to form?

Modeling question: does KdV require flow to be irrotational?

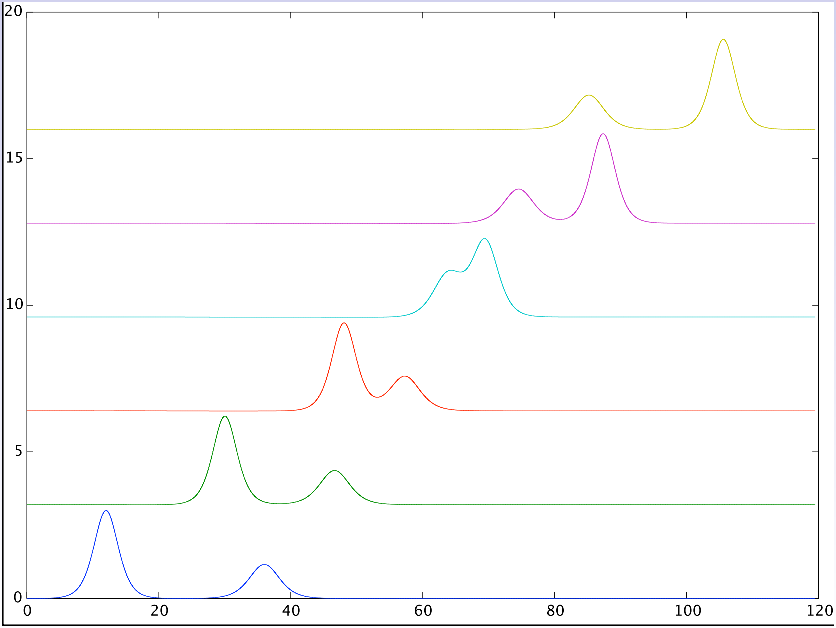
There are many other types of solutions to KdV/BBM.

- soliton interactions
- dispersion
- compare: no dispersion
- dispersive shock waves [EKL12]

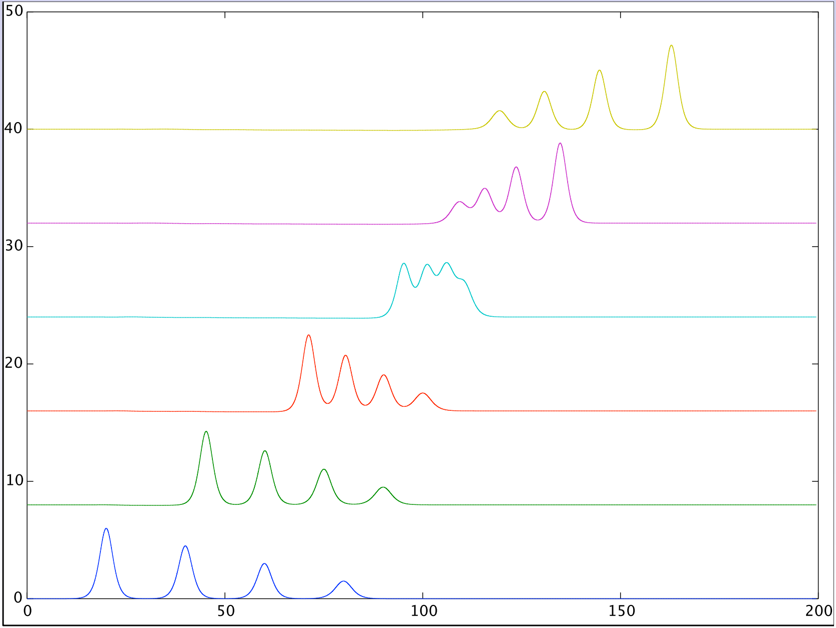
Exercise: explore different initial states

Compare with data [Gre61].

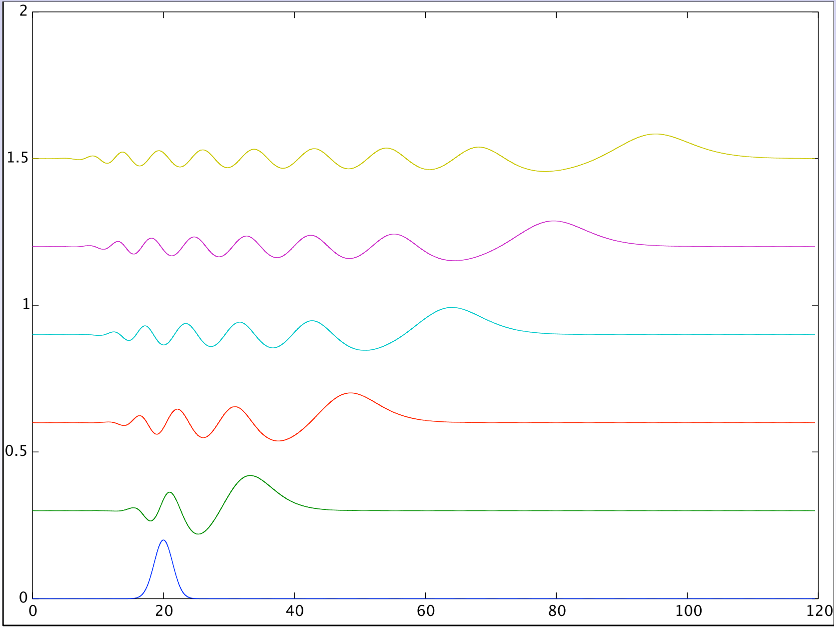
Soliton interaction (BBM)



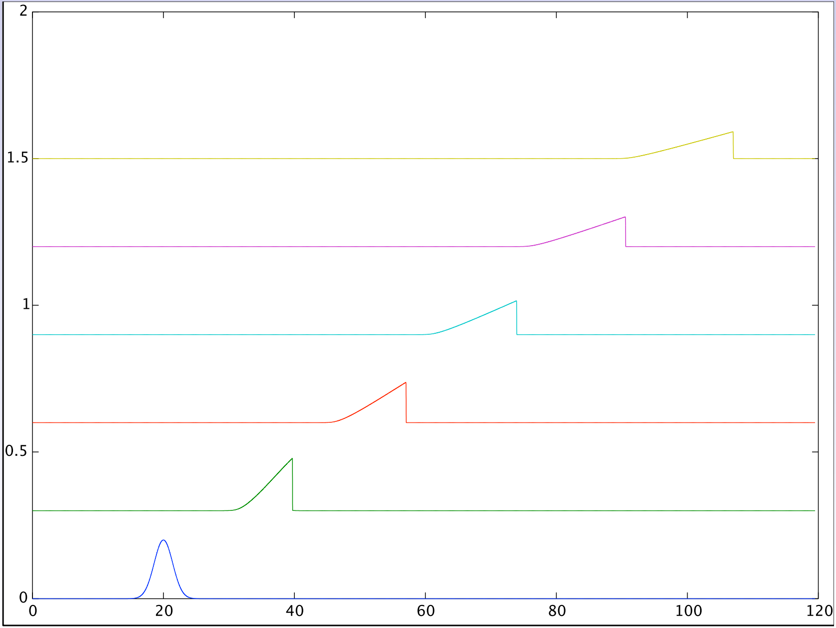
Multi-soliton interaction (BBM)



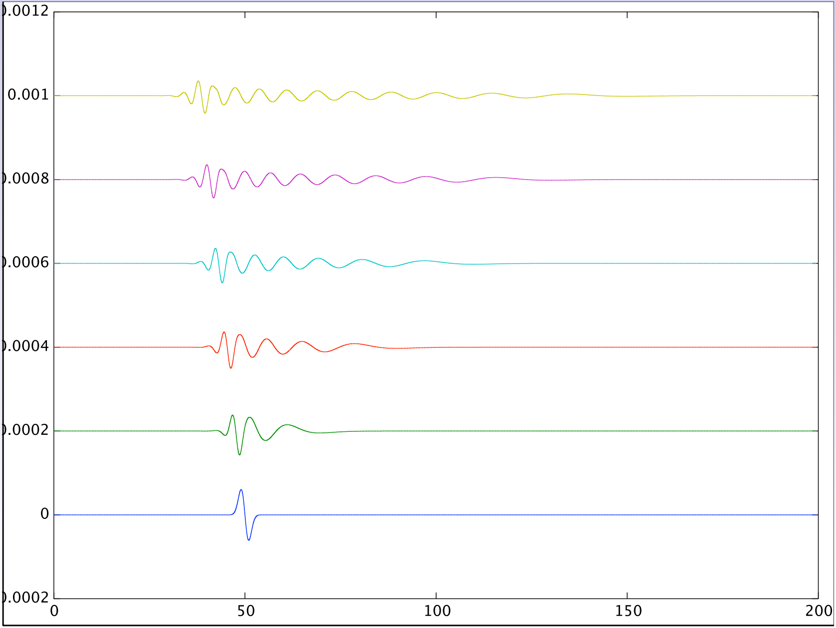
Gaussian dispersion (BBM)



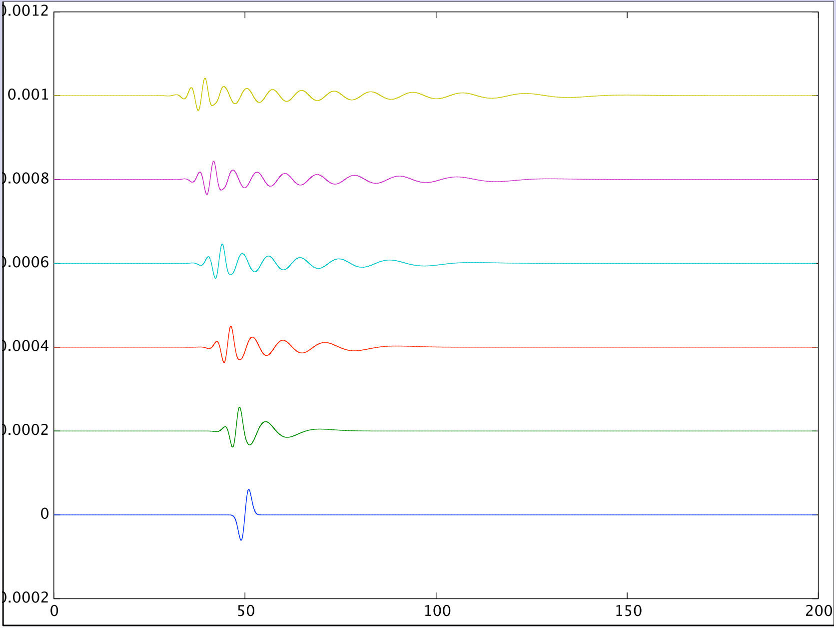
Compare Gaussian with no dispersion



Leading depression

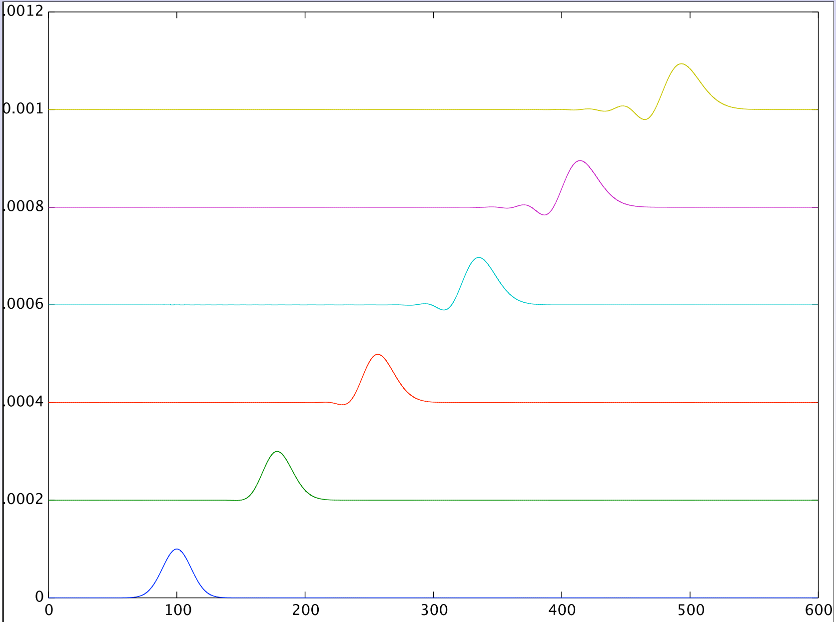


Trailing depression=-leading depression



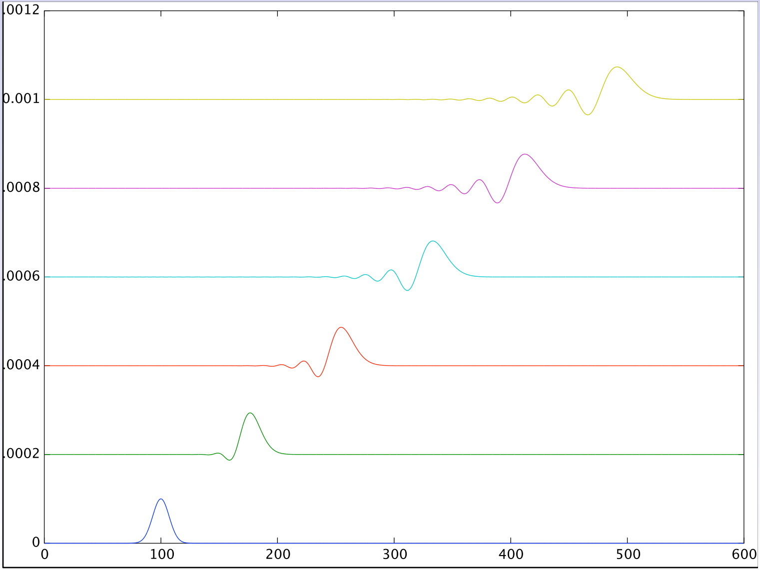
Very long waves are mostly linear

$y_o = a * (exp(-c * (r - s)^2)), a = .0001, c = .004$



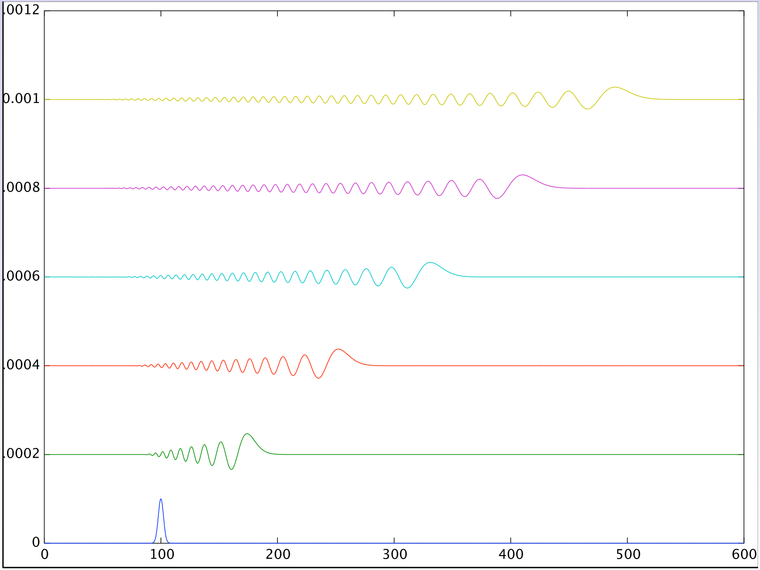
Less long waves are more dispersive

$y_o = a * (exp(-c * (r - s)^2)), a = .0001, c = .01$



Shorter waves are very dispersive

$$y_o = a * (exp(-c * (r - s)^2)), a = .0001, c = .1$$



Software issues

One-D problems are simple,

so you can use simple software systems, e.g., Matlab/octave.

Consider the time-stepping scheme for the advection problem

$$0 = u_t + f(u)_x$$

given by

$$u_{i+1,j} = u_{i,j} - \frac{\Delta t}{\Delta x}\left(f(u)_{i,j} - f(u)_{i,j-1}\right)$$

Tricks with octave: filter

The “filter” command performs finite difference specified by vectors “b” and “a”:

```
b=[ +1 -1 ];
a=[ 1 ];
xr=dx*[1:1000000];
yu=exp(-(.05*(xr-50)).^ 2);
...
cfl=dt/dx
for k=1:nts
yu=yu-cfl*filter(b,a,yu + yu .* yu);
end
```

Details about filter

Typing “help filter” in octave produces

- - Loadable Function: y = filter (B, A, X)

Return the solution to the following linear, time-invariant difference equation:

$$\sum_{k=0}^{N} a(k+1)y(n-k) = \sum_{k=0}^{M} b(k+1)x(n-k)$$

where N=length(a)-1 and M=length(b)-1.

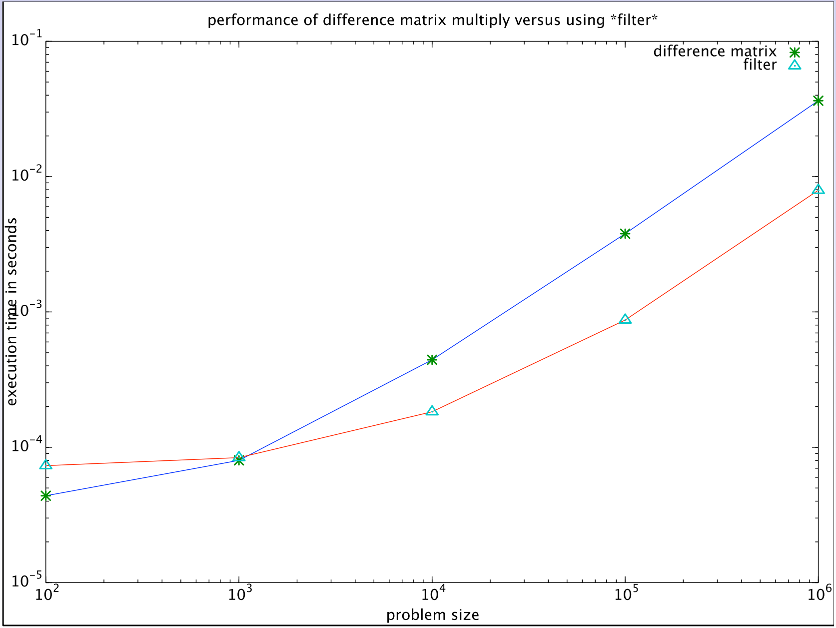
Equivalent difference matrix

Using “filter” is equivalent to multiplying by the sparse matrix “fod” defined as follows:

```
tdx=2*dx;
kc=0;
for k=2:nr;
kc=kc+1; hiv(kc)=k; hjv(kc)=k-1; hsv(kc)=-(1/tdx);
end
for k=1:nr;
kc=kc+1; hiv(kc)=k; hjv(kc)=k; hsv(kc)=+(1/tdx);
end
fod=sparse(hiv,hjv,hsv);
```

Key is to create a sparse matrix.

Performance of difference matrix vs. filter



Using filter for boundary value problems

There are some challenges is using “filter” to solve two-point boundary value problems.
Suppose we want to solve

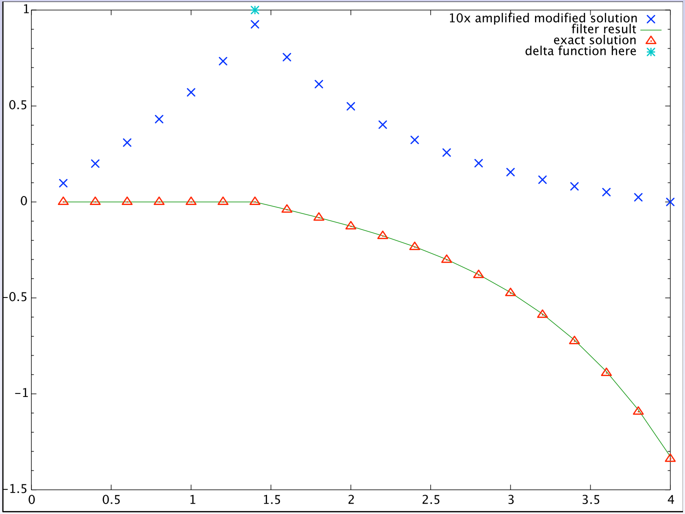
$$\alpha u - u_{xx} = f \text{ on } x_0 < x < x_1, \quad u(x_i) = 0 \text{ for } i = 0, 1.$$

We can do this via

```
b=[ 0 1 ];
a=[0 alfa 0]+(1/(dx*dx))*[-1 2 -1 ];
u=filter(b,a,f);
```

However, filter assumes a boundary condition $u(x_0) = u'(x_0) = 0$.

Behavior of filter



$u(x_0) = u'(x_0) = 0$ gives different solution
need to modify it by a homogeneous solution to get the correct boundary conditions.

BBM model has been tested against laboratory experiments [BPS81]

Key parameter for model is the Stokes number

$$S = \frac{a\lambda^2}{d^3},$$

where

- a is the wave amplitude,
- λ is the wave length, and
- d is the water depth.

Example: $a = 1$, $\lambda = 10^6$, $d = 10^4$ (meters) $\implies S = 1$.

Modeling issues

For $S < 1$, the data in [BPS81] suggest that the linear dispersive model is as accurate as nonlinear dispersive

For larger $S > 10$ the model experiences greater than 10% errors.

Question: how important is dispersion in such simulations?

The results in [BPS81] also suggest the importance of dissipation due to bottom friction for small values of depth d.

Comparing two models

What about the different nonlinear, dispersive models: KdV versus BBM?

Possible to give analytic comparisons [BPS83].

Compare the model [BC99]

$$u_t + u_x + 2uu_x + u_{xtt} = 0$$

Time scales

The time scale for these models is

$$t = \sqrt{\frac{d}{g}}$$

where d is the water depth and g is the acceleration due to gravity:

$$g = 9.81 \text{ meters/second}^2 \approx 32.2 \text{ feet/second}^2$$

For $d \approx 10^4$ meters, this means $t \approx \frac{1}{2}$ minute.
For $d \approx 10$ meters, this means $t \approx$ one second.
Thus we can think that the time scale of interest is a small number of seconds, less than a minute.

Wave speeds

For small amplitude waves, the wave speed in nondimensional coordinates is essentially 1.

That means the wave speed is the length scale divided by the time scale.

Therefore the speed c is given by

$$c \approx \frac{d}{t} = \frac{d}{\sqrt{d/g}} = \sqrt{dg}$$

For $d \approx 10^4$ meters, this means

$c \approx 313$ meters/second ≈ 700 miles/hour.

(speed of sound at sea level is 343.2 m/s)

For $d \approx 10$ m, $c \approx 9.9$ m/s ≈ 22 miles/hour.

For reference, Usain Bolt has run 100 meters at an average speed of 10.44 meters per second

Comparing nonlinearity and dispersion

Suppose we have a wave of amplitude $\alpha = a/d$ and wave length $\lambda = L/d$.

That is, $u(x) \approx \alpha\phi(x/\lambda)$. Then KdV looks like

$$u_t + u_x(1 + \alpha + \lambda^{-2}) = 0 \tag{5}$$

We have seen that tsunamis have small amplitude: $\alpha \approx 10^{-4}$.

This suggests that nonlinearity has little effect.

But how big can the wave length be?

Known inundation by tsunamis places a limit on L.

Wave length

The character of the wave propagation depends on wave length.

For a fixed mass of water, a smaller amplitude requires a longer wave length.

For a 1 meter wave, a length $L = 10$ kilometers ($\lambda = L/d = 1$) yields a catastrophic wave.

Historic 10 meter tsunamis might have $\lambda/d = 100$.

Hilo Bay, Big Island, Hawaii

Compare the 1960 Chilean-generated tsunami effect on Hilo

Hilo, HI – Google Maps 11/14/12 6:58 AM

To see all the details that are visible on the screen, use the "Print" link next to the map.

Influence of depth

The KdV/BBM models do not include varying depth.
However, we can get a sense of how effects change as d becomes smaller.
Recall our notation: wave amplitude $\alpha = a/d$ and wave length $\lambda = L/d$.

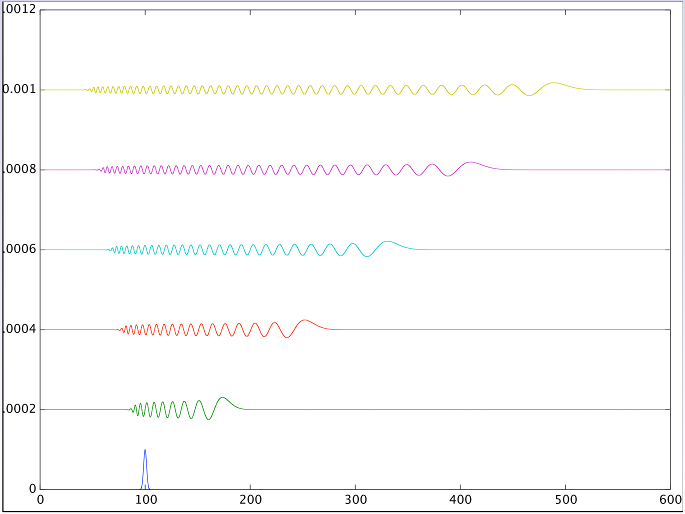
That is, $u(x) \approx \alpha\phi(x/\lambda)$. Then KdV looks like

$$u_t + u_x(1 + \alpha + \lambda^{-2}) = 0$$

As d decreases, the nonlinear term increases and the dispersion term decreases.

Long-time integration of Gaussian

Here we take typical amplitude (1 meter) and a wave length of about 25 kilometers in a depth of 10^4 meters.



Wave volume

Exercise: compute what the area under the curve is for the first wave in the dispersive wave train.

Exercise: compute the evolution of a wave that has a negative Gaussian (depression) initially.

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