Wall penetration

Notes on Quantum Mechanics

http://quantum.bu.edu/notes/QuantumMechanics/WallPenetration.pdf
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In a one-dimensional forbidden region of infinite extent extending from $x = 0$ rightward, the wavefunction has the form

$$\psi(x) = e^{-\kappa x},$$

where

$$\kappa = \sqrt{2\, m\, |E - V(x)|} \Big/ \hbar.$$

> Show that this expression is correct. Hint: Show that $\psi(x)$ is an eigenfunction of momentum, and then use the eigenvalue to express kinetic energy in terms of κ.

This means that penetration goes down with mass, m, and also with the height of the wall, $|E - V(x)|$. We can quantitfy this effect by calculating how far into the forbidden region the wavefunction must extend for its value to be reduced to $1/e$ of its value at entry into the forbidden region (at $x = 0$). The result is

$$x_0 = \frac{\hbar}{\sqrt{2\, m\, |E - V(x)|}}.$$

> Show that this expression is correct.

Here is a *Mathematica* function to evaluate this penetration distance.

```
x0[height_, m_] := ħ / Sqrt[2 m height] /. {
    ħ → PlanckConstantReduced} // Evaluate
```

Here are the penetrations of an electron into a wall of height 1, 10, and 100 eV.

```
x0[{1, 10, 100} eV, m] //. {
   m → ElectronMass,
   eV → ElectronCharge × Volt,
   Volt → Joule / Coulomb,
   Joule → Kilogram Meter^2 Second^-2,
   Meter → 10^10 Å
  } // PowerExpand
```

```
{1.95192 Å, 0.61725 Å, 0.195192 Å}
```

> Plot, on the same set of axes, $\psi(x)$ corresponding to these three wall heights, from $x = 0$ to $x = 3$ Å.

Here are the penetrations of a proton into a wall of height 1, 10, and 100 eV.

```
x0[{1, 10, 100} eV, m] //. {
   m → ProtonMass,
   eV → ElectronCharge × Volt,
   Volt → Joule / Coulomb,
   Joule → Kilogram Meter^2 Second^-2,
   Meter → 10^10 Å
  } // PowerExpand
```

```
{0.045552 Å, 0.0144048 Å, 0.0045552 Å}
```

> Evaluate the penetrations of a hydrogen molecule into a wall of height 1, 10, and 100 eV.
> Answer: 0.0322 Å, 0.0102 Å, and 0.00322 Å